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Time discretisation of monotone nonlinear evolution problems by the discontinuous Galerkin method

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Abstract A class of discontinuous Galerkin methods is studied for the time discretisation of the initial-value problem for a nonlinear first-order evolution equation that is governed by a monotone, coercive, and hemicontinuous operator. The numerical solution is shown to converge towards the weak solution of the original problem. Furthermore, well-posedness of the time-discrete problem as well as a priori error estimates for sufficiently smooth exact solutions are studied.

Keywords Nonlinear evolution equation · monotone operator · time discretisation · discontinuous Galerkin method · convergence · a priori error estimate

Mathematics Subject Classification (2000) 65M12 · 65M15 · 47J35 · 35K55 · 47H05

1 Introduction

In the last two decades, the discontinuous Galerkin method attained more and more attention and has become a widely used numerical method (see the monographs [20, 28] for an overview). Besides its application to the spatial approximation, the discontinuous Galerkin method has also been studied for the discretisation in time (see the initial work [12] and, for an overview, the monographs [28, 33], see, e.g., also [31, 32] for the *hp*-version and [3] for the discretisation of an integro-differential equation).

Our interest lies in the study of nonlinear evolution problems and their time discretisation. Methods other than the discontinuous Galerkin approximation in time have been considered by many authors. For quasilinear problems, we refer to, e.g., [16, 21, 22, 24, 35]. Stability and error estimates for nonlinear evolution equations are proven in [25] relying on a linearisation. Fully nonlinear problems have been dealt

Dedicated to Professor Lutz Tobiska on the occasion of his 60th birthday

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with, again by linearisation, in [15,26,27]. Evolution equations governed by a maximal monotone operator have been studied in [17–19,30].

A space-time finite element method with basis functions that are discontinuous in time has been analysed in [13] for a semilinear problem. In [10], the discontinuous Galerkin method in time with polynomial order zero (reducing then to the backward Euler or Rothe method) has been studied for a particular class of non-degenerate scalar quasilinear problems (the governing operator possesses a coercive and angle-bounded Fréchet derivative), see also [11]. The quasilinear problems studied in [1] can be described by a time-independent strongly positive, linear bounded operator perturbed by a locally Lipschitz continuous operator. Besides well-posedness of the h-version of the discontinuous Galerkin method in time (employing the concept of strongly monotone Lipschitz continuous operators, which corresponds to the special case $p = 2$ in our setting), error estimates are derived for sufficiently regular exact solutions. An essential aspect in [1] is the control over the $L^2(0,T;H)$-norm of the time discrete solution.

The aim of this paper is to provide an analysis for a much larger class of nonlinear problems (including degenerate and non-autonomous problems), to incorporate the hp-variant with variable polynomial degree, to prove convergence results even for exact solutions not possessing additional smoothness, and to give estimates of the approximation and discretisation error in a L^p-setting.

The main focus in the aforementioned work is on error estimates (thus requiring smoothness of the exact solution) rather than on convergence only. In this paper, however, we study the convergence for a class of discontinuous Galerkin methods in time covering the h- as well as hp-version, on non-uniform time grids, for the general class of nonlinear evolution problems

$$u' + Au = f \quad \text{in } (0,T), \quad u(0) = u_0, \tag{1.1}$$

governed by a time-dependent monotone and coercive operator A. More precisely, the operator A is assumed to be the Nemytskii operator corresponding to a family of hemicontinuous operators $A(t) : V \to V^*$ $(t \in [0,T])$ acting on a Gelfand triple $V \subseteq H \subseteq V^*$ such that $A(t) + \kappa I : V \to V^*$ (with I being the identity) is p-coercive and monotone for some $\kappa \geq 0$, uniformly in $t \in [0,T]$. Moreover, $A(t) : V \to V^*$ is assumed to fulfill a certain $(p-1)$-growth condition. This framework allows to consider partial differential equations with more involved nonlinearities compared to the results known so far, including degenerate problems.

The numerical method under consideration is as follows: With the variable time grid

$$\begin{cases} \mathbb{I} : 0 = t_0 < t_1 < \cdots < t_N = T \quad (N \in \mathbb{N}) \quad \text{ with } I_n := (t_{n-1}, t_n), \\ \tau_n := t_n - t_{n-1} \; (n = 1,2,\ldots,N), \; \tau_{\max} := \max\limits_{n=1,2,\ldots,N} \tau_n, \end{cases} \tag{1.2}$$

we associate the vector $\boldsymbol{q} = [q_1, q_2, \ldots q_N]^\mathsf{T} \in \mathbb{N}^N$ of polynomial degrees and the linear space

$$\mathscr{W}_{\mathbb{I}} := \left\{ v : (0,T) \to V \, : \, v_{|I_n} \in \mathscr{P}^{q_n}(I_n;V), \, n = 1,2,\ldots,N \right\},$$

where $\mathscr{P}^q(I;V)$ denotes the space of polynomials of highest degree q on the interval I taking values in V. We set for $v_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ and $n = 0,1,\dots,N-1$

$$v_{\mathbb{I}}(t_n^+) := \lim_{\substack{t\to t_n\\ t>t_n}} v_{\mathbb{I}}(t)\,, \quad v_{\mathbb{I}}(t_n^-) := \lim_{\substack{t\to t_n\\ t<t_n}} v_{\mathbb{I}}(t)\,, \quad [\![v_{\mathbb{I}}(t_n)]\!] := v_{\mathbb{I}}(t_n^+) - v_{\mathbb{I}}(t_n^-)\,,$$

where $v_{\mathbb{I}}(0^-)$ is prescribed, as well as $v_{\mathbb{I}}(t_N^-) := \lim_{\substack{t\to T\\ t<T}} v_{\mathbb{I}}(t)$. The method then reads: *For given f and $u_{\mathbb{I}}^0$, find $u_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ such that for all $v_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$*

$$\begin{aligned} &\sum_{n=1}^{N} \int_{I_n} (u_{\mathbb{I}}'(t), v_{\mathbb{I}}(t))dt + \sum_{n=1}^{N} \big([\![u_{\mathbb{I}}(t_{n-1})]\!], v_{\mathbb{I}}(t_{n-1}^+)\big) \\ &+ \int_0^T \langle A(t)u_{\mathbb{I}}(t), v_{\mathbb{I}}(t)\rangle dt = \int_0^T \langle f(t), v_{\mathbb{I}}(t)\rangle dt \quad \text{with} \quad u_{\mathbb{I}}(0^-) := u_{\mathbb{I}}^0\,. \end{aligned} \tag{1.3}$$

In practice, a suitable composite quadrature will be employed in order to approximate the terms $\int_0^T \langle A(t)u_{\mathbb{I}}(t), v_{\mathbb{I}}(t)\rangle dt$ and $\int_0^T \langle f(t), v_{\mathbb{I}}(t)\rangle dt$, which shall not be taken into account here.

We prove the convergence of the sequence of numerical solutions of (1.3), corresponding to an appropriate sequence of time grids and associated degree vectors, towards the weak solution of (1.1). The proof of convergence essentially relies upon a priori estimates (which follow from the coercivity assumption) as well as upon compactness arguments and the theory of monotone operators.

The advantage of our convergence result lies in the fact that it does not require any additional regularity of the weak solution. Remember that results on higher regularity of solutions of nonlinear evolution problems are very rare and always restricted to special situations. The convergence result is complementary to error estimates that are of importance for situations where a smooth solution is at hand. Note that, without additional assumptions such as the smoothness of the exact solution, one cannot expect convergence "better" than the weak convergence provided here. We provide, however, a result on the strong convergence under an additional monotonicity assumption. We also remark that the consideration of variable time grids as studied here is a prerequisite for any analysis of adaptive methods. It turns out that, in contrast to other methods (see, e.g., [7,8]), there are no severe restrictions on the sequence of variable time grids.

We also present results on the existence, uniqueness, and stability of the numerical solution as well as a priori error estimates in the case of a sufficiently regular exact solution. These error estimates rely on estimates of the approximation error in appropriate norms of Bochner–Lebesgue spaces for vector-valued functions taking values in a Banach space. To our best knowledge, also these error estimates are new.

Convergence results analogous to those obtained here can be found in, e.g., [29, Ch. 8.2] for the backward Euler method. For other time discretisation methods, we have recently been able to prove similar results (see [5,6] for the two-step backward differentiation formula (BDF) on an equidistant grid, [7] for the two-step BDF on a variable time grid, [8] for the ϑ-scheme on a variable time grid, and [9] for a class of stiffly accurate Runge–Kutta methods) although the assumptions on the underlying operator as well as the convergence results themselves differ from method to method. Moreover, in contrast to the techniques used there, the analysis of the discontinuous Galerkin method is indeed more involved and requires additional techniques due to their non-conforming character and the external approximation of the standard solution space for (1.1) by $\mathscr{W}_{\mathbb{I}}$. This is also the reason why we do not consider non-monotone perturbations of the principle part, which require more intrigued compactness arguments based on additional a priori estimates for the numerical solution.

The paper is organised as follows: In Section 2, we introduce the necessary notation and recapitulate the functional analytical framework. The solvability of the numerical scheme (1.3) and a priori estimates for its solution are studied in Section 3. The main convergence result is then formulated and proven in Section 4. In Section 5, we finally present results on the stability of (1.3) and a priori error estimates for smooth exact solutions together with estimates of the approximation error.

2 Notation and time continuous problem

Let $V \subseteq H \subseteq V^*$ be a Gelfand triple with $(V, \|\cdot\|)$ being a reflexive, separable, real Banach space that is dense and continuously embedded in the Hilbert space $(H, (\cdot,\cdot), |\cdot|)$. The dual V^* of V is equipped with the norm $\|f\|_* := \sup_{v\in V\setminus\{0\}} \langle f, v\rangle / \|v\|$, where $\langle \cdot,\cdot\rangle$ denotes the duality pairing.

For a Banach space X and the time interval $[0,T]$, let $L^r(0,T;X)$ ($r \in [1,\infty]$) denote the Banach space of Bochner integrable (for $r = \infty$ Bochner measurable and essentially bounded) abstract functions equipped with the standard norm denoted by $\|\cdot\|_{L^r(0,T;X)}$. Let $p \in (1,\infty)$ and set $p^* := p/(p-1)$. The function space

$$\mathscr{X} := L^p(0,T;V) \cap L^2(0,T;H)\,, \quad \|v\|_{\mathscr{X}} := \|v\|_{L^p(0,T;V)} + \|v\|_{L^2(0,T;H)}\,,$$

is a reflexive, separable Banach space. Its dual $\mathscr{X}^* = L^{p^*}(0,T;V^*) + L^2(0,T;H)$ is equipped with the norm

$$\|f\|_{\mathscr{X}^*} := \inf_{\substack{f_1 \in L^{p^*}(0,T;V^*),\, f_2 \in L^2(0,T;H) \\ f = f_1 + f_2}} \max\left(\|f_1\|_{L^{p^*}(0,T;V^*)}, \|f_2\|_{L^2(0,T;H)} \right).$$

If f allows the representation $f = f_1 + f_2$ with $f_1 \in L^{p^*}(0,T;V^*)$, $f_2 \in L^2(0,T;H)$ then the duality pairing between $f \in \mathscr{X}^*$ and $v \in \mathscr{X}$ is given by

$$\langle f, v\rangle_{\mathscr{X}^*\times\mathscr{X}} = \int_0^T \left(\langle f_1(t), v(t)\rangle_{V^*\times V} + (f_2(t), v(t)) \right) dt = \int_0^T \langle f(t), v(t)\rangle_{V^*\times V}\, dt\,,$$

see, e.g., [14] for more details. Note that $\mathscr{X} \subseteq L^2(0,T;H) \subseteq \mathscr{X}^*$ is again a Gelfand triple. In the case $p \geq 2$, we can just take $\mathscr{X} = L^p(0,T;V)$.

