External approximation of nonlinear operator equations

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Based upon an external approximation scheme for the underlying Banach space, a nonlinear operator equation is approximated by a sequence of coercive problems. The equation is supposed to be governed by the sum of two nonlinear operators acting between a reflexive Banach space and its dual. Under suitable stability assumptions and if the underlying operators can be approximated consistently, weak convergence of a subsequence of approximate solutions is shown. This also proves existence of solutions to the original equation.

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1. Introduction

Let V be a real reflexive Banach space with its dual V^*. We are concerned with the approximate solution of the following problem: *For* $f \in V^*$ *find* $u \in V$ *such that*

$$Au + Bu = f \quad \textit{in } V^* . \tag{1}$$

Here, $A : V \to V^*$ and $B : V \to V^*$ are given operators. A standard situation is A being monotone and hemicontinuous, B being a strongly continuous perturbation of A and $A + B$ being coercive.

Many boundary value problems for quasilinear partial differential equations arising in physics, fluid mechanics and other areas of application can be formulated as (1) (see, e.g., the monographs [6, 8, 19] and the references therein). For their approximate solution, often Galerkin-type or finite difference methods are employed. The study of all these methods can be unified by considering so-called external approximation schemes. This concept covers in particular finite differences, nonconforming finite elements as well as fully discrete finite element methods with quadrature (see [15, 16] for several examples).

External approximation schemes have been studied in [13] in the context of the convergence of finite difference approximations for quasilinear partial differential equations and A-proper operators (see [16], [19, Ch. 35] for introductions into the concept of external approximation schemes). Later on, extensions of the results obtained in [13] have been studied in [17, 18], the focus being on the equivalence of unique solvability of the original and approximate problem.

External approximations have also been studied for eigenvalue problems and the phenomenon of superconvergence (see [10–12]) and, more recently, for the approx-

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imation of variational problems in spaces of piecewise constant functions (see [5]). The concept of external approximation has further been employed in [15] for studying different numerical methods for solving the Navier-Stokes problem that is, in the stationary case, indeed of the type (1) with A being linear and strongly positive, B being strongly continuous and $A + B$ being coercive.

The assumptions on the approximating operators studied so far in [13, 17, 18] (in particular, an inverse stability that would follow from stronger uniform monotonicity assumptions) are, however, different from the assumptions on which our studies are based. Indeed, we try to apply the concept of external approximation in order to generalise the often found standard situation for (1) with A being monotone and hemicontinuous, B being strongly continuous and $A + B$ being coercive. It is well-known that in this situation Brézis' theorem on pseudomonotone operators provides existence of solutions to (1). So, our approach is not based upon the concept of A-properness (see [9] and the references cited therein for a discussion of this concept) but on weaker assumptions yielding then only convergence of a subsequence in a weak sense. In particular, we do not assume well-posedness of the approximate problems but only solvability and, in general, we do not have a continuous inverse of the approximate operator (for this case, see also the exhaustive work [3]).

Another approach for studying approximations of monotone operator equations employs projection methods as in [1, 2, 6], which can be interpreted as internal approximation schemes. The essential advantage of external approximation schemes, however, is that the function spaces approximating V need not to be subspaces of V. This allows much more flexibility in the choice of the numerical method and often simplifies the numerical analysis.

We should mention that, besides the concept of external approximation, also the concept of discrete convergence and discrete approximation as introduced in [14], which goes without prolongation and restriction operators, provides a frame for considering rather different numerical methods in a unifying way. This concept has been applied in [7] to the study of nonlinear operator equations based upon the notion of approximation-regular operators, which is a generalisation of A-properness. The results in [7] also cover quasilinear elliptic problems (leading to a coercive and strictly monotone operator equation of the type (1)) under perturbation of the domain or coefficients.

The main result in this paper will be a general convergence result for the case that there is a stable and admissible external approximation scheme and that A, B, f in (1) can be approximated in a consistent way. The required assumption on the sequence of approximations of A can be seen as a discrete analogue of the property (M). Moreover, our convergence result is based upon an a priori estimate for the sequence of approximate solutions which follows from uniform coercivity. The Galerkin method in the standard situation as described above is shown to be a special case of our approximation scheme.

The paper is organised as follows: In Section 2, we describe the external approximation scheme, present the necessary notation, and prove some results on the consistency and solvability of the approximation. The main result is stated and proven in Section 3. Section 4 provides an example.

