Fair partitions and normal tilings

Dirk Frettlöh

Joint work with Christian Richter, Alexey Glazyrin, Zsolt Lángi

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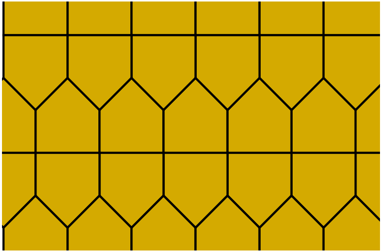
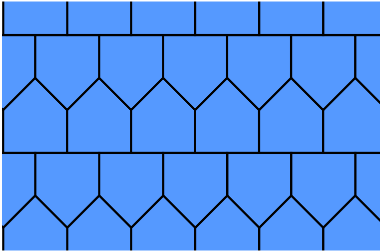
Part 0

Introduction

A *tiling* is a covering of the plane which is a packing of the plane as well.





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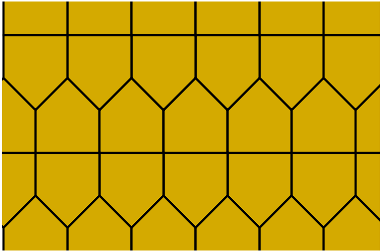

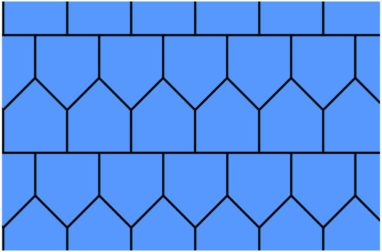

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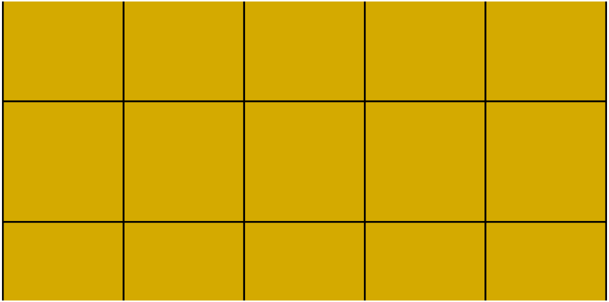
A tiling is called *vertex-to-vertex* if the intersection of any two tiles is a full edge, or a vertex, or empty.

(Left tiling: yes, right tiling: no)

A tiling is called *normal* if there are $r > 0, R > 0$ such that

- Each tile contains in a disk of radius r
- Each tile is contained in a disk of radius R



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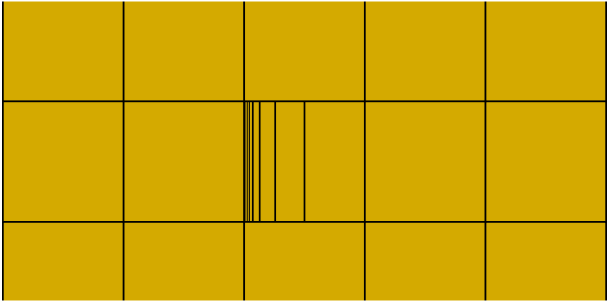


Not normal.

Discrete geoemtry provides some (seemingly) elementary problems that sometimes can (only?) be solved by heavy machinery.

Read *Cannons at Sparrows* by Günter Ziegler:

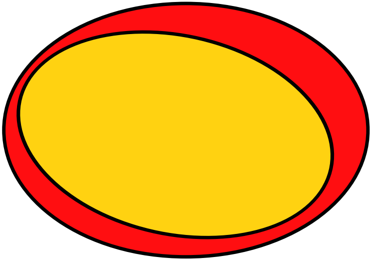
`https://www.ems-ph.org/journals/newsletter/pdf/2015-03-95.pdf`

For instance:

- Ham Sandwich Theorem
- Spicy chicken Theorem

Ham Sandwich Problem:

Can we divide two convex sets in $\mathbb{R}^2$ by one line into equal halves each?



Can we divide d convex sets in $\mathbb{R}^d$ by one line hyperplane into equal halves each?