The solution of (1.1) will be sought in the Banach space

$$\mathscr{W} := \{v \in \mathscr{X} : v' \in \mathscr{X}^*\}\,, \quad \|v\|_{\mathscr{W}} := \|v\|_{\mathscr{X}} + \|v'\|_{\mathscr{X}^*}\,,$$

which is continuously embedded in the space $\mathscr{C}([0,T];H)$ of uniformly continuous functions with values in H. Here, v' denotes the distributional time derivative of v.

The structural properties we always assume for A read as follows:

Assumption A. *$\{A(t)\}_{t\in[0,T]}$ is a family of hemicontinuous operators $A(t) : V \to V^*$, such that for all $v \in V$ the mapping $t \mapsto A(t)v : [0,T] \to V^*$ is Bochner integrable on $(0,T)$. There is a constant $\kappa \geq 0$ such that $A(t) + \kappa I : V \to V^*$ is monotone. For a suitable $p \in (1,\infty)$, there are constants $\mu > 0$, $\lambda \geq 0$ such that for all $t \in [0,T]$ and $v \in V$*

$$\langle (A(t) + \kappa I)v, v\rangle \geq \mu \|v\|^p - \lambda\,.$$

There exists $\alpha > 0$ such that for all $t \in [0,T]$ and $v \in V$

$$\|A(t)v\|_* \leq \alpha\left(1 + \|v\|^{p-1}\right)\,.$$

With $\{A(t)\}_{t\in[0,T]}$, we associate the Nemytskii operator A that is defined by $(Av)(t) := A(t)v(t)$ $(t \in [0,T])$ for a function $v : [0,T] \to V$.

Under Assumption A, the Nemytskii operator A maps $L^p(0,T;V)$ into $(L^p(0,T;V))^* = L^{p^*}(0,T;V^*)$ and is hemicontinuous and bounded. Moreover, $A + \kappa I : \mathscr{X} \to \mathscr{X}^*$ is monotone and satisfies for all $v \in \mathscr{X}$

$$\langle (A + \kappa I)v, v\rangle \geq \mu \|v\|^p_{L^p(0,T;V)} - \lambda T\,.$$

Problem (1.1) then possesses for any $u_0 \in H$ and $f \in \mathscr{X}^*$ a unique solution $u \in \mathscr{W}$ such that the evolution equation holds in $\mathscr{X}^*$ (see, e.g., [29, Thm. 8.28], [2, Thm. 4.2 on p. 167], [34, Thm. 30.A], [14, Satz 1.1 on p. 201, Bem. 1.5 on p. 210]).

A standard example that fits into the above framework is the weak formulation of the evolutionary p-Laplacian on a bounded domain Ω with locally Lipschitz continuous boundary $\partial\Omega$, endowed with homogeneous Dirichlet boundary conditions. The corresponding initial-boundary value problem reads as

$$\begin{aligned} \partial_t u - \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f \quad \text{in } \Omega\times(0,T) \\ u &= 0 \quad \text{on } \partial\Omega\times(0,T) \\ u(\cdot,0) &= u_0 \quad \text{in } \Omega\,. \end{aligned} \tag{2.1}$$

The underlying function spaces are then the standard Sobolev space $V = W_0^{1,p}(\Omega)$ and the Hilbert space $H = L^2(\Omega)$. The time-independent operator $A : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega)$ is given by

$$\langle Av, w\rangle = \int_\Omega |\nabla v(x)|^{p-2}\nabla v(x)\cdot\nabla w(x)\,dx, \quad v, w \in W_0^{1,p}(\Omega).$$

Another example is the fluid flow through a porous medium when working with the Sobolev space $H = H^{-1}(\Omega)$ as the pivot space and with $V = L^p(\Omega)$ (with $p > 1$ if $d \in \{1,2\}$ and $p \geq 2d/(d+2)$ if $d > 3$, where d denotes the dimension of Ω) in the underlying Gelfand triple (see [23, pp. 191 ff.], [14, pp. 72 f.] for more details). This choice of function spaces is appropriate for the very weak formulation of the corresponding initial-boundary value problem

$$\begin{aligned} \partial_t u - \Delta\sigma(u) &= f \quad \text{in } \Omega\times(0,T) \\ \sigma(u) &= g \quad \text{on } \partial\Omega\times(0,T) \\ u(\cdot,0) &= u_0 \quad \text{in } \Omega\,, \end{aligned} \tag{2.2}$$

where the boundary $\partial\Omega$ is of class $\mathscr{C}^{1,1}$ and f, g, u_0 are suitably given functions. The function $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous, monotonically increasing, coercive, and to fulfill an appropriate growth condition. A typical choice is $\sigma(u) = |u|^{p-2}u$. The operator $A : L^p(\Omega) \to L^{p^*}(\Omega)$ is now given by

$$\langle Av, w\rangle = \int_\Omega \sigma(v(x)) \cdot w(x) dx\,, \quad v, w \in L^p(\Omega)\,.$$

More examples for operators possessing the above properties can be found, e.g., in [14, pp. 68 ff., 215 ff.], [23], [29, pp. 232 ff.], and [34, pp. 567 ff., 590 ff., 779 ff.]). In particular, initial-boundary value problems for systems of quasilinear partial differential equations of the type

$$\partial_t - \nabla \cdot (\sigma(x,t,u,\nabla u)\nabla u) = f \quad \text{ in } \Omega \times (0,T)$$

fit into our framework if the function σ fulfills appropriate assumptions. The corresponding operator A then depends, via the function σ, explicitly on time t. But also quasilinear problems of higher differentiation order in space are included. We should remark that the Navier–Stokes problem or generalised Newtonian fluid flow problems as studied in [6] are not included because of the appearance of the convection term as a non-monotone perturbation of the principle part.

In what follows, we restrict our considerations to the case $\kappa = 0$. This is always possible by a suitable transformation (see [8, Remark 1], [14, Satz 1.3 on p. 211]).

3 Solvability and a priori estimates for the time discrete problem

We commence with a statement on the solvability of the time discrete problem.

In what follows let $q_{\min} := \min_{n=1,2,\ldots,N} q_n$, $q_{\max} := \max_{n=1,2,\ldots,N} q_n$.

Theorem 3.1 *Let Assumption A be fulfilled. For given $f \in L^1(0,T;V^*)$ and $u_{\mathbb{I}}^0 \in H$, problem (1.3) possesses a solution $u_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$. If $q_{\max} \le 1$ or if $A(t) : V \to V^*$ ($t \in [0,T]$) is strictly monotone then the solution is unique.*

Proof We first observe that (1.3) is equivalent to the N ($n = 1,\ldots,N$) problems *For given $f_{|I_n} \in L^1(I_n;V^*)$ and $u_{\mathbb{I}}(t_{n-1}^-) \in H$, find $u_{\mathbb{I}|I_n} \in \mathscr{P}^{q_n}(I_n;V)$ such that for all $v \in \mathscr{P}^{q_n}(I_n;V)$*

$$\begin{aligned}\int_{I_n} (u_{\mathbb{I}|I_n}'(t), v(t))dt + \big(u_{\mathbb{I}|I_n}(t_{n-1}^+), v(t_{n-1}^+)\big) + \int_{I_n} \langle A(t)u_{\mathbb{I}|I_n}(t), v(t)\rangle dt \\ = \int_{I_n} \langle f_{|I_n}(t), v(t)\rangle dt + \big(u_{\mathbb{I}}(t_{n-1}^-), v(t_{n-1}^+)\big)\,. \end{aligned} \tag{3.1}$$

This can be seen from testing (1.3) by functions $v_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ that vanish in all but one subinterval.

Let $\{\varphi_j\}_{j=0,1,\ldots,q_n}$ be a basis of the finite dimensional space $\mathscr{P}^{q_n}(I_n;\mathbb{R})$. Then each element $v = v(t) \in \mathscr{P}^{q_n}(I_n;V)$ can be represented as

$$v(t) = \sum_{j=0}^{q_n} v_j \varphi_j(t)\,, \quad t \in I_n\,, \tag{3.2a}$$

where $v_j \in V$ $(j = 0, 1, \ldots, q_n)$, and we have $\mathscr{P}^{q_n}(I_n;V) \cong V^{q_n+1}$ via the mapping

$$\mathscr{P}^{q_n}(I_n;V) \ni v \mapsto \boldsymbol{v} = [v_0, v_1, \ldots, v_{q_n}]^\mathsf{T} \in V^{q_n+1}\,. \tag{3.2b}$$

Moreover,

$$\|\boldsymbol{v}\|_{V^{q_n+1}} := \left(\int_{I_n} \|v(t)\|^p dt\right)^{1/p}$$

then defines a norm on V^{q_n+1}.

Let now $s : V^{q_n+1} \times V^{q_n+1} \to \mathbb{R}$ be defined by

$$s(\boldsymbol{u},\boldsymbol{v}) := \int_{I_n} (u'(t), v(t))dt + \left(u(t_{n-1}^+), v(t_{n-1}^+)\right) + \int_{I_n} \langle A(t)u(t), v(t)\rangle dt\,,$$

where again u and v correspond to $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, via (3.2). One may show that, for fixed $\boldsymbol{u} \in V^{q_n+1}$, the mapping $\boldsymbol{v} \mapsto s(\boldsymbol{u},\boldsymbol{v})\,, V^{q_n+1} \to \mathbb{R}$ is linear and continuous. For doing so, we employ an inverse inequality combined with the continuous embedding of V into H,

$$|v(t_{n-1}^+)| \le \|v\|_{L^\infty(I_n;H)} \le \tau_n^{-1/p}\|v\|_{L^p(I_n;H)} \le c\tau_n^{-1/p}\|v\|_{L^p(I_n;V)} = c\tau_n^{-1/p}\|\boldsymbol{v}\|_{V^{q_n+1}}\,, \tag{3.3}$$

which can be derived from the transformation of I_n onto $[0,1]$, the equivalence of the norms $\|\cdot\|_{L^\infty(0,1;H)}$ and $\|\cdot\|_{L^p(0,1;H)}$ on $\mathscr{P}^{q_n}(0,1;H)$, and the inverse transformation. This gives rise to the definition of the associated operator $S : V^{q_n+1} \to (V^{q_n+1})^*$ via $\langle S\boldsymbol{u},\boldsymbol{v}\rangle = s(\boldsymbol{u},\boldsymbol{v})$. Since also $\boldsymbol{g} : \boldsymbol{v} \mapsto \int_{I_n}\langle f_{|I_n}(t), v(t)\rangle dt + \left(u_{\mathbb{I}}(t_{n-1}^-), v(t_{n-1}^+)\right), V^{q_n+1} \to \mathbb{R}$ is an element of $(V^{q_n+1})^*$ (use again (3.3)), we can rewrite (3.1) as the operator equation

$$S\boldsymbol{u} = \boldsymbol{g} \quad \text{in} \quad (V^{q_n+1})^*\,.$$

We now wish to apply the Browder–Minty theorem (see, e.g., [34, Thm. 26.A]) in order to prove existence. We thus have to show that $S : V^{q_n+1} \to (V^{q_n+1})^*$ is coercive, monotone and hemicontinuous. Uniqueness will then follow from the strict monotonicity of S.