2. The external approximation scheme

For a normed space X, we always denote its norm by $\|\cdot\|_X$, its dual space by X^* with standard norm $\|\cdot\|_{X^*}$ and the dual pairing by $\langle\cdot,\cdot\rangle$.

Let V be a given real reflexive Banach space. Let $\mathscr{H}$ be a countable infinite sequence of indices and let $\{(V_h, p_h, r_h)\}_{h\in\mathscr{H}}$ be a sequence of real normed spaces V_h, prolongation operators $p_h : V_h \to F$ and restriction operators $r_h : V \to V_h$. The prolongation operators are assumed to be linear and bounded. Here, F is a suitably chosen real reflexive Banach space such that there is a so-called synchronisation operator $\omega : V \to F$ that is linear, bounded and injective. The family $\{(V_h, p_h, r_h)\}_{h\in\mathscr{H}}$ then is said to be an *external approximation scheme for* V. In some situations, the restriction operators are only defined on a dense subset of V but can be extended on V (see [15, Prop. 3.1 on p. 30], [16, Prop. 4 on p. 28]).

Definition 2.1: An external approximation scheme $\{(V_h, p_h, r_h)\}_{h\in\mathscr{H}}$ for V is said to be *stable* iff there is a constant $c > 0$ such that for all $h \in \mathscr{H}$

$$\|p_h v_h\|_F \le c\|v_h\|_{V_h} \quad \forall v_h \in V_h\,.$$

It is said to be *admissible* iff it fulfills
(i) the compatibility condition:

$$p_h r_h v \to \omega v \text{ in } F\ (h \in \mathscr{H}) \quad \forall v \in V\,,$$

(ii) the synchronisation condition: for any subsequence $\mathscr{H}' \subseteq \mathscr{H}$ of indices and $\{v_h\}_{h\in\mathscr{H}'} \in \{V_h\}_{h\in\mathscr{H}'}$, $g \in F$ with $p_h v_h \rightharpoonup g$ in F $(h \in \mathscr{H}')$ there is an element $v \in V$ such that $\omega v = g$.

Note that the use of the foregoing notions is not consistent in the literature. We now consider the sequence of approximate problems: *For* $f_h \in V_h^*$ *find* $u_h \in V_h$ *such that*

$$A_h u_h + B_h u_h = f_h \quad \textit{in } V_h^*\,. \tag{2}$$

Here, $\{A_h\}_{h\in\mathscr{H}}$, $\{B_h\}_{h\in\mathscr{H}}$ and $\{f_h\}_{h\in\mathscr{H}}$ are sequences of operators and functionals approximating A, B and f, respectively.

Definition 2.2: Let $\{(V_h, p_h, r_h)\}_{h\in\mathscr{H}}$ be a stable and admissible external approximation scheme for V. A sequence $\{(A_h, B_h, f_h)\}_{h\in\mathscr{H}}$ of operators $A_h : V_h \to V_h^*$, $B_h : V_h \to V_h^*$ and functionals $f_h \in V_h^*$ is said to be a *consistent approximation of* (A, B, f) iff for any subsequence $\mathscr{H}' \subseteq \mathscr{H}$ and any $\{v_h\}_{h\in\mathscr{H}'} \in \{V_h\}_{h\in\mathscr{H}'}$, $v \in V$ with $p_h v_h \rightharpoonup \omega v$ in F $(h \in \mathscr{H}')$ there holds

(i) if there is an element $g \in F$ such that

$$\langle A_h v_h, r_h w\rangle \to \langle g, w\rangle \quad \forall w \in V\,, \quad \limsup_{h\in\mathscr{H}'} \langle A_h v_h, v_h\rangle \le \langle g, v\rangle$$

then $Av = g$ in V^*;

(ii) $\quad \langle B_h v_h, r_h w\rangle \to \langle Bv, w\rangle \quad \forall w \in V\,, \quad \liminf_{h\in\mathscr{H}'} \langle B_h v_h, v_h\rangle \ge \langle Bv, v\rangle\,;$

(iii) $\quad \langle f_h, r_h w\rangle \to \langle f, w\rangle \quad \forall w \in V\,, \quad \limsup_{h\in\mathscr{H}'} \langle f_h, v_h\rangle \le \langle f, v\rangle\,.$

