Fair partitions and normal tilings

Dirk Frettlöh

Joint work with Christian Richter, Alexey Glazyrin, Zsolt Lángi

Technische Fakultät
Universität Bielefeld

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Part 0

Introduction
A *tiling* is a covering of the plane which is a packing of the plane as well.

Here all tiles are convex polygons.
A *tiling* is a covering of the plane which is a packing of the plane as well.

Here all tiles are convex polygons.

A tiling is called *vertex-to-vertex* if the intersection of any two tiles is a full edge, or a vertex, or empty.

(Left tiling: yes, right tiling: no)
A tiling is called *normal* if there are $r > 0, R > 0$ such that

- Each tile contains in a disk of radius $r$
- Each tile is contained in a disk of radius $R$

Normal.
A tiling is called *normal* if there are \( r > 0, R > 0 \) such that

- Each tile contains in a disk of radius \( r \)
- Each tile is contained in a disk of radius \( R \)

Not normal.
Discrete geometry provides some (seemingly) elementary problems that sometimes can (only?) be solved by heavy machinery.

Read *Cannons at Sparrows* by Günter Ziegler:


For instance:

- Ham Sandwich Theorem
- Spicy chicken Theorem
Ham Sandwich Problem:

Can we divide two convex sets in $\mathbb{R}^2$ by one line into equal halves each?

Can we divide $d$ convex sets in $\mathbb{R}^d$ by one line hyperplane into equal halves each?
Ham Sandwich Problem—Theorem:

Can we divide two convex sets in $\mathbb{R}^2$ by one line into equal halves each?

Can we divide $d$ convex sets in $\mathbb{R}^d$ by one line hyperplane into equal halves each?

Proof via the Borsuk-Ulam Theorem
Spicy Chicken Theorem

Can we divide any convex set in $\mathbb{R}^2$ into $n$ convex sets of the same area and the same perimeter?

$n=2$  $n=3$  $n=4$  $n=5$
Spicy Chicken Theorem

Can we divide any convex set in $\mathbb{R}^2$ into $n$ convex sets of the same area and the same perimeter?

$n=2$  $n=3$  $n=4$  $n=5$
Part 1
(with Christian Richter)
A rich source of interesting problems:

nandacumar.blogspot.com

**Question:** Is there a tiling of the plane by pairwise noncongruent triangles of equal area and equal perimeter?

(I.e. a spicy chicken theorem for $\mathbb{R}^2$ where all pieces are triangles)
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**Weaker question:** Is there a tiling of the plane by pairwise noncongruent triangles of equal area?
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**Question:** Is there a tiling of the plane by pairwise noncongruent triangles of equal area and equal perimeter?

(I.e. a spicy chicken theorem for \( \mathbb{R}^2 \) where all pieces are triangles)

**Answer:** No


**Weaker question:** Is there a tiling of the plane by pairwise noncongruent triangles of equal area?

**Answer:** Yes.
...but this tiling is not normal.
...and this tiling is not normal either.
Slightly harder question: Is there a normal tiling of the plane by pairwise noncongruent triangles of equal area?
Slightly harder question: Is there a normal tiling of the plane by pairwise noncongruent triangles of equal area?

Answer: Yes.


Even harder question: Is there a normal vertex-to-vertex tiling of the plane by pairwise noncongruent triangles of the same area?
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Answer: Yes.

Even harder question: Is there a normal vertex-to-vertex tiling of the plane by pairwise noncongruent triangles of the same area?

Answer: Yes.


Idea: distort
**Even harder question:** *Is there a normal vertex-to-vertex tiling of the plane by pairwise noncongruent triangles of the same area?*

**Answer:** Yes.


**Idea:** distort

![Diagram of vertex-to-vertex tiling of the plane by pairwise noncongruent triangles of the same area.](image)
Stack sheared copies of the strip tiling:

(7 pages of computation show: all triangles are incongruent)
Variations of the questions for $n$-gons ($3 \leq n \leq 6$)

Is there a normal equal area tiling by....

<table>
<thead>
<tr>
<th></th>
<th>vtv</th>
<th>not vtv</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Triangles</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>normal</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>equal perimeter</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td><strong>Quadrangles</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>normal</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>equal perimeter</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td><strong>Pentagons</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>normal</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>equal perimeter</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td><strong>Hexagons</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>normal</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>equal perimeter</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
Quadrangles, pentagons, hexagons are easier. E.g.:
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Triangles seem to be the "limiting" case (wrt degrees of freedom)
Current work:

- There are tilings of $\mathbb{R}^2$ by unit area quadrangles with equal perimeter.
- There are normal triangle tilings of $\mathbb{R}^2$ by unit area quadrangles which are arbitrarily close to the regular triangle tiling.
- There are tilings of $\mathbb{R}^2$ by unit area pentagons with equal perimeter.

Paper containing the first two is ready and soon on arXiv.org.
Part 2
(with Alexey Glazyrin and Zsolt Lángi)
Usually "not vertex-to-vertex" is less restrictive than "vertex-to-vertex".
But for hexagons it is the other way around: it is harder to find non-vertex-to-vertex tilings by hexagons than vertex-to-vertex ones.

Here: only two non-vertex-to-vertex situations. This raises the...
**Question:** How many non-vertex-to-vertex situations can a tiling by convex hexagons have?
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Very similar question: How many heptagons can a tiling by convex \( n \)-gons have, if \( n \geq 6 \)?
**Question:** How many non-vertex-to-vertex situations can a tiling by convex hexagons have?

Very similar question: How many heptagons can a tiling by convex $n$-gons have, if $n \geq 6$?

**Answer:** a lot.
Question: How many heptagons can a normal tiling by convex $n$-gons have, if $n \geq 6$?
Question: How many heptagons can a *normal* tiling by convex $n$-gons have, if $n \geq 6$?

Problem:
Question: How many heptagons can a *normal* tiling by convex $n$-gons have, if $n \geq 6$?

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Partial answer: at most finitely many.


**Question:** How many heptagons can a *normal* tiling by convex $n$-gons have, if $n \geq 6$?

Partial answer: at most finitely many.


Akopyan provides an upper bound:

$$\# \text{ heptagons} \leq \frac{2\pi D}{A} - 6$$

$D$: maximal diameter, $A$: minimal area.

(so $D/A$ is a measure for how ”normal” the tiling is)
Answer: Arbitrarily many. (Even of unit area)

D.F., Alexey Glazyrin, Zsolt Lángi: Hexagon tilings of the plane that are not edge-to-edge, submitted, arxiv:1911:xxxxx
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**Corollary:** A hexagon tiling can have arbitrarily many non-vertex-to-vertex situations (but not infinitely many)
How to obtain "arbitrary many"
We can do the maths in order to compare with Akopyan’s bound: This construction achieves 3/4 of his bound, hence his bound is asymptotically tight (linear in $D/A$). Thank you!
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We can do the maths in order to compare with Akopyan’s bound: This construction achieves $3/4$ of his bound, hence his bound is asymptotically tight (linear in $D/A$).

*Thank you!*