Regarding the coercivity, we observe (employing integration by parts and the coercivity of $A(t)$) that for any $\boldsymbol{v} \in V^{q_n+1}$ (and corresponding function $v \in \mathscr{P}^{q_n}(I_n;V)$, see (3.2))

$$\begin{aligned}\langle S\boldsymbol{v},\boldsymbol{v}\rangle &= \int_{I_n} (v'(t), v(t))dt + |v(t_{n-1}^+)|^2 + \int_{I_n}\langle A(t)v(t), v(t)\rangle dt \\ &\ge \frac{1}{2}|v(t_n^-)|^2 + \frac{1}{2}|v(t_{n-1}^+)|^2 + \mu\int_{I_n}\|v(t)\|^p dt - \lambda\tau_n \\ &\ge \mu\|\boldsymbol{v}\|^p_{V^{q_n+1}} - \lambda\tau_n\,,\end{aligned}$$

which shows indeed the coercivity of $S : V^{q_n+1} \to (V^{q_n+1})^*$.

With respect to the monotonicity, we similarly find (employing again integration by parts and the monotonicity of $A(t)$) for any $\boldsymbol{v},\boldsymbol{w} \in V^{q_n+1}$ (and corresponding functions $v, w \in \mathscr{P}^{q_n}(I_n;V)$, see (3.2))

$$\langle S\boldsymbol{v} - S\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle \ge \frac{1}{2}|v(t_n^-) - w(t_n^-)|^2 + \frac{1}{2}|v(t_{n-1}^+) - w(t_{n-1}^+)|^2\,.$$

This shows the monotonicity of $S : V^{q_n+1} \to (V^{q_n+1})^*$. If $q_n \le 1$ $(n = 1,2,\ldots,N)$ then $\boldsymbol{v} \ne \boldsymbol{w}$ implies $v(t_n^-) \ne w(t_n^-)$ or $v(t_n^+) \ne w(t_n^+)$, and $S : V^{q_n+1} \to (V^{q_n+1})^*$ becomes strictly monotone. Moreover, $S : V^{q_n+1} \to (V^{q_n+1})^*$ is strictly monotone if $A(t) : V \to V^*$ $(t \in [0,T])$ is. In both cases, uniqueness of a solution of the time discrete problem follows.

The hemicontinuity of $S : V^{q_n+1} \to (V^{q_n+1})^*$ can easily be derived from the bilinearity and boundedness of the first two terms in $\boldsymbol{s}$ together with (3.3), and the hemicontinuity of $A(t)$ $(t \in [0,T])$ together with the growth condition (applying Lebesgue's theorem of dominated convergence). □

We shall remark that the foregoing result covers the existence and uniqueness result given in [1] for the h-version in the situation of a strongly monotone Lipschitz continuous operator $A(t)$ $(t \in [0,T])$, which corresponds here to $p = 2$ and a stronger monotonicity assumption.

An essential prerequisite for proving convergence is an appropriate a priori estimate for the numerical solution. This is provided by the following theorem.

Theorem 3.2 *Let Assumption A be fulfilled. Let $u_{\mathbb{I}}^0 \in H$ and $f \in L^{p^*}(0,T;V^*)$. Any solution $u_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ to (1.3) then satisfies the a priori estimate*

$$\max_{n=1,2,\ldots,N} |u_{\mathbb{I}}(t_n^-)|^2 + \max_{n=1,2,\ldots,N} |u_{\mathbb{I}}(t_{n-1}^+)|^2 + \sum_{n=1}^{N} |[\![u_{\mathbb{I}}(t_{n-1})]\!]|^2 + \|u_{\mathbb{I}}\|^p_{L^p(0,T;V)}$$
$$\le c\left(|u_{\mathbb{I}}^0|^2 + \|f\|^{p^*}_{L^{p^*}(0,T;V^*)} + \lambda T\right),$$

where $c > 0$ only depends on μ and p.

Proof Taking $v_{\mathbb{I}} = u_{\mathbb{I}}\chi_{[0,t_k]}$ $(k = 1,2,\ldots,N$; χ_I denotes the characteristic function on the interval I) in (1.3) leads to

$$\sum_{n=1}^{k} \int_{I_n} (u_{\mathbb{I}}'(t), u_{\mathbb{I}}(t))dt + \sum_{n=1}^{k} \left([\![u_{\mathbb{I}}(t_{n-1})]\!], u_{\mathbb{I}}(t_{n-1}^+)\right) + \int_0^{t_k} \langle A(t)u_{\mathbb{I}}(t), u_{\mathbb{I}}(t)\rangle dt$$
$$= \int_0^{t_k} \langle f(t), u_{\mathbb{I}}(t)\rangle dt\,.$$

From integration by parts, we find for $n = 1,2,\ldots,k$

$$\int_{I_n} (u_{\mathbb{I}}'(t), u_{\mathbb{I}}(t))dt = \frac{1}{2}\left(|u_{\mathbb{I}}(t_n^-)|^2 - |u_{\mathbb{I}}(t_{n-1}^+)|^2\right). \tag{3.4}$$

Moreover, the binomic formula shows that

$$\left([\![u_{\mathbb{I}}(t_{n-1})]\!], u_{\mathbb{I}}(t_{n-1}^+)\right) = \frac{1}{2}\left(|u_{\mathbb{I}}(t_{n-1}^+)|^2 - |u_{\mathbb{I}}(t_{n-1}^-)|^2 + |[\![u_{\mathbb{I}}(t_{n-1})]\!]|^2\right). \tag{3.5}$$

This, together with the coercivity of $A(t)$ $(t \in [0,T])$ and Young's inequality, shows that

$$\frac{1}{2}|u_{\mathbb{I}}(t_k^-)|^2 + \frac{1}{2}\sum_{n=1}^{k} |[\![u_{\mathbb{I}}(t_{n-1})]\!]|^2 + \frac{\mu}{2}\int_0^{t_k} \|u_{\mathbb{I}}(t)\|^p dt$$
$$\le \frac{1}{2}|u_{\mathbb{I}}(t_0^-)|^2 + c\int_0^{t_k} \|f(t)\|_*^{p^*} dt + \lambda t_k\,,$$

where $c > 0$ only depends on μ and p. Finally, we observe that for all $n = 1, 2, \ldots, N$

$$|u_{\mathbb{I}}(t_{n-1}^{+})|^2 \leq 2\left(|u_{\mathbb{I}}(t_{n-1}^{-})|^2 + |[\![u_{\mathbb{I}}(t_{n-1})]\!]|^2\right).$$

This immediately proves the assertion. □

It should be noted that we have no $L^\infty(0,T;H)$-bound on the solution at hand unless the polynomial degree is less or equal 1. We, therefore, were not able to deal with more general right-hand sides in $\mathscr{X}^*$. Furthermore, we have no bound on the discrete counterpart of the time derivative (that would be analogous to $u' \in \mathscr{X}^*$ for the continuous problem).

4 Convergence towards a weak solution

In what follows, we often write, e.g., $\tau_{\max}(\mathbb{I})$ in order to emphasise the dependence of a quantity on the time grid $\mathbb{I}$.

From now on, we consider a sequence $\{\mathbb{I}_\ell\}_{\ell\in\mathbb{N}}$ of time grids (1.2) with corresponding degree vectors $\boldsymbol{q}(\mathbb{I}_\ell)$ such that

$$\tau_{\max}(\mathbb{I}_\ell) \to 0 \text{ as } \ell \to \infty, \quad q_{\min}(\mathbb{I}_\ell) \geq 1 \quad \text{and} \quad \mathscr{W}_{\mathbb{I}_\ell} \subseteq \mathscr{W}_{\mathbb{I}_{\ell+1}},\ \ell \in \mathbb{N}. \tag{4.1}$$

The last condition is fulfilled if the time grid at level $\ell+1$ contains all abscissae of the time grid at level ℓ (i.e., the time grid at level $\ell+1$ is not a coarsening of the foregoing one) and if the polynomial degree on a subinterval is not decreasing.

The main result of the paper now reads as follows.

Theorem 4.1 *Let Assumption A be fulfilled. Let $u_0 \in H$ and $f \in L^{p^*}(0,T;V^*)$ be given. Let $\{\mathbb{I}_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of time grids (1.2) with a corresponding sequence of degree vectors such that (4.1) is fulfilled. Moreover, let $\{u^0_{\mathbb{I}_\ell}\}_{\ell\in\mathbb{N}} \subseteq H$ be such that*

$$u^0_{\mathbb{I}_\ell} \to u_0 \text{ in } H \text{ as } \ell \to \infty. \tag{4.2}$$

The sequence $\{u_{\mathbb{I}_\ell}\}_{\ell\in\mathbb{N}}$ of solutions of the discontinuous Galerkin method (1.3) then converges weakly in $L^p(0,T;V)$ towards the exact solution $u \in \mathscr{W}$ to (1.1).

Proof Theorem 3.2, together with the growth condition for $A(t)$ ($t \in [0,T]$) and standard compactness arguments, immediately shows the existence of a subsequence (denoted by ℓ') and elements $\xi \in H$, $u \in L^p(0,T;V)$, $a \in L^{p^*}(0,T;V^*)$ such that

$$u_{\mathbb{I}_{\ell'}}(T^-) \rightharpoonup \xi \text{ in } H,\ u_{\mathbb{I}_{\ell'}} \rightharpoonup u \text{ in } L^p(0,T;V),\ Au_{\mathbb{I}_{\ell'}} \rightharpoonup a \text{ in } L^{p^*}(0,T;V^*) \text{ as } \ell' \to \infty. \tag{4.3}$$

We will show that u is indeed in $\mathscr{W}$ and solves the original problem (i.e., fulfills the differential equation and the initial condition in (1.1)). Moreover, $a = Au$ and $\xi = u(T)$.