We remark that condition (i) in the foregoing definition is a discrete counterpart of the property (M) (see [8, p. 173]). A standard example is given by

Proposition 2.3: *Let* $\{V_h\}_{h\in\mathscr{H}}$ *be a Galerkin scheme for* V, $p_h : V_h \to V$ *be the identity and* $r_h : V \to V_h$ *such that* $r_h v$ *is a best approximation of* $v \in V$ *in* V_h. *Then* $\{(V_h, p_h, r_h)\}_{h\in\mathscr{H}}$ *is a stable and admissible external approximation scheme with* $F = V$. *Let* $A_h = p_h^* A p_h$ *(with* $p_h^* : V^* \to V_h^*$ *denoting the dual operator of* p_h*),* $B_h = p_h^* B p_h$ *and* $f_h = p_h^* f$. *If* $A : V \to V^*$ *is monotone and* *hemicontinuous and* $B : V \to V^*$ *is strongly continuous then* $(A_h, B_h, f_h)_{h\in\mathscr{H}}$ *is a consistent approximation of* (A, B, f).

Proof: Stability and admissibility of the external approximation scheme built by a Galerkin scheme is evident. For proving consistency, let $\mathscr{H}' \subseteq \mathscr{H}$ and $\{v_h\}_{h\in\mathscr{H}'} \in \{V_h\}_{h\in\mathscr{H}'}$, $v \in V$ with $p_h v_h \rightharpoonup \omega v = v$ in $F = V$ $(h \in \mathscr{H}')$ be given. In what follows, all convergence is meant for $h \in \mathscr{H}'$.

(i) With $A_h = p_h^* A p_h$, we find from the definition of the dual operator

$$\langle A_h v_h, w_h\rangle = \langle A p_h v_h, p_h w_h\rangle \quad \forall w_h \in V_h\,.$$

Let $w \in V$ be arbitrary. The monotonicity of $A : V \to V^*$ then provides

$$\begin{aligned}\langle A_h v_h, v_h\rangle &= \langle A p_h v_h, p_h v_h\rangle \\ &\ge \langle A p_h v_h, p_h v_h\rangle - \langle A p_h v_h - Aw, p_h v_h - w\rangle \\ &= \langle Aw, p_h v_h - w\rangle + \langle A p_h v_h, p_h r_h w\rangle + \langle A p_h v_h, w - p_h r_h w\rangle\,.\end{aligned} \tag{3}$$

The first term on the right-hand side of (3) converges towards $\langle Aw, v - w\rangle$ since $p_h v_h \rightharpoonup v$ in V. The second term $\langle A p_h v_h, p_h r_h w\rangle = \langle A_h v_h, r_h w\rangle$ converges towards $\langle g, w\rangle$ by the assumption in Definition 2.2 (i). For the third term, we observe that $\{A p_h v_h\}_{h\in\mathscr{H}'}$ is bounded in V^*, which follows (see, e.g., [6, Folg. 1.2 on p. 65]) from the boundedness of $\{\langle A p_h v_h, p_h v_h\rangle\}_{h\in\mathscr{H}'}$ (by the assumption in Definition 2.2 (i)), the boundedness of $\{p_h v_h\}_{h\in\mathscr{H}'}$ in V (the sequence is weakly convergent) and the monotonicity of $A : V \to V^*$. The third term thus vanishes in the limit since $p_h r_h w \to w$ in V (compatibility). In the limit, we finally obtain from (3) together with the assumption in Definition 2.2 (i)

$$\begin{aligned}\langle g, v\rangle &\ge \limsup_{h\in\mathscr{H}'} \langle A_h v_h, v_h\rangle \\ &\ge \lim_{h\in\mathscr{H}'} \Big(\langle Aw, p_h v_h - w\rangle + \langle A p_h v_h, p_h r_h w\rangle + \langle A p_h v_h, w - p_h r_h w\rangle\Big) \\ &= \langle Aw, v - w\rangle + \langle g, w\rangle\,.\end{aligned} \tag{4}$$

Taking $w = v \pm sz$ for arbitrary $z \in V$ and $s \in (0, 1]$ yields

$$\langle A(v + sz), z\rangle \ge \langle g, z\rangle \text{ and } \langle A(v - sz), z\rangle \le \langle g, z\rangle\,,$$

and with $s \to 0+$, the hemicontinuity of $A : V \to V^*$ shows $\langle Av, z\rangle = \langle g, z\rangle$ and hence $Av = g$.