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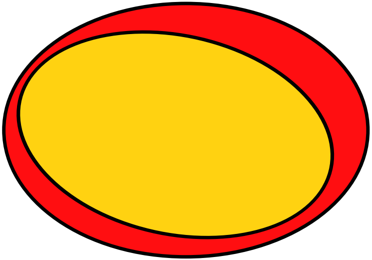
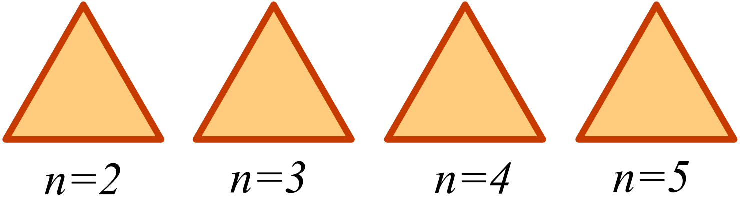


Can we divide d convex sets in $\mathbb{R}^d$ by one ~~line~~ hyperplane into equal halves each?

Proof via the Borsuk-Ulam Theorem

Spicy Chicken Theorem

Can we divide any convex set in $\mathbb{R}^2$ into n convex sets of the same area and the same perimeter?



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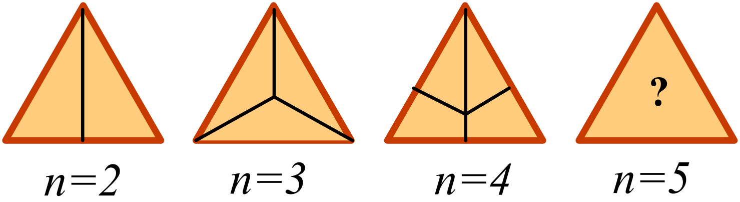

Part 1

(with Christian Richter)

A rich source of interesting problems:

`nandacumar.blogspot.com`

Question: *Is there a tiling of the plane by pairwise noncongruent triangles of equal area and equal perimeter?*

(I.e. a spicy chicken theorem for $\mathbb{R}^2$ where all pieces are triangles)

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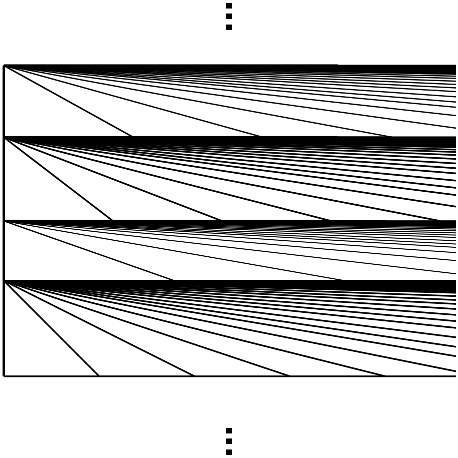
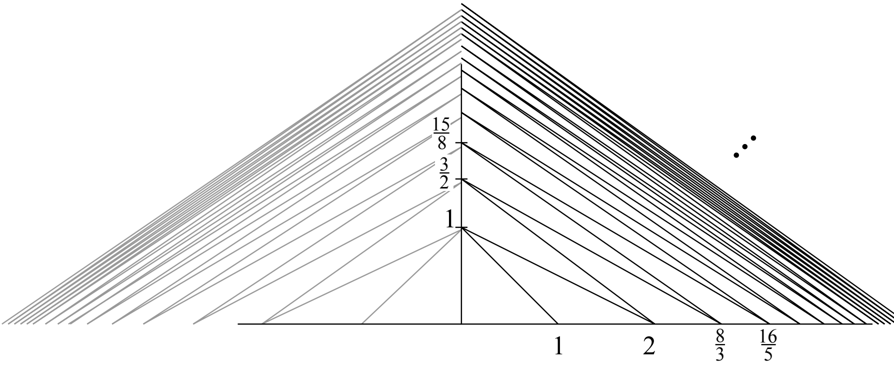


...but this tiling is not normal.



...and this tiling is not normal either.

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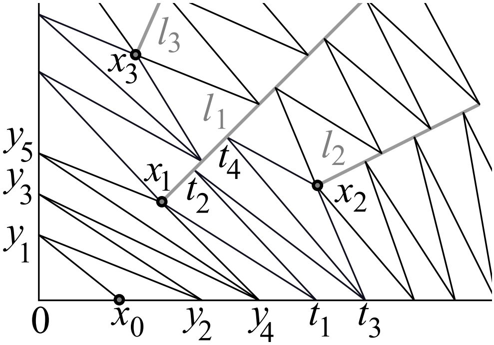

D.F.: Noncongruent equidissections of the plane, *Discrete Geometry and Symmetry*, Springer (2018)