Let $m \in \mathbb{N}$ be arbitrary but fixed. Because of (4.1), we then find from (1.3) for all $v_{\mathbb{I}_m} \in \mathscr{W}_{\mathbb{I}_m} \cap \mathscr{W} \subseteq \mathscr{C}([0,T];H)$ with $v_{\mathbb{I}_m}(0) = v_{\mathbb{I}_m}(T) = 0$ and all $\ell' \geq m$

$$\begin{aligned}
\int_0^T \langle f(t), v_{\mathbb{I}_m}(t)\rangle dt &= \sum_{n=1}^{N(\mathbb{I}_{\ell'})} \int_{I_n(\mathbb{I}_{\ell'})} (u'_{\mathbb{I}_{\ell'}}(t), v_{\mathbb{I}_m}(t)) dt \\
&\quad + \sum_{n=1}^{N(\mathbb{I}_{\ell'})} \big([\![u_{\mathbb{I}_{\ell'}}(t_{n-1}(\mathbb{I}_{\ell'}))]\!], v_{\mathbb{I}_m}(t_{n-1}(\mathbb{I}_{\ell'})) \big) \\
&\quad + \int_0^T \langle A(t) u_{\mathbb{I}_{\ell'}}(t), v_{\mathbb{I}_m}(t)\rangle dt \\
&= \text{(with integration by parts and } v_{\mathbb{I}_m}(0) = v_{\mathbb{I}_m}(T) = 0\text{)} \\
&= -\int_0^T \langle v'_{\mathbb{I}_m}(t), u_{\mathbb{I}_{\ell'}}(t)\rangle dt + \int_0^T \langle A(t) u_{\mathbb{I}_{\ell'}}(t), v_{\mathbb{I}_m}(t)\rangle dt \\
&\to -\int_0^T \langle v'_{\mathbb{I}_m}(t), u(t)\rangle dt + \int_0^T \langle a(t), v_{\mathbb{I}_m}(t)\rangle dt \qquad \text{as } \ell' \to \infty .
\end{aligned} \tag{4.4}$$

Let $\varepsilon > 0$. Then for each $v \in \mathscr{C}^1([0,T];V)$ there is a sufficiently fine partition $\mathbb{I}_{m(\varepsilon)}$ ($m(\varepsilon) \in \mathbb{N}$) such that the corresponding piecewise linear interpolant $\pi^1_{\mathbb{I}_{m(\varepsilon)}} v$ of v, which is an element of $\mathscr{W}_{\mathbb{I}_{m(\varepsilon)}} \cap \mathscr{W}$ if $q_{\min}(\mathbb{I}_{m(\varepsilon)}) \geq 1$, fulfills

$$\| v - \pi^1_{\mathbb{I}_{m(\varepsilon)}} v \|_{\mathscr{W}} < \varepsilon .$$

Therefore, we conclude from (4.4) that for all $v \in \mathscr{C}^1([0,T];V)$ with compact support

$$-\int_0^T \langle v'(t), u(t)\rangle dt = \int_0^T \langle f(t) - a(t), v(t)\rangle dt . \tag{4.5}$$

This, however, shows that the distributional time derivative of $u \in L^p(0,T;V)$ equals $f - a \in L^{p^*}(0,T;V^*)$ (which already implies $u \in \mathscr{C}([0,T];H)$) and thus we have $u \in \mathscr{W}$. Moreover, from integration by parts (which is now allowed because of $u \in \mathscr{W}$) and by the density of functions $v \in \mathscr{C}^1([0,T];V)$ with compact support in $L^p(0,T;V)$, we find from (4.5) that

$$u' + a = f \text{ in } L^{p*}(0,T;V^*) . \tag{4.6}$$

We now prove that $u(0) = u_0$ and $u(T) = \xi$. Let $v \in V$ and φ be a linear polynomial on $[0,T]$. Then $v\varphi \in \mathscr{W} \cap \mathscr{W}_{\mathbb{I}_\ell}$ for all $\ell \in \mathbb{N}$. For readability, we omit denoting the subsequence by ℓ' for a moment. From integration by parts, inserting (4.6) and (1.3), and employing again integration by parts, we find

$$
\begin{aligned}
&(u(T),v\varphi(T))-(u(0),v\varphi(0)) = \int_0^T \big(\langle u'(t),v\varphi(t)\rangle + \langle v\varphi'(t),u(t)\rangle\big)\,dt\\
&= \int_0^T \big(\langle f(t)-a(t),v\varphi(t)\rangle + \langle v\varphi'(t),u(t)\rangle\big)\,dt\\
&= \sum_{n=1}^N \int_{I_n} \big(u_{\mathbb{I}}'(t),v\varphi(t)\big)\,dt + \sum_{n=1}^N \big([\![u_{\mathbb{I}}(t_{n-1})]\!],v\varphi(t_{n-1})\big)\\
&\quad + \int_0^T \langle A(t)u_{\mathbb{I}}(t)-a(t),v\varphi(t)\rangle dt + \int_0^T \langle v\varphi'(t),u(t)\rangle dt\\
&= \sum_{n=1}^N \big((u_{\mathbb{I}}(t_n^-),v\varphi(t_n)) - (u_{\mathbb{I}}(t_{n-1}^+),v\varphi(t_{n-1}))\big) + \sum_{n=1}^N \big([\![u_{\mathbb{I}}(t_{n-1})]\!],v\varphi(t_{n-1})\big)\\
&\quad + \int_0^T \langle A(t)u_{\mathbb{I}}(t)-a(t),v\varphi(t)\rangle dt + \int_0^T \langle v\varphi'(t),u(t)-u_{\mathbb{I}}(t)\rangle dt\\
&= (u_{\mathbb{I}}(T^-),v\varphi(T)) - (u_{\mathbb{I}}^0,v\varphi(0))\\
&\quad + \int_0^T \langle A(t)u_{\mathbb{I}}(t)-a(t),v\varphi(t)\rangle dt + \int_0^T \langle v\varphi'(t),u(t)-u_{\mathbb{I}}(t)\rangle dt\,.
\end{aligned}
$$

Taking the limit then leads, because of (4.2) and (4.3), to

$$(u(T),v\varphi(T))-(u(0),v\varphi(0)) = (\xi,v\varphi(T))-(u_0,v\varphi(0))\,.$$

Choosing now $\varphi(T)=0$ and $\varphi(0)=0$ proves $u(0)=u_0$ and $u(T)=\xi$, respectively.

For proving $a=Au$, we employ the monotonicity of $A(t)$ $(t\in[0,T])$. We again omit denoting the subsequence by ℓ'. Taking $v_{\mathbb{I}}=u_{\mathbb{I}}$ in (1.3) and integrating by parts employing (3.4) and (3.5) yields for arbitrary $w\in L^p(0,T;V)$

$$
\begin{aligned}
&\frac{1}{2}|u_{\mathbb{I}}(T^-)|^2 - \frac{1}{2}|u_{\mathbb{I}}^0|^2 + \frac{1}{2}\sum_{n=1}^N |[\![u_{\mathbb{I}}(t_{n-1})]\!]|^2\\
&= \sum_{n=1}^N \int_{I_n} (u_{\mathbb{I}}'(t),u_{\mathbb{I}}(t))dt + \sum_{n=1}^N \big([\![u_{\mathbb{I}}(t_{n-1})]\!],u_{\mathbb{I}}(t_{n-1}^+)\big)\\
&= \int_0^T \langle f(t)-A(t)u_{\mathbb{I}}(t),u_{\mathbb{I}}(t)\rangle dt\\
&\le \int_0^T \langle f(t)-A(t)u_{\mathbb{I}}(t),u_{\mathbb{I}}(t)\rangle dt + \int_0^T \langle A(t)u_{\mathbb{I}}(t)-A(t)w(t),u_{\mathbb{I}}(t)-w(t)\rangle dt\\
&= \int_0^T \langle f(t),u_{\mathbb{I}}(t)\rangle dt - \int_0^T \langle A(t)u_{\mathbb{I}}(t),w(t)\rangle dt - \int_0^T \langle A(t)w(t),u_{\mathbb{I}}(t)-w(t)\rangle dt\,.
\end{aligned}
\tag{4.7}
$$

Taking the limit on both sides of the foregoing inequality and taking into account (4.2), (4.3) and $\xi=u(T)$, the weak lower semicontinuity of the norm in H as well as

(4.6) then shows for all $w \in L^p(0,T;V)$

$$\begin{aligned}
&\int_0^T \langle u'(t), u(t)\rangle dt = \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u_0|^2 \\
&\le \int_0^T \langle f(t), u(t)\rangle dt - \int_0^T \langle a(t), w(t)\rangle dt - \int_0^T \langle A(t)w(t), u(t) - w(t)\rangle dt \\
&= \int_0^T \langle u'(t), u(t)\rangle dt + \int_0^T \langle a(t), u(t) - w(t)\rangle dt - \int_0^T \langle A(t)w(t), u(t) - w(t)\rangle dt\,.
\end{aligned}$$

We, hence, come up with

$$\int_0^T \langle A(t)w(t) - a(t), w(t) - u(t)\rangle dt \ge 0\,.$$

With $w = u \pm sv$ ($v \in L^p(0,T;V)$, $s \in (0,1]$), the hemicontinuity of A (together with the growth condition and Lebesgue's theorem of dominated convergence) implies the assertion $a = Au$.

After all, we have shown the convergence asserted at least for a subsequence. Since the solution $u \in \mathscr{W}$ to (1.1) is unique, the whole sequence $\{u_{\mathbb{I}_\ell}\}_{\ell\in\mathbb{N}}$ converges weakly in $L^p(0,T;V)$ towards u. This is proven by contradiction. □

We shall emphasise here that we have no $L^\infty(0,T;H)$-bound on the sequence of numerical solutions and, in particular, no control over the time derivative of the numerical solutions (which, in general, does not exist in the weak sense). We, therefore, cannot make any statement about strong convergence due to the lack of compactness except under additional structural assumptions.

Corollary 4.1 *In addition to the assumptions of Theorem 4.1 let V be uniformly convex. Moreover, assume that there is a constant $\mu_0 > 0$ such that for all $v, w \in V$ and all $t \in [0,T]$*

$$\langle A(t)v - A(t)w, v - w\rangle \ge \mu_0 \left(\|v\|^{p-1} - \|w\|^{p-1}\right)\left(\|v\| - \|w\|\right). \tag{4.8}$$

Then $\{u_{\mathbb{I}_\ell}\}_{\ell\in\mathbb{N}}$ converges strongly in $L^p(0,T;V)$ towards the exact solution $u \in \mathscr{W}$.

Proof With V also $L^p(0,T;V)$ is a uniformly convex Banach space (see, e.g., [14, Satz 1.15]). In order to prove strong convergence in $L^p(0,T;V)$ of the already weakly convergent sequence of numerical solutions, it thus suffices to show that the sequence of the corresponding norms converges towards the norm of the exact solution.