(ii) With $B_h = p_h^* B p_h$, we find for all $w \in V$

$$\langle B_h v_h, r_h w\rangle = \langle B p_h v_h, p_h r_h w\rangle\,.$$

Since $B : V \to V^*$ is strongly continuous and $p_h v_h \rightharpoonup v$ in V, it follows $B p_h v_h \to Bv$ in V^*. Because of $p_h r_h w \to w$ in V, we come up with

$$\langle B_h v_h, r_h w\rangle \to \langle Bv, w\rangle\,.$$

Moreover, we have

$$\langle B_h v_h, v_h\rangle = \langle B p_h v_h, p_h v_h\rangle \to \langle Bv, v\rangle\,.$$

(iii) With respect to the approximation of $f \in V^*$, we observe for all $w \in V$

$$\langle f_h, r_h w \rangle = \langle f, p_h r_h w \rangle \to \langle f, w \rangle$$

as well as (remember $p_h v_h \rightharpoonup v$ in V)

$$\langle f_h, v_h \rangle = \langle f, p_h v_h \rangle \to \langle f, v \rangle .$$

□

Note that a best approximation $r_h v$ of $v \in V$ in V_h always exists if $\dim V_h < \infty$ but r_h might be nonlinear. Instead of the best approximation, one may also take a suitable projection. In [16, p. 28], the restriction operator on V is constructed from its definition on the dense subset $\bigcup_{h \in \mathscr{H}} V_h$.

It arises the question, from which assumptions one can derive condition (i) in Definition 2.2. An answer is given by

Proposition 2.4: *Let $\{(V_h, p_h, r_h)\}_{h \in \mathscr{H}}$ be a stable and admissible external approximation scheme for V and let $A : V \to V^*$ be hemicontinuous. Assume that all $A_h : V_h \to V_h^*$ $(h \in \mathscr{H})$ are monotone and that for any subsequence $\mathscr{H}' \subseteq \mathscr{H}$ and any $\{v_h\}_{h \in \mathscr{H}'} \in \{V_h\}_{h \in \mathscr{H}'}$, $v \in V$ with $p_h v_h \rightharpoonup \omega v$ in F $(h \in \mathscr{H}')$*

$$\limsup_{h \in \mathscr{H}'} \langle A_h r_h w, v_h \rangle \geq \langle Aw, v \rangle \quad \forall w \in V . \tag{5}$$

Then condition (i) in Definition 2.2 is fulfilled.

Proof: The monotonicity of A_h $(h \in \mathscr{H}')$ yields for arbitrary $w \in V$

$$\begin{aligned} \langle A_h v_h, v_h \rangle &\geq \langle A_h v_h, v_h \rangle - \langle A_h v_h - A_h r_h w, v_h - r_h w \rangle \\ &= \langle A_h r_h w, v_h - r_h w \rangle + \langle A_h v_h, r_h w \rangle . \end{aligned}$$

In the limit, we thus obtain by the assumption in Definition 2.2 (i) and with (5)

$$\begin{aligned} \langle g, v \rangle &\geq \limsup_{h \in \mathscr{H}'} \langle A_h v_h, v_h \rangle \\ &\geq \limsup_{h \in \mathscr{H}'} \langle A_h r_h w, v_h - r_h w \rangle + \lim_{h \in \mathscr{H}'} \langle A_h v_h, r_h w \rangle \\ &\geq \langle Aw, v - w \rangle + \langle g, w \rangle . \end{aligned}$$

Here, we have employed that $p_h v_h \rightharpoonup \omega v$, $p_h r_h w \to \omega w$ in F $(h \in \mathscr{H}')$. The hemicontinuity of A implies $Av = g$ in V^* as in (4). □

We will also make use of the following notion.

Definition 2.5: Let $\{V_h\}_{h \in \mathscr{H}}$ be a sequence of normed spaces. A sequence $\{T_h\}_{h \in \mathscr{H}}$ of operators $T_h : V_h \to V_h^*$ is said to be *coercive uniformly in h* iff there is a function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}$ with $\gamma(z) \to \infty$ as $z \to \infty$ such that for all $h \in \mathscr{H}$

$$\langle T_h v_h, v_h \rangle \geq \gamma(\|v_h\|_{V_h}) \|v_h\|_{V_h} \quad \forall v_h \in V_h .$$

We end this section by presenting a criterion for the existence of solutions to the approximate problem.