A. Kupavskii, J. Pach, G. Tardos: Tilings of the plane with unit area triangles of bounded diameter, *Acta Math. Hungar.* 155 (2018)

Even harder question: *Is there a normal vertex-to-vertex tiling of the plane by pairwise noncongruent triangles of the same area?*

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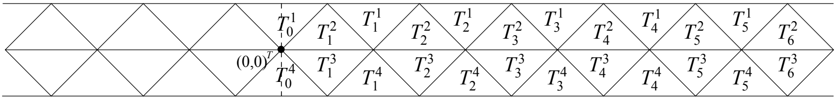
Idea: distort



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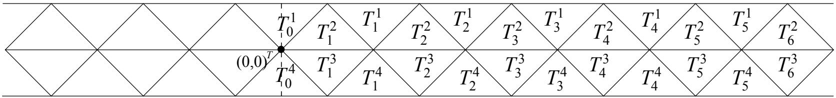
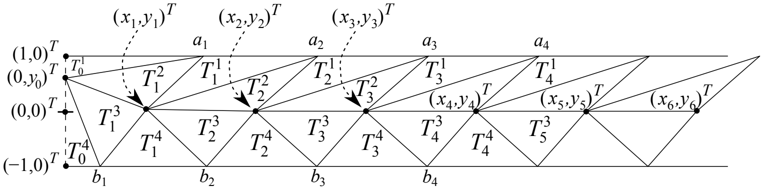
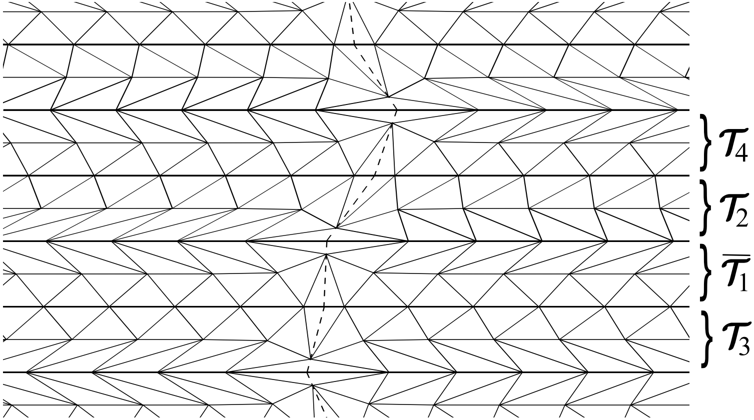




Stack sheared copies of the strip tiling:



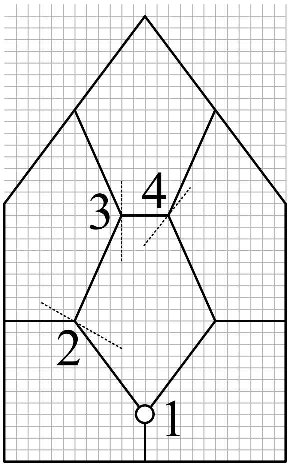
(7 pages of computation show: all triangles *are* incongruent)

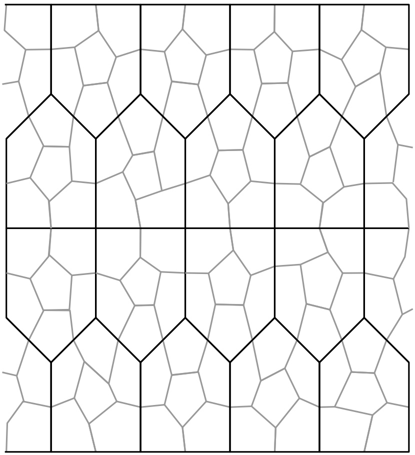
Variations of the questions for n-gons ($3 \leq n \leq 6$)

Is there a normal equal area tiling by....

Triangles	vtv	not vtv
normal	Yes	Yes
equal perimeter	No	No
Quadrangles	**vtv**	**not vtv**
normal	Yes	Yes
equal perimeter	?	?
Pentagons	**vtv**	**not vtv**
normal	Yes	Yes
equal perimeter	?	?
Hexagons	**vtv**	**not vtv**
normal	Yes	Yes
equal perimeter	?	?