With Hölder's inequality, it is easy to show that (4.8) implies

$$\begin{aligned}
&\int_0^T \langle A(t)v(t) - A(t)w(t), v(t) - w(t)\rangle\, dt \\
&\ge \mu_0 \left(\|v\|^{p-1}_{L^p(0,T;V)} - \|w\|^{p-1}_{L^p(0,T;V)}\right)\left(\|v\|_{L^p(0,T;V)} - \|w\|_{L^p(0,T;V)}\right)
\end{aligned}$$

for all $v, w \in L^p(0,T;V)$.

Let u be the exact solution and $\{u_{\mathbb{I}_\ell}\}_{\ell\in\mathbb{N}}$ be the sequence of numerical solutions. With (4.7), (3.4), and (3.5), we then find

$$\begin{aligned}
&\mu_0 \left(\|u_{\mathbb{I}_\ell}\|^{p-1}_{L^p(0,T;V)} - \|u\|^{p-1}_{L^p(0,T;V)} \right) \left(\|u_{\mathbb{I}_\ell}\|_{L^p(0,T;V)} - \|u\|_{L^p(0,T;V)} \right) \\
&\leq \int_0^T \langle A(t)u_{\mathbb{I}_\ell}(t) - A(t)u(t), u_{\mathbb{I}_\ell}(t) - u(t) \rangle \, dt \\
&= \int_0^T \langle A(t)u_{\mathbb{I}_\ell}(t), u_{\mathbb{I}_\ell}(t) \rangle \, dt - \int_0^T \langle A(t)u_{\mathbb{I}_\ell}(t), u(t) \rangle \, dt - \int_0^T \langle A(t)u(t), u_{\mathbb{I}_\ell}(t) - u(t) \rangle \, dt \\
&= -\frac{1}{2}|u_{\mathbb{I}_\ell}(T^-)|^2 + \frac{1}{2}|u^0_{\mathbb{I}_\ell}|^2 - \frac{1}{2}\sum_{n=1}^N \left| [\![u_{\mathbb{I}_\ell}(t_{n-1})]\!] \right|^2 + \int_0^T \langle f(t), u_{\mathbb{I}_\ell}(t) \rangle \, dt \\
&\qquad - \int_0^T \langle A(t)u_{\mathbb{I}_\ell}(t), u(t) \rangle \, dt - \int_0^T \langle A(t)u(t), u_{\mathbb{I}_\ell}(t) - u(t) \rangle \, dt \, .
\end{aligned}$$

Because of (4.2), (4.3) (with $\xi = u(T)$ and $a = Au$), the weak lower semicontinuity of the norm in H, and (4.6), the limes superior of the right-hand side of the foregoing estimate is bounded by

$$\begin{aligned}
-\frac{1}{2}|u(T)|^2 + \frac{1}{2}|u_0|^2 + \int_0^T \langle f(t), u(t) \rangle \, dt - \int_0^T \langle A(t)u(t), u(t) \rangle \, dt \\
= \int_0^T \langle -u'(t) + f(t) - A(t)u(t), u(t) \rangle dt = 0 \, .
\end{aligned}$$

Since $z \mapsto z^{p-1}$ is strictly monotonically increasing, this proves

$$\|u_{\mathbb{I}_\ell}\|_{L^p(0,T;V)} \to \|u\|_{L^p(0,T;V)}$$

and hence the assertion. □

Note that a monotone operator fulfilling (5.1) below always fulfills (4.8). An example is given in the following section. Moreover, the function spaces $W^{1,p}_0(\Omega)$ and $L^p(\Omega)$ ($p \in (1,\infty)$) from the examples in Section 2 are uniformly convex.

5 Stability and smooth-data error estimates

In addition to Assumption A, let us suppose that the operators $A(t) : V \to V^*$ ($t \in [0,T]$) are uniformly monotone in the sense that there is a constant $\mu_0 > 0$ such that for all $t \in [0,T]$ and $v, w \in V$

$$\langle A(t)v - A(t)w, v - w \rangle \geq \mu_0 \|v - w\|^p \, . \tag{5.1}$$

Note that this restricts the range for the Lebesgue exponent p to the interval $[2,\infty)$ as there is no nontrivial monotone operator fulfilling (5.1) with $1 < p < 2$. Furthermore, uniform monotonicity implies the coercivity condition as well as strict monotonicity (and thus uniqueness of a solution of the time discrete problem). We could also allow uniform monotonicity up to a shift $+\kappa I$. However, the problem can then again be transformed into a problem with $\kappa = 0$.

For deriving error estimates, we will also require the following Hölder-like condition on bounded subsets: there are $\beta > 0$, $\gamma \in (0,1]$ such that for all $t \in [0,T]$ and $v, w \in V$

$$\|A(t)v - A(t)w\|_* \leq \beta(1 + \|v\| + \|w\|)^{p-1-\gamma}\|v - w\|^\gamma . \tag{5.2}$$

Taking $\gamma = 1$, both the conditions (5.1) and (5.2) are, e.g., fulfilled for the p-Laplacian (see (2.1)) and the porous medium equation (see (2.2)) with $\sigma(u) = |u|^{p-2}u$. This is an immediate consequence of the elementary fact that there are constants $\mu_0, \beta > 0$ such that for all $x, y \in \mathbb{R}$

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq \mu_0(x-y)^p$$

and

$$\big||x|^{p-2}x - |y|^{p-2}y\big| \leq \beta(1 + |x| + |y|)^{p-2}|x-y| .$$

A simple example with $\gamma = 1/2$ and $p = 2$ is given by the weak formulation of the initial-boundary value problem

$$\begin{aligned} \partial_t u - \partial_x \rho(\partial_x u) &= f \quad \text{in } (a,b) \times (0,T) \\ u(a,\cdot) = u(b,\cdot) &= 0 \quad \text{in } (0,T) \\ u(\cdot,0) &= u_0 \quad \text{in } (a,b) , \end{aligned}$$

where f, u_0 are suitably given functions and

$$\rho(z) = \begin{cases} 0 & \text{if } z = 0 , \\ |z|^{-1/2}z & \text{if } 0 < |z| \leq 1 , \\ z & \text{else.} \end{cases}$$

It is elementary to show that for all $x, y \in \mathbb{R}$

$$(\rho(x) - \rho(y))(x-y) \geq \frac{1}{2}|x-y|^2$$

and

$$|\rho(x) - \rho(y)| \leq \sqrt{2}(1 + |x| + |y|)^{1/2}|x-y|^{1/2} .$$

This implies the properties of the corresponding time-independent operator A when choosing $H = L^2(a,b)$ and $V = H_0^1(a,b)$. Here, the operator $A : H_0^1(a,b) \to H^{-1}(a,b)$ is given by

$$\langle Av, w \rangle = \int_a^b \rho(\partial_x v)\,\partial_x w\,dx , \quad v, w \in H_0^1(a,b) .$$

Let us turn back to the abstract setting. We are now in the position to prove the continuous dependence of the numerical solution on the problem's data.

Theorem 5.1 *Let Assumption A be fulfilled with $p \geq 2$ and assume (5.1). Let $u_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ and $v_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ be the solution of (1.3) corresponding to $u^0_{\mathbb{I}} \in H$, $f \in L^{p^*}(0,T;V^*)$ and $v^0_{\mathbb{I}} \in H$, $g \in L^{p^*}(0,T;V^*)$, respectively. Then*

$$\max_{n=1,2,\ldots,N} |(u_{\mathbb{I}} - v_{\mathbb{I}})(t_n^-)|^2 + \max_{n=1,2,\ldots,N} |(u_{\mathbb{I}} - v_{\mathbb{I}})(t_{n-1}^+)|^2 + \sum_{n=1}^{N} |[\![(u_{\mathbb{I}} - v_{\mathbb{I}})(t_{n-1})]\!]|^2$$
$$+ \|u_{\mathbb{I}} - v_{\mathbb{I}}\|^p_{L^p(0,T;V)} \leq c \left(|u^0_{\mathbb{I}} - v^0_{\mathbb{I}}|^2 + \|f - g\|^{p^*}_{L^{p^*}(0,T;V^*)} \right),$$

where $c > 0$ only depends on μ_0 and p.

Proof We subtract the two equations (1.3) corresponding to $u_{\mathbb{I}}$ and $v_{\mathbb{I}}$, test with $(u_{\mathbb{I}} - v_{\mathbb{I}})\chi_{[0,t_k]}$ ($k = 1,2,\ldots,N$) and employ the uniform monotonicity of $A(t)$ ($t \in [0,T]$). The rest of the proof follows the same arguments as that of Theorem 3.2. □

Unfortunately, we cannot derive a priori error estimates (in the case of sufficiently regular exact solutions) directly from the foregoing stability estimate, since we cannot test (1.3) by $u - u_{\mathbb{I}}$ as the discretisation error is not an element of $\mathscr{W}_{\mathbb{I}}$. We, therefore, introduce the interpolation $\pi^{\boldsymbol{q}}_{\mathbb{I}} u \in \mathscr{W}_{\mathbb{I}}$ of $u \in \mathscr{W}$ (see also [31, p. 842], [33, p. 207]) defined by

$$\begin{aligned} (\pi^{\boldsymbol{q}}_{\mathbb{I}} u)(t_n^-) &= u(t_n) \quad \text{in } H \text{ for } n = 1,2,\ldots,N, \\ \int_{I_n} (\pi^{\boldsymbol{q}}_{\mathbb{I}} u)(t)\varphi(t)dt &= \int_{I_n} u(t)\varphi(t)dt \quad \text{in } V \text{ for all } \varphi \in \mathscr{P}^{q_n-1}(I_n;\mathbb{R}),\ n = 1,2,\ldots,N. \end{aligned} \tag{5.3}$$

Remember here that $\mathscr{W} \hookrightarrow \mathscr{C}([0,T];H)$ and note that the number of degrees of freedom for $\pi^{\boldsymbol{q}}_{\mathbb{I}} u \in \mathscr{W}_{\mathbb{I}}$ coincides with the number of conditions above, namely $\sum_{n=1}^N (q_n + 1)$. Relying on a Hilbert space theory, existence and uniqueness of $\pi^{\boldsymbol{q}}_{\mathbb{I}} u \in \mathscr{W}_{\mathbb{I}}$ is shown in [31,33] (for slightly different definitions). For the sake of completeness, we show in the sequel existence and uniqueness also for our setting.

The determination of $\pi^{\boldsymbol{q}}_{\mathbb{I}} u \in \mathscr{W}_{\mathbb{I}}$ can be split into that of $(\pi^{\boldsymbol{q}}_{\mathbb{I}} u)_{|I_n} \in \mathscr{P}^{q_n}(I_n;V)$ ($n = 1,2,\ldots,N$) fulfilling the conditions of (5.3) for the current n. We may now transform I_n on $(-1,1)$ and may introduce the local interpolation operator $\hat{\pi}^q$ defined by

$$(\hat{\pi}^q \hat{u})(1) = \hat{u}(1), \quad \int_{-1}^{1} (\hat{\pi}^q \hat{u})(t)\varphi(t)dt = \int_{-1}^{1} \hat{u}(t)\varphi(t)dt \ \text{ for all } \varphi \in \mathscr{P}^{q-1}(-1,1;\mathbb{R}),$$

where $\hat{u}$ denotes the transform of the restriction $u_{|\overline{I_n}}$ on the interval $[-1,1]$ and $q = q_n$. Note that $\hat{\pi}^q \hat{u} = \widehat{(\pi^{\boldsymbol{q}}_{\mathbb{I}} u)_{|\overline{I_n}}}$.