Lemma 2.6: *Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be continuous. If there is $R > 0$ such that $\Phi(\mathbf{v}) \cdot \mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^N$ with $\|\mathbf{v}\|_{\mathbb{R}^N} = R$ then there exists $\mathbf{u} \in \mathbb{R}^N$ with $\|\mathbf{u}\|_{\mathbb{R}^N} \leq R$ and $\Phi(\mathbf{u}) = 0$.*

Proof: The proof follows by contradiction from Brouwer's fixed point theorem (see, e.g., [6, Lemma 2.1 on p. 74]). □

Theorem 2.7: *Let V_h be a normed space with $\dim V_h = N < \infty$ and let $A_h, B_h : V_h \to V_h^*$ be continuous operators such that $A_h + B_h$ is coercive. For any $f_h \in V_h^*$, equation (2) then possesses a solution.*

Proof: Let $\{e_i\}_{i=1}^N$ be a basis in V_h. Then there is a bijective mapping between V_h and $\mathbb{R}^N$ given by the representation

$$v_h = \sum_{i=1}^{N} v_i e_i \in V_h \,, \quad \mathbf{v} = [v_1, \ldots, v_N] \in \mathbb{R}^N \,.$$

On $\mathbb{R}^N$, we define the norm $\|\mathbf{v}\|_{\mathbb{R}^N} := \|v_h\|_{V_h}$ and the mapping $\Phi(\mathbf{v}) = [\Phi_1, \ldots, \Phi_N]$ with

$$\Phi_i(\mathbf{v}) := \langle A_h v_h + B_h v_h - f_h, e_i \rangle \quad (i = 1, \ldots, N) \,.$$

Obviously, $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ is continuous if $A_h, B_h : V_h \to V_h$ are continuous.

Because of the coercivity of $A_h + B_h$, we find (with a function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}$ with $\gamma(z) \to \infty$ as $z \to \infty$)

$$\Phi(\mathbf{v}) \cdot \mathbf{v} = \langle A_h v_h + B_h v_h - f_h, v_h \rangle \geq \gamma(\|v_h\|_{V_h}) \|v_h\|_{V_h} - \|f_h\|_{V_h^*} \|v_h\|_{V_h} \geq 0$$

if $\|v_h\|_{V_h} = \|\mathbf{v}\|_{\mathbb{R}^N}$ is sufficiently large.

Lemma 2.6 now yields the existence of $\mathbf{u} \in \mathbb{R}^N$ and thus of $u_h \in V_h$ such that $\Phi(\mathbf{u}) = 0$. But then u_h solves (2). □

In applications, the continuity of the approximate operators in a finite dimensional space often follows already from the hemicontinuity of the operators A, B.

3. Convergence

The main result can be formulated as follows.

Theorem 3.1: *Suppose there is a consistent approximation of (A, B, f). Assume further that (2) possesses a solution $u_h \in V_h$ for any $h \in \mathscr{H}$, that the operators $A_h + B_h : V_h \to V_h^*$ $(h \in \mathscr{H})$ are coercive uniformly in h and that the sequence $\{\|f_h\|_{V_h^*}\}_{h \in \mathscr{H}}$ is bounded. Then there is a subsequence $\mathscr{H}' \subseteq \mathscr{H}$ and an element $u \in V$ such that*

$$p_h u_h \rightharpoonup \omega u \text{ in } F \ (h \in \mathscr{H}') \,;$$

the limit u satisfies (1).

Proof: With the coercivity assumption, we immediately find

$$\gamma(\|u_h\|_{V_h}) \|u_h\|_{V_h} \leq \langle A_h u_h + B_h u_h, u_h \rangle = \langle f_h, u_h \rangle \leq \|f_h\|_{V_h^*} \|u_h\|_{V_h} \,.$$

Since $\gamma(z) \to z$ as $z \to \infty$ and since $\{\|f_h\|_{V_h^*}\}_{h\in\mathscr{H}}$ is bounded, this shows also the boundedness of $\{\|u_h\|_{V_h}\}_{h\in\mathscr{H}}$.