Denoting by

$$L_i(t) := \frac{(-1)^i}{2^i\, i!} \frac{d^i}{dt^i} (1-t^2)^i \in \mathscr{P}^i(-1,1;\mathbb{R}), \quad i = 0,1,\ldots,q,$$

the Legendre polynomials, which form a basis in $\mathscr{P}^q(-1,1;\mathbb{R})$, we can make the ansatz

$$(\hat{\pi}^q \hat{u})(t) = \sum_{i=0}^{q} \hat{u}_i L_i(t) \tag{5.4}$$

and have to determine the coefficients $\hat{u}_i \in V$. Employing the relations

$$L_i(1) = 1\,, \quad \int_{-1}^{1} L_i(t)L_j(t)dt = \frac{2}{2i+1}\,\delta_{ij}\,,$$

there is obviously a unique solution of the corresponding linear system of equations, which is given by

$$\hat{u}_i = \frac{2i+1}{2}\int_{-1}^{1}\hat{u}(t)L_i(t)dt \ (i = 0,1,\ldots,q-1)\,, \quad \hat{u}_q = \hat{u}(1) - \sum_{i=0}^{q-1}\hat{u}_i\,, \tag{5.5}$$

which proves the existence and uniqueness of the local interpolant and thus also of the interpolant $\pi_{\mathbb{I}}^{\boldsymbol{q}} u$.

Unfortunately, the results in [32] on the interpolation error cannot be applied here as these results rely upon a setting in $L^2(0,T;V)$ with V being a separable Hilbert space. We, therefore, provide in the following an estimate of the approximation and interpolation error. The dependence on the polynomial degree and order of regularity in our estimate might, however, not be optimal as it is in [32, Prop. 3.9]. Nevertheless, we obtain the same order of convergence.

We commence by proving an approximation result.

Lemma 5.1 *Let $(X,\|\cdot\|_X)$ be a real Banach space and let $v : [-1,1] \to X$ be a given function with $v, v', \ldots, v^{(q+1)} \in L^r(-1,1;X)$, where $q \in \mathbb{N}$ and $r \in [1,\infty]$. Then there is a polynomial $Qv \in \mathscr{P}^q(-1,1;X)$ such that for all $s \in [1,\infty]$*

$$\begin{aligned}\|v - Qv\|_{L^s(-1,1;X)} &\le C(j,r,s)\|v^{(j+1)}\|_{L^r(-1,1;X)}\,, \quad j = 0,1,\ldots,q\,,\\ \|v' - (Qv)'\|_{L^s(-1,1;X)} &\le C(j-1,r,s)\|v^{(j+1)}\|_{L^r(-1,1;X)}\,, \quad j = 1,2,\ldots,q\,,\end{aligned}$$

where (with the convention $1/0 := \infty$, $1/\infty := 0$, $\infty^0 := 1$)

$$C(j,r,s) = \frac{2^{j+1+2/s-1/r}}{j!(j+1+j/(r-1))^{1-1/r}(j+1+1/s-1/r)^{1/s}s^{1/s}}\,. \tag{5.6}$$

Proof The proof employs, as in the classical setting for deriving estimates of the approximation error in Sobolev spaces, an averaged Taylor expansion.

The assumption on v implies that $v, v', \ldots, v^{(q)} : [-1,1] \to X$ are absolutely continuous. We, therefore, have for all $t, y \in [-1,1]$ the Taylor expansion

$$v(t) = \sum_{k=0}^{j}\frac{(t-y)^k}{k!}v^{(k)}(y) + \int_y^t \frac{(t-z)^j}{j!}v^{(j+1)}(z)dz\,.$$

Let $\varphi \in \mathscr{C}_0^\infty(-1,1)$ be nonnegative with $\int_{-1}^1 \varphi(y)dy = 1$ and define

$$(Qv)(t) := \int_{-1}^{1}\left(\sum_{k=0}^{j}\frac{(t-y)^k}{k!}v^{(k)}(y)\right)\varphi(y)dy\,.$$

It is clear that $Qv \in \mathscr{P}^j(-1,1;X) \subseteq \mathscr{P}^q(-1,1;X)$. We then find

$$\begin{aligned} v(t)-(Qv)(t) &= \int_{-1}^{1}\left(\int_{y}^{t}\frac{(t-z)^j}{j!}v^{(j+1)}(z)dz\right)\varphi(y)dy \\ &= \int_{-1}^{t}\left(\int_{-1}^{z}\varphi(y)dy\right)\frac{(t-z)^j}{j!}v^{(j+1)}(z)dz \\ &\quad + \int_{t}^{1}\left(\int_{z}^{1}\varphi(y)dy\right)\frac{(t-z)^j}{j!}v^{(j+1)}(z)dz. \end{aligned}$$

Let $r^* = r/(r-1)$ be the exponent conjugated to $r>1$ with $r^* := 1$ for $r=\infty$ (and the usual modifications). With the properties of φ and Hölder's inequality, we obtain for $r>1$

$$\begin{aligned} \|v(t)-(Qv)(t)\|_X &\le \int_{-1}^{1}\frac{|t-z|^j}{j!}\|v^{(j+1)}(z)\|_X dz \\ &\le \left(\int_{-1}^{1}\left[\frac{|t-z|^j}{j!}\right]^{r^*}dz\right)^{1/r^*}\|v^{(j+1)}\|_{L^r(-1,1;X)}, \end{aligned}$$

where

$$\int_{-1}^{1}\left[\frac{|t-z|^j}{j!}\right]^{r^*}dz = \frac{(t+1)^{jr^*+1}}{(j!)^{r^*}(jr^*+1)} + \frac{(1-t)^{jr^*+1}}{(j!)^{r^*}(jr^*+1)}.$$

If $r=1$, we have analogously

$$\|v(t)-(Qv)(t)\|_X \le \frac{\max(t+1,1-t)^j}{j!}\|v^{(j+1)}\|_{L^1(-1,1;X)}.$$

We thus come up with

$$\|v-Qv\|_{L^s(-1,1;X)} \le C(j,r,s)\|v^{(j+1)}\|_{L^r(-1,1;X)},$$

where for $r>1$ and $s<\infty$

$$\begin{aligned} &\left(\int_{-1}^{1}\left[\frac{(t+1)^{jr^*+1}}{(j!)^{r^*}(jr^*+1)} + \frac{(1-t)^{jr^*+1}}{(j!)^{r^*}(jr^*+1)}\right]^{s/r^*}dt\right)^{1/s} \\ &\le \frac{2^{j+1/r^*+2/s}}{j!(jr^*+1)^{1/r^*}(js+s/r^*+1)^{1/s}} =: C(j,r,s) \end{aligned}$$

and

$$C(j,1,s) = \frac{2^{j+2/s}}{j!(js+1)^{1/s}}\ (s<\infty),\ C(j,1,\infty) = \frac{2^j}{j!},\ C(j,r,\infty) = \frac{2^{j+1/r^*}}{j!(jr^*+1)^{1/r^*}}\ (r>1).$$

This, together with some elementary calculations, proves the first assertion.

For the second estimate, we observe that

$$v'(t)-(Qv)'(t) = \int_{-1}^{1}\int_{y}^{t}\frac{(t-z)^{(j-1)}}{(j-1)!}v^{(j+1)}(z)dz\varphi(y)dy.$$

The rest of the proof follows the same arguments as before but with j being replaced by $j-1$ at the appropriate places. □

Note that we have, in particular,

$$C(j,p^*,p) = \frac{2^{j+3/p}}{j!\sqrt[p]{(jp+1)(jp+2)}}\,.$$

The following result provides the boundedness of the local interpolation operator.

Lemma 5.2 *Let $(X,\|\cdot\|_X)$ be a real Banach space and let $v:[-1,1]\to X$ be a given function with $v,v'\in L^s(-1,1;X)$ ($s\in[1,\infty]$). Then (with the convention $(2q-1):=0$ for $q=0$ and denoting by s^* the exponent conjugated to s)*

$$\begin{aligned}\|\hat{\pi}^q v\|_{L^s(-1,1;X)} \le \Big((2q-1)^{3/2-1/\max(s,s^*)} + (2q+1)^{-1/\max(2,s)}\Big)\,\|v\|_{L^s(-1,1;X)}\\ + 2(2q+1)^{-1/\max(2,s)}\|v'\|_{L^s(-1,1;X)}\,.\end{aligned}$$

Proof We restrict our considerations to the case $q\ge 1$; the proof for $q=0$ immediately follows from (5.7) below. As in (5.4) and (5.5), we have

$$\begin{aligned}(\hat{\pi}^q v)(t) &= \sum_{i=0}^{q-1} v_i\,(L_i(t)-L_q(t)) + v(1)L_q(t)\,,\\ v_i &= \frac{2i+1}{2}\int_{-1}^{1} v(t)L_i(t)dt\ (i=0,1,\ldots,q-1)\,.\end{aligned}$$

We, therefore, find

$$\|\hat{\pi}^q v\|_{L^s(-1,1;X)} \le \sum_{i=0}^{q-1}\|v_i\|_X\|L_i-L_q\|_{L^s(-1,1)} + \|v(1)\|_X\|L_q\|_{L^s(-1,1)}\,,$$

where for $i=0,1,\ldots,q-1$

$$\|v_i\|_X \le \frac{2i+1}{2}\|v\|_{L^s(-1,1;X)}\|L_i\|_{L^{s^*}(-1,1)}\,.$$

Because of the continuous embedding of the space $W^{1,s}(0,T;X):=\{v\in L^s(-1,1;X): v'\in L^s(-1,1;X)\}$ into $\mathscr{C}([-1,1];X)$, there holds

$$\|v(1)\|_X \le 2^{-1/s}\|v\|_{L^s(-1,1;X)} + 2^{1/s^*}\|v'\|_{L^s(-1,1;X)}\,. \tag{5.7}$$

Moreover, we have

$$\|L_i\|_{L^s(-1,1)} \le 2^{1/s}(2i+1)^{-1/\max(2,s)}\ (i=0,1,\ldots,q)\,,$$

which can be shown by employing the properties of the Legendre polynomials (in particular, $\|L_i\|_{L^\infty(-1,1)}=1$, $\|L_i\|^2_{L^2(-1,1)}=2/(2i+1)$). The assertion follows from some simple but tedious estimates showing in particular that

$$\sum_{i=0}^{q-1}\frac{2i+1}{2}\|L_i\|_{L^{s^*}(-1,1)}\|L_i-L_q\|_{L^s(-1,1)} \le (2q-1)^{3/2-1/\max(s,s^*)}\,.$$

□

An immediate consequence is the following stability of the interpolation operator.

Lemma 5.3 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $v : [0,T] \to X$ be a given function with $v, v' \in L^r(I_n;X)$ $(n = 1,2,\dots,N)$, where $q_{min} \geq 1$ and $r \in [1,\infty]$. Then for all $s \in [1,\infty]$ and $j_n = 1,2,\dots,q_n$ $(n = 1,2,\dots,N)$*

$$\begin{aligned}\|\pi_{\mathbb{I}}^{\boldsymbol{q}} v\|_{L^s(0,T;X)} \leq \Big((2q_{\max}-1)^{3/2-1/\max(s,s^*)} + (2q_{\min}+1)^{-1/\max(2,s)} \Big) \|v\|_{L^s(0,T;X)} \\ +\ 2(2q_{\min}+1)^{-1/\max(2,s)} \tau_{\max} \|v'\|_{L^s(0,T;X)}\,.\end{aligned}$$

Proof Since

$$\|\pi_{\mathbb{I}}^{\boldsymbol{q}} v\|_{L^s(0,T;X)}^s = \sum_{n=1}^{N} \|\pi_{\mathbb{I}}^{\boldsymbol{q}} v\|_{L^s(I_n;X)}^s\,,$$

the assertion follows immediately from a transformation of I_n onto $(-1,1)$, estimating the local interpolation on $(-1,1)$, and an inverse transformation. Remember here that $\hat{\pi}^q \hat{v} = \widehat{\pi_{\mathbb{I}}^{\boldsymbol{q}} v}$ (with $\hat{}$ denoting the transformation). We only have to employ that

$$\begin{aligned}&\max_{n=1,\dots,N} \Big((2q-1)^{3/2-1/\max(s,s^*)} + (2q+1)^{-1/\max(2,s)} \Big) \\ &\leq (2q_{\max}-1)^{3/2-1/\max(s,s^*)} + (2q_{\min}+1)^{-1/\max(2,s)}\end{aligned}$$

and

$$\max_{n=1,\dots,N} 2(2q+1)^{-1/\max(2,s)} \leq 2(2q_{\min}+1)^{-1/\max(2,s)}\,.$$

□

We are now in the position to prove an estimate for the interpolation error.

Lemma 5.4 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $v : [0,T] \to X$ be a given function with $v, v', \dots, v^{(q_n+1)} \in L^r(I_n;X)$ $(n = 1,2,\dots,N)$, where $q_{min} \geq 1$ and $r \in [1,\infty]$. Then for all $s \in [1,\infty]$ and $j_n = 1,2,\dots,q_n$ $(n = 1,2,\dots,N)$*

$$\|v - \pi_{\mathbb{I}}^{\boldsymbol{q}} v\|_{L^s(0,T;X)} \leq \left(\sum_{n=1}^{N} \tau_n^{(j_n+1)s} D_n(j_n,r,s)^s \|v^{(j_n+1)}\|_{L^r(I_n;X)}^s \right)^{1/s},$$

where (see (5.6))

$$\begin{aligned}D_n(j_n,r,s) := \Big(1 + (2q_n-1)^{3/2-1/\max(s,s^*)} + (2q_n+1)^{-1/\max(2,s)} \Big) C(j_n,r,s) \\ +\ 2(2q_n+1)^{-1/\max(2,s)} C(j_n-1,r,s)\,.\end{aligned}$$

Proof Since

$$\|v - \pi_{\mathbb{I}}^{\boldsymbol{q}} v\|_{L^s(0,T;X)}^s = \sum_{n=1}^{N} \|v - \pi_{\mathbb{I}}^{\boldsymbol{q}} v\|_{L^s(I_n;X)}^s\,,$$

the assertion will follow again from a transformation of I_n onto $(-1,1)$, estimating the local interpolation error on $(-1,1)$, and an inverse transformation. Remember that $\hat{\pi}^q \hat{v} = \widehat{\pi_{\mathbb{I}}^{\boldsymbol{q}} v}$ (with $\hat{}$ denoting the transformation).

For the local interpolation error (with $q = q_n$ when transforming I_n on $(-1,1)$), we have

$$\hat{v} - \hat{\pi}^q \hat{v} = \hat{v} - \hat{\pi}^q Q\hat{v} + \hat{\pi}^q Q\hat{v} - \hat{\pi}^q \hat{v} = \hat{v} - Q\hat{v} - \hat{\pi}^q \left(\hat{v} - Q\hat{v}\right),$$

and thus with Lemma 5.2

$$\begin{aligned}
&\|\hat{v} - \hat{\pi}^q \hat{v}\|_{L^s(-1,1;X)} \\
&\le \left(1 + (2q-1)^{3/2-1/\max(s,s^*)} + (2q+1)^{-1/\max(2,s)}\right) \|\hat{v} - Q\hat{v}\|_{L^s(-1,1;X)} \\
&\quad + 2(2q+1)^{-1/\max(2,s)} \|\hat{v}' - (Q\hat{v})'\|_{L^s(-1,1;X)} .
\end{aligned}$$

Lemma 5.1 now yields

$$\|\hat{v} - \hat{\pi}^q \hat{v}\|_{L^s(-1,1;X)} \le D_n(j,r,s) \|\hat{v}^{(j+1)}\|_{L^r(-1,1;X)} ,$$

from which the assertion follows. □

The error estimate now reads as follows.

Theorem 5.2 *Let Assumption A be fulfilled with $p \ge 2$ and assume (5.1) as well as (5.2). Moreover, let $u \in \mathscr{W} \hookrightarrow \mathscr{C}([0,T];H)$ be the solution of (1.1) with $u_0 \in H$ and $f \in \mathscr{C}([0,T];V^*)$. If $u' \in L^p(0,T;V)$ and $u^{(r)} \in L^{p^*}(I_n;V^*)$ ($r = 2,\ldots,j_n+1$, $1 \le j_n \le q_n$, $j_{\min} := \min_{n=1,\ldots,N} j_n$) then the solution $u_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}}$ to (1.3) fulfills for $k = 1,2,\ldots,N$ the error estimates*

$$\begin{aligned}
&|u(t_k) - u_{\mathbb{I}}(t_k^-)| + \left|u_0 - u_{\mathbb{I}}(0^+)\right| \\
&\le c\left|u_0 - u_{\mathbb{I}}^0\right| + cD_1(j_1,p^*,p)^{1/2} \tau_1^{(j_1 p+2p-1)/(2p)} \|u'\|_{\mathscr{C}(\overline{I_1};V^*)}^{1/2} \|u^{(j_1+1)}\|_{L^{p^*}(I_1;V^*)}^{1/2} \\
&\quad + c\left(1 + \left(1 + (2q_{\max}-1)^{3/2-1/p}\right) \|u\|_{L^p(0,t_k;V)} + \tau_{\max} \|u'\|_{L^p(0,t_k;V)}\right)^{p(p-1-\gamma)/(2(p-1))} \times \\
&\quad \times \left(\sum_{n=1}^{k} \tau_n^{(j_n+1)p} D_n(j_n,p^*,p)^p \|u^{(j_n+1)}\|_{L^{p^*}(I_n;V^*)}^p\right)^{\gamma/(2(p-1))} ,
\end{aligned}$$

$$\begin{aligned}
&\|u - u_{\mathbb{I}}\|_{L^p(0,t_k;V)} \\
&\le c\left|u_0 - u_{\mathbb{I}}^0\right|^{2/p} + cD_1(j_1,p^*,p)^{1/p} \tau_1^{(j_1 p+2p-1)/p^2} \|u'\|_{\mathscr{C}(\overline{I_1};V^*)}^{1/p} \|u^{(j_1+1)}\|_{L^{p^*}(I_1;V^*)}^{1/p} \\
&\quad + c\left(1 + \left(1 + (2q_{\max}-1)^{3/2-1/p}\right) \|u\|_{L^p(0,t_k;V)} + \tau_{\max} \|u'\|_{L^p(0,t_k;V)}\right)^{(p-1-\gamma)/(p-1)} \times \\
&\quad \times \left(\sum_{n=1}^{k} \tau_n^{(j_n+1)p} D_n(j_n,p^*,p)^p \|u^{(j_n+1)}\|_{L^{p^*}(I_n;V^*)}^p\right)^{\gamma/(p(p-1))} ,
\end{aligned}$$

and thus

$$|u(t_k) - u_{\mathbb{I}}(t_k^-)| + \left|u_0 - u_{\mathbb{I}}(0^+)\right| \le c\left|u_0 - u_{\mathbb{I}}^0\right| + \bar{c}_1 \tau_1^{(j_1 p+2p-1)/(2p)} + \bar{c}\, \tau_{\max}^{(j_{\min}+1)\gamma p/(2(p-1))} ,$$

$$\|u - u_{\mathbb{I}}\|_{L^p(0,t_k;V)} \le c\left|u_0 - u_{\mathbb{I}}^0\right|^{2/p} + \bar{c}_1 \tau_1^{(j_1 p+2p-1)/p^2} + \bar{c}\, \tau_{\max}^{(j_{\min}+1)\gamma/(p-1)} .$$

Here $c > 0$ is independent of u and $\boldsymbol{q}$ but $\bar{c}_1$ may depend on $\boldsymbol{q}$ as well as on $\|u'\|_{\mathscr{C}(\overline{I_1};V^*)}$, $\|u^{(j_1+1)}\|_{L^{p^*}(I_1;V^*)}$, and $\bar{c}$ may depend on $\boldsymbol{q}$ as well as on $\|u\|_{L^p(0,T;V)}$, $\|u'\|_{L^p(0,T;V)}$, $\|u^{(j_{\min}+1)}\|_{L^{p^*}(0,T;V^*)}$.

Remark 5.1 Let the initial error be of appropriate order and let τ_1 be chosen appropriately. The above theorem then provides a global error estimate of maximum order $(q_{\min}+1)\gamma p/(2(p-1))$ for the left values of the numerical solution in each of the subintervals, measured in the H-norm, and of order $(q_{\min}+1)\gamma/(p-1)$ for the error measured in the $L^p(0,T;V)$-norm.

The order of the error here coincides with the result known from [1] for the special situation $p=2$, $\gamma=1$, and $q_n \equiv q$ dealt with there; the order then is $q+1$.

We do not claim that the given orders are optimal with respect to the dependence on p. It is, e.g., known for the evolutionary p-Laplacian and similar problems with p-structure that the concept of quasi-norms allows to derive convergence rates of optimal first order for the implicit Euler method (see [4] and the references given there). Quasi-norms are depending on the exact solution and take into account the possible degeneracy of the underlying problem. It remains open for future research whether analogous results can be derived for the discontinuous Galerkin approximation in time.

We are also not going to quantify the dependence on $\boldsymbol{q}$ in more detail as we would do for a hp-variant method since, in general, we cannot expect a very high regularity of the exact solution of the nonlinear problem.