Because of the stability of the external approximation scheme, then also the sequence $\{p_h u_h\}_{h\in\mathscr{H}} \subseteq F$ is bounded. In view of the reflexivity of F, there is a subsequence $\mathscr{H}' \subseteq \mathscr{H}$ such that $\{p_h u_h\}_{h\in\mathscr{H}'}$ is weakly convergent in F (see, e.g., [4, Thm. III.27]). Together with the synchronisation condition in Definition 2.1 (ii), there is an element $u \in V$ such that

$$p_h u_h \rightharpoonup \omega u \text{ in } F \ (h \in \mathscr{H}') .$$

With Definition 2.2 (ii), (iii), we now find for all $w \in V$

$$\langle f_h - B_h u_h, r_h w\rangle \to \langle f - Bu, w\rangle , \quad \limsup_{h\in\mathscr{H}'} \langle f_h - B_h u_h, u_h\rangle \le \langle f - Bu, u\rangle .$$

With (2), it follows for all $w \in V$

$$\langle A_h u_h, r_h w\rangle = \langle f_h - B_h u_h, r_h w\rangle \to \langle f - Bu, w\rangle$$

as well as

$$\limsup_{h\in\mathscr{H}'} \langle A_h u_h, u_h\rangle = \limsup_{h\in\mathscr{H}'} \langle f_h - B_h u_h, u_h\rangle \le \langle f - Bu, u\rangle .$$

Definition 2.2 (i) now provides $Au = f - Bu$ in V^*. □

4. Example

In order to keep the presentation short, we only consider a somewhat simple example: a linear finite element method with quadrature for a one-dimensional quasilinear Dirichlet problem without perturbation such that $B = B_h \equiv 0$.

Let $\psi = \psi(x,t) : [0,1] \times \mathbb{R}_0^+ \to \mathbb{R}$ be a given continuous function. We suppose that $t \mapsto \psi(x,t)t$ is monotonically increasing for all $x \in [0,1]$ and that there is a number $p \in (1,\infty)$ and constants $\mu, c > 0$ such that

$$\psi(x,t)t^2 \ge \mu t^p , \quad |\psi(x,t)| \le c(1 + t^{p-2}) \quad \forall (x,t) \in [0,1] \times \mathbb{R}_0^+ .$$

This setting covers, e.g., the one-dimensional p-Laplacian.

For a given right-hand side $f \in L^{p^*}(0,1)$ $(1/p + 1/p^* = 1)$, then consider the problem

$$-\Big(\psi(x, |u'(x)|)u'(x)\Big)' = f(x) \ \ (x \in (0,1)) , \quad u(0) = u(1) = 0 . \tag{6}$$

The weak formulation of (6) leads to the operator equation (1) with the standard Sobolev space $V = W_0^{1,p}(0,1)$ with norm $\|v\|_V := \Big(\int_0^1 |v'(x)|^p dx\Big)^{1/p}$ and an operator $A : V \to V^*$, defined for $v, w \in V$ via

$$\langle Av, w\rangle := \int_0^1 \psi(x, |v'(x)|)v'(x)w'(x)dx .$$

The growth condition for ψ ensures that A maps V into V^*. The hemicontinuity of A is a direct consequence of the continuity of ψ. The right-hand side in (1) is the functional $v \mapsto \int_0^1 f(x)v(x)dx$.

We partition $[0,1]$ equidistantly into $M \in \mathbb{N}$ subintervals $[x_i, x_{i+1}]$ ($x_i = i/M$, $i = 0, \ldots, M$) of length $h = 1/M$ and employ linear finite elements. When also applying a simple rectangular rule (taking the right value) for the numerical evaluation of the appearing integrals, we end up with a fully discrete approximation of (6) that can be written in the form (2). This finite element method with quadrature is equivalent to a finite difference scheme.

We take V_h as the space of grid functions $v_h = [v_{h,0}, \ldots, v_{h,M}]^T \in \mathbb{R}^{M+1}$, $v_{h,0} = v_{h,M} = 0$, and endow it with the norm

$$\|v_h\|_{V_h} := \left(h \sum_{i=1}^{M} |D_i^- v_h|^p \right)^{1/p} ,$$

where $D_i^- v_h := (v_i - v_{i-1})/h$. The dual V_h^* can be identified with the $(M-1)$-dimensional space of grid functions $g_h = [g_{h,1}, \ldots, g_{h,M-1}]^T \in \mathbb{R}^{M-1}$ such that $g_i = -D_i^+ w_h := -(w_{h,i+1} - w_{h,i})/h$ ($i = 1, \ldots, M-1$) for some $w_h \in V_h$, the dual pairing is given by

$$\langle g_h, v_h \rangle = h \sum_{i=1}^{M-1} g_{h,i} v_{h,i} = h \sum_{i=1}^{M-1} w_{h,i} D_i^- v_h \,.$$