Proof (of Theorem 5.2) We test (1.1) as well as (1.3) by an arbitrary function $v_{\mathbb{I}} \in \mathscr{W}_{\mathbb{I}} \subseteq \mathscr{X}$ and subtract the two equations. This gives (with the convention $u(0^-) := u_0$, $(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-) := (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)$) the relation

$$\sum_{n=1}^{N} \int_{I_n} \big(u'(t) - u_{\mathbb{I}}'(t), v_{\mathbb{I}}(t)\big)\, dt + \sum_{n=1}^{N} \big([\![u(t_{n-1}) - u_{\mathbb{I}}(t_{n-1})]\!], v_{\mathbb{I}}(t_{n-1}^+)\big) \\ + \int_0^T \langle A(t)u(t) - A(t)u_{\mathbb{I}}(t), v_{\mathbb{I}}(t)\rangle\, dt = 0\,, \tag{5.8}$$

which resembles the classical Galerkin orthogonality. (Remember $[\![u(t_{n-1})]\!] \equiv 0$ since u is continuous.)

With (5.1), integration by parts (employing (3.4), (3.5)), taking $v_{\mathbb{I}} = (u_{\mathbb{I}} - \pi_{\mathbb{I}}^{\boldsymbol{q}} u)\chi_{[0,t_k]}$ ($k = 1, 2, \ldots, N$) in (5.8) and integration by parts again, we find

$$\begin{aligned} &\frac{1}{2}|u_{\mathbb{I}}(t_k^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-)|^2 - \frac{1}{2}|u_{\mathbb{I}}(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-)|^2 \\ &\quad + \frac{1}{2}\sum_{n=1}^{k} \big|[\![u_{\mathbb{I}}(t_{n-1}) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1})]\!]\big|^2 + \mu \int_0^{t_k} \|u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|^p dt \\ &\le \sum_{n=1}^{k} \int_{I_n} \big(u_{\mathbb{I}}'(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)'(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big)\, dt \\ &\quad + \sum_{n=1}^{k} \big([\![u_{\mathbb{I}}(t_{n-1}) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1})]\!], u_{\mathbb{I}}(t_{n-1}^+) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1}^+)\big) \\ &\quad + \int_0^{t_k} \big\langle A(t)u_{\mathbb{I}}(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big\rangle\, dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{k} \int_{I_n} \big(u'(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)'(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big)\, dt \\
&\quad + \sum_{n=1}^{k} \big([\![u(t_{n-1}) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1})]\!], u_{\mathbb{I}}(t_{n-1}^+) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1}^+)\big) \\
&\quad + \int_0^{t_k} \big\langle A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big\rangle\, dt \\
&= - \sum_{n=1}^{k} \int_{I_n} \big(u(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t), u_{\mathbb{I}}'(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)'(t)\big)\, dt \\
&\quad - \sum_{n=1}^{k} \big(u(t_{n-1}^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1}^-), [\![u_{\mathbb{I}}(t_{n-1}) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1})]\!]\big) \\
&\quad + \big(u(t_k^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-), u_{\mathbb{I}}(t_k^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-)\big) \\
&\quad - \big(u(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-), u_{\mathbb{I}}(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-)\big) \\
&\quad + \int_0^{t_k} \big\langle A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big\rangle\, dt\,.
\end{aligned}
$$

The first term of the right-hand side of the foregoing estimate vanishes in view of the definition of $\pi_{\mathbb{I}}^{\boldsymbol{q}}$ since the restriction of $u_{\mathbb{I}}' - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)'$ on I_n is in $\mathscr{P}^{q_n - 1}(I_n; V)$. The second (except for the initial error with $n = 1$) and the third term vanish also because of the definition of $\pi_{\mathbb{I}}^{\boldsymbol{q}}$. Hence, we find

$$
\begin{aligned}
&\frac{1}{2}|u_{\mathbb{I}}(t_k^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-)|^2 - \frac{1}{2}|u_{\mathbb{I}}(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-)|^2 \\
&\quad + \frac{1}{2}\sum_{n=1}^{k} \big|[\![u_{\mathbb{I}}(t_{n-1}) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_{n-1})]\!]\big|^2 + \mu \int_0^{t_k} \|u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|^p dt \\
&\le - \big(u(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-), [\![u_{\mathbb{I}}(0) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0)]\!]\big) \\
&\quad - \big(u(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-), u_{\mathbb{I}}(0^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-)\big) \\
&\quad + \int_0^{t_k} \big\langle A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big\rangle\, dt\,.
\end{aligned}
$$

Recalling $u(0^-) := u_0$, $u_{\mathbb{I}}(0^-) := u_{\mathbb{I}}^0$, $(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^-) := (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)$ and employing Cauchy–Schwarz's and Young's inequality yields

$$
\begin{aligned}
&\frac{1}{2}|u_{\mathbb{I}}(t_k^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-)|^2 + \frac{1}{2}\big|u_{\mathbb{I}}(0^+) - u_{\mathbb{I}}^0\big|^2 + \mu \int_0^{t_k} \|u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|^p dt \\
&\le \frac{1}{2}|u_{\mathbb{I}}^0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)|^2 - \big(u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+), u_{\mathbb{I}}(0^+) - u_{\mathbb{I}}^0 + u_{\mathbb{I}}^0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)\big) \\
&\quad + \int_0^{t_k} \big\langle A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t), u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\big\rangle\, dt \\
&\le |u_{\mathbb{I}}^0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)|^2 + \frac{3}{2}\big|u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)\big|^2 + \frac{1}{4}\big|u_{\mathbb{I}}(0^+) - u_{\mathbb{I}}^0\big|^2 \\
&\quad + c\int_0^{t_k} \|A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|_*^{p^*} dt + \frac{\mu}{2}\int_0^{t_k} \|u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|^p dt\,,
\end{aligned}
$$

where $c > 0$ only depends on μ and p. It follows

$$
\begin{aligned}
&|u_{\mathbb{I}}(t_k^-) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-)|^2 + \left|u_{\mathbb{I}}(0^+) - u_{\mathbb{I}}^0\right|^2 + \mu \int_0^{t_k} \|u_{\mathbb{I}}(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|^p dt \\
&\le c\left(|u_{\mathbb{I}}^0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)|^2 + \left|u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)\right|^2 + \int_0^{t_k} \|A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|_*^{p^*} dt\right) \\
&\le c\left(|u_{\mathbb{I}}^0 - u_0|^2 + \left|u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)\right|^2 + \int_0^{t_k} \|A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|_*^{p^*} dt\right).
\end{aligned}
\tag{5.9}
$$

With (5.2), Hölder's inequality and Lemma 5.2, we also have

$$
\begin{aligned}
&\int_0^{t_k} \|A(t)u(t) - A(t)(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|_*^{p^*} dt \\
&\le c\left(1 + \|u\|_{L^p(0,t_k;V)} + \|\pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_k;V)}\right)^{p^*(p-1-\gamma)} \|u - \pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_k;V)}^{p^*\gamma}.
\end{aligned}
\tag{5.10}
$$

Recalling that $u \in \mathscr{C}([0,T];H)$ and $u(t_k) = (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-)$ $(k = 1,2,\ldots,N)$, we find with (5.9) and (5.10)

$$
\begin{aligned}
&|u(t_k) - u_{\mathbb{I}}(t_k^-)|^2 + \left|u_0 - u_{\mathbb{I}}(0^+)\right|^2 + \mu \int_0^{t_k} \|u(t) - u_{\mathbb{I}}(t)\|^p dt \\
&\le c\Big(|(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_k^-) - u_{\mathbb{I}}(t_k^-)|^2 + \left|u_0 - u_{\mathbb{I}}^0\right|^2 + \left|u_{\mathbb{I}}^0 - u_{\mathbb{I}}(0^+)\right|^2 \\
&\qquad + \mu \int_0^{t_k} \|u(t) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t)\|^p dt + \mu \int_0^{t_k} \|(\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t) - u_{\mathbb{I}}(t)\|^p dt\Big) \\
&\le c\left(\left|u_0 - u_{\mathbb{I}}^0\right|^2 + |u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)|^2 + \|u - \pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_k;V)}^p\right) \\
&\qquad + c\left(1 + \|u\|_{L^p(0,t_k;V)} + \|\pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_k;V)}\right)^{p^*(p-1-\gamma)} \|u - \pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_k;V)}^{p^*\gamma}.
\end{aligned}
$$

The final error estimate now follows from the foregoing estimate together with Lemma 5.3 and 5.4 upon noting that $p \ge p^*\gamma$ and that with (5.3), integration by parts, and $u' \in W^{1,p^*}(0,T;V^*) \hookrightarrow \mathscr{C}([0,T];V^*)$

$$
\begin{aligned}
|u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)|^2 &= |u_0 - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(0^+)|^2 - |u(t_1) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(t_1^-)|^2 \\
&= -\int_0^{t_1} \langle u'(s) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)'(s), u(s) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(s)\rangle ds \\
&= -\int_0^{t_1} \langle u'(s), u(s) - (\pi_{\mathbb{I}}^{\boldsymbol{q}} u)(s)\rangle ds \\
&\le \|u'\|_{L^{p^*}(0,t_1;V^*)} \|u - \pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_1;V)} \\
&\le \tau_1^{1/p^*} \|u'\|_{\mathscr{C}([0,t_1];V^*)} \|u - \pi_{\mathbb{I}}^{\boldsymbol{q}} u\|_{L^p(0,t_1;V)}.
\end{aligned}
$$

Indeed, in order to get a local error estimate, we apply Lemma 5.3 and 5.4 only on the subintervals I_1 and $(0,t_k)$, which is clear from the corresponding proofs.

Altogether, we obtain

$$
\begin{aligned}
&|u(t_k) - u_{\mathbb{I}}(t_k^-)|^2 + \left|u_0 - u_{\mathbb{I}}(0^+)\right|^2 + \mu \int_0^{t_k} \|u(t) - u_{\mathbb{I}}(t)\|^p dt \\
&\le c\left|u_0 - u_{\mathbb{I}}^0\right|^2 + cD_1(j_1, p^*, p)\tau_1^{j_1+1+1/p^*} \|u'\|_{\mathscr{C}(\overline{I_1};V^*)} \|u^{(j_1+1)}\|_{L^{p^*}(I_1;V^*)} \\
&\quad + c\left(1 + \left(1 + (2q_{\max} - 1)^{3/2-1/p}\right)\|u\|_{L^p(0,t_k;V)} + \tau_{\max}\|u'\|_{L^p(0,t_k;V)}\right)^{p(p-1-\gamma)/(p-1)} \times \\
&\quad \times \left(\sum_{n=1}^{k} \tau_n^{(j_n+1)p} D_n(j_n, p^*, p)^p \|u^{(j_n+1)}\|^p_{L^{p^*}(I_n;V^*)}\right)^{\gamma/(p-1)},
\end{aligned}
$$

which proves the assertion. □

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