The prolongation $p_h v_h$ of $v_h \in V_h$ is the piecewise linear interpolation of the points $(x_0, v_{h,0}), \ldots, (x_M, v_{h,M})$. The restriction is defined via $(r_h v)_i := v(x_i)$ ($i = 0, \ldots, M$) which is well-defined since $W_0^{1,p}(0,1)$ is continuously embedded in $\mathscr{C}([0,1])$. It is easy to show that the sequence $\{(V_h, p_h, r_h)\}_{M=1/h \in \mathbb{N}}$ builds a stable and admissible external approximation scheme for V with $F = V$. In particular, we find for all $v_h \in V_h$

$$\|p_h v_h\|_V = \|v_h\|_{V_h} ,$$

and the compatibility condition, i.e. $p_h r_h v \to v$ in V, is easily shown by density.

If we would take a simpler prolongation such as the piecewise constant interpolation of the values of v_h as well as of their divided differences then we would come up with $\omega v = (v, v') \in F = L^p(0,1) \times L^p(0,1)$.

A simple calculation shows for $v_h, w_h \in V_h$

$$\langle A_h v_h, w_h \rangle = h \sum_{i=1}^{M} \psi(x_i, |D_i^- v_h|) D_i^- v_h D_i^- w_h \,,$$

and, loosely written, we have $(A_h v_h)_i = -D_i^+ \left(\psi(x_i, |D_i^- v_h|) D_i^- v_h \right)$ for $i = 1, \ldots, M-1$. The assumptions on ψ allow to prove in particular that A_h maps V_h into V_h^* and is monotone.

In view of Theorem 2.7, the discrete problem is solvable: The continuity of $A_h : V_h \to V_h^*$ is a direct consequence of the continuity of ψ. The coercivity (uniform in h) follows from

$$\langle A_h v_h, v_h\rangle = h\sum_{i=1}^{M}\psi(x_i,|D_i^- v_h|)(D_i^- v_h)^2 \geq \mu h\sum_{i=1}^{M}|D_i^- v_h|^p = \mu\|v_h\|_{V_h}^p\,,\; v_h\in V_h\,.$$

We now prove condition (5). Let $p_h v_h \rightharpoonup v$ in V (for an arbitrary null sequence of mesh sizes h with $1/h = M\in\mathbb{N}$) and let $w\in V$ be arbitrary. We set $D_i^- w := (w(x_i)-w(x_{i-1}))/h$ $(i=1,\dots,M)$ (remember $w\in V = W_0^{1,p}(0,1)\hookrightarrow\mathscr{C}([0,1])$). A straightforward calculation shows

$$\begin{aligned}&\langle A_h r_h w, v_h\rangle - \langle Aw, v\rangle = \\ &\sum_{i=1}^{M}\int_{x_{i-1}}^{x_i}\Big(\psi(x_i,|D_i^- w|)D_i^- w - \psi(x,|w'(x)|)w'(x)\Big)dx\, D_i^- v_h \\ &+\int_0^1\psi(x,|w'(x)|)w'(x)\left((p_h v_h)'(x)-v'(x)\right)dx =: a_{1,h}+a_{2,h}\,.\end{aligned}\tag{7}$$

Denoting by ψ_h the w.r.t. the first argument piecewise constant approximation of ψ such that $\psi_h(x,t)=\psi(x_i,t)$ for $x\in(x_{i-1},x_i]$ $(i=1,\dots,M)$ and $t\in\mathbb{R}_0^+$ and upon noting that $(p_h r_h w)'(x) = D_i^- w$ for $x\in(x_{i-1},x_i)$ $(i=1,\dots,M)$, we find for the first term $a_{1,h}$ by applying Hölder's inequality with $1/p+1/p^*=1$

$$\begin{aligned}&|a_{1,h}| \\ &\leq\left(h\sum_{i=1}^{M}\left|\frac{1}{h}\int_{x_{i-1}}^{x_i}\Big(\psi(x_i,|D_i^- w|)D_i^- w - \psi(x,|w'(x)|)w'(x)\Big)dx\right|^{p^*}\right)^{1/p^*}\|v_h\|_{V_h} \\ &\leq\left(\sum_{i=1}^{M}\int_{x_{i-1}}^{x_i}\left|\psi(x_i,|D_i^- w|)D_i^- w - \psi(x,|w'(x)|)w'(x)\right|^{p^*}dx\right)^{1/p^*}\|v_h\|_{V_h} \\ &=\left(\int_0^1\left|\psi_h(x,|(p_h r_h w)'(x)|)(p_h r_h w)'(x) - \psi(x,|w'(x)|)w'(x)\right|^{p^*}dx\right)^{1/p^*}\|v_h\|_{V_h}\,.\end{aligned}$$

Since the sequence $\{p_h v_h\}$ is weakly convergent in V it is also bounded in V such that $\|v_h\|_{V_h} = \|p_h v_h\|_V \leq c$ for some $c>0$. It remains to analyse the integral on the right-hand side of the foregoing estimate. This integral, however, converges towards zero, which follows from the continuity of ψ, the continuity of the Nemyzkii operator corresponding to ψ as a mapping from $L^p(0,1)$ into $L^{p^*}(0,1)$ (see [19, Prop. 26.6 on p. 561] and remember the growth condition for ψ) as well as $p_h r_h w\to w$ in $V=W_0^{1,p}(0,1)$, i.e. $(p_h r_h w)'\to w'$ in $L^p(0,1)$.

For the second term $a_{2,h}$ in (7), we immediately have (in view of $p_h v_h\rightharpoonup v$ in V)

$$a_{2,h} = \langle Aw, p_h v_h - v\rangle \to 0\,.$$

With respect to the right-hand side, we have to be somewhat careful as $f\in L^{p^*}(0,1)$ does not allow to take point values. Instead, we may take

$$f_{h,i} = \frac{1}{h}\int_{x_i}^{x_{i+1}} f(x)dx\,,\quad i=1,\dots,M-1\,.$$

One can easily prove that f_h is in V_h^*. Moreover, for arbitrary $v_h \in V_h$, we have

$$\langle f_h, v_h \rangle = \sum_{i=1}^{M-1} \int_{x_i}^{x_{i+1}} f(x)dx\, v_{h,i} = h \sum_{i=1}^{M} \int_{x_i}^{1} f(x)dx\, D_i^- v_h$$

and thus (with $1/p + 1/p^* = 1$)

$$\|f_h\|_{V_h^*} = \sup_{v_h \in V_h \setminus \{0\}} \frac{\langle f_h, v_h \rangle}{\|v_h\|_{V_h}} \leq \left(h \sum_{i=1}^{M} \left| \int_{x_i}^{1} f(x)dx \right|^{p^*} \right)^{1/p^*} \leq \|f\|_{L^{p^*}(0,1)}\,,$$

which shows the boundedness of the sequence $\{\|f_h\|_{V_h}\}$ as required in Theorem 3.1. It remains to prove condition (iii) in Definition 2.2. Let $p_h v_h \rightharpoonup v$ in V, which implies that the sequence $\{p_h v_h\}$ is bounded in V. Since

$$\begin{aligned}\langle f_h, v_h \rangle - \langle f, p_h v_h \rangle &= \sum_{i=1}^{M-1} \int_{x_{i-1}}^{x_i} f(x)(v_{h,i} - (p_h v_h)(x))dx \\ &= \sum_{i=1}^{M-1} \int_{x_{i-1}}^{x_i} f(x)(p_h v_h)'(x)(x_i - x)dx\end{aligned}$$

and thus

$$|\langle f_h, v_h \rangle - \langle f, p_h v_h \rangle| \leq h \int_0^1 |f(x)(p_h v_h)'(x)|dx \leq h\|f\|_{L^{p^*}(0,1)} \|p_h v_h\|_V \to 0\,,$$

we obtain

$$\langle f_h, v_h \rangle - \langle f, v \rangle = \langle f_h, v_h \rangle - \langle f, p_h v_h \rangle + \langle f, p_h v_h - v \rangle \to 0\,.$$

After all, Proposition 2.4 and Theorem 3.1 can be applied. This shows the existence of a weak solution $u \in V = W_0^{1,p}(0,1)$ to (6) and a subsequence, denoted by h', such that the piecewise linear interpolations of the discrete solutions $u_{h'}$ converge weakly in V towards u.

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