Quadrangles, pentagons, hexagons are easier. E.g.:





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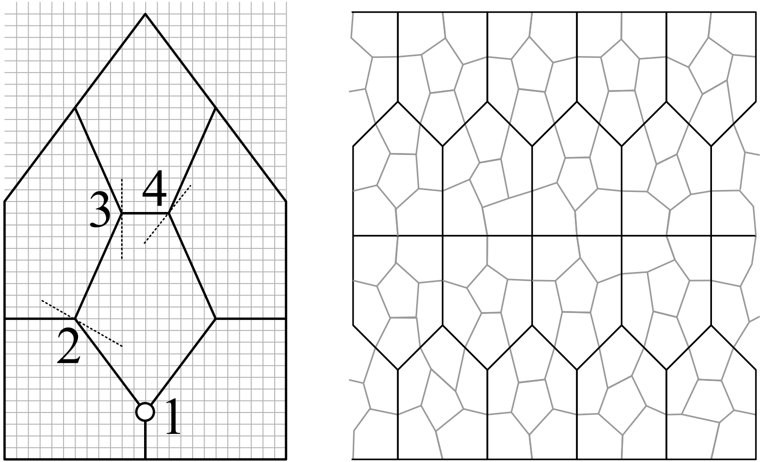


Triangles seem to be the "limiting" case (wrt degrees of freedom)

Current work :

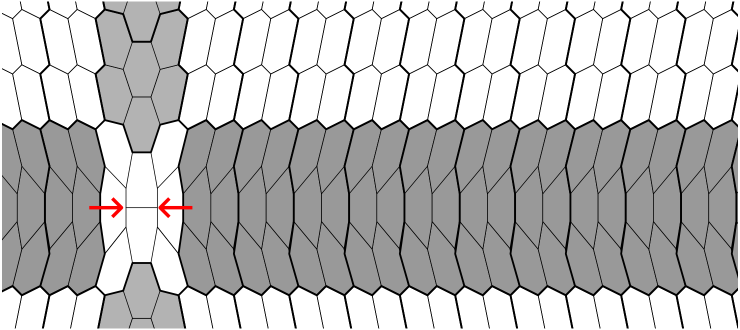
- There *are* tilings of $\mathbb{R}^2$ by unit area quadrangles with equal perimeter
- There are normal triangle tilings of $\mathbb{R}^2$ by unit area quadrangles which are arbitrarily close to the regular triangle tiling
- There are tilings of $\mathbb{R}^2$ by unit area pentagons with equal perimeter

Paper containing the first two is ready and soon on `arXiv.org`.

Part 2

(with Alexey Glazyrin and Zsolt Lángi)

Usually "not vertex-to-vertex" is less restrictive than "vertex-to-vertex".
But for hexagons it is the other way around: it is harder to find non-vertex-to-vertex tilings by hexagons than vertex-to-vertex ones.



Here: only two non-vertex-to-vertex situations. This raises the...

Question: How many non-vertex-to-vertex situations can a tiling by convex hexagons have?

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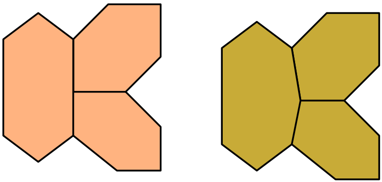
Very similar question: How many heptagons can a tiling by convex n-gons have, if $n \geq 6$?



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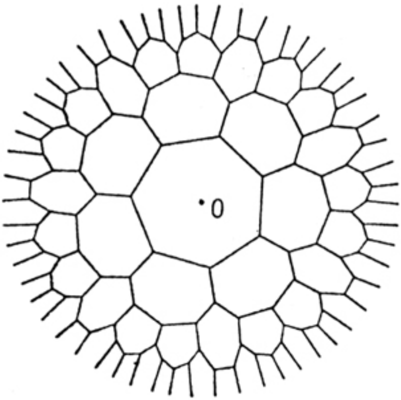
Answer: a lot.



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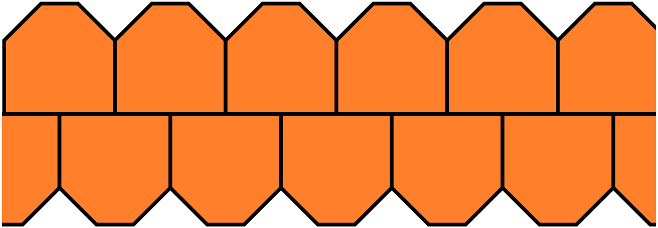
Problem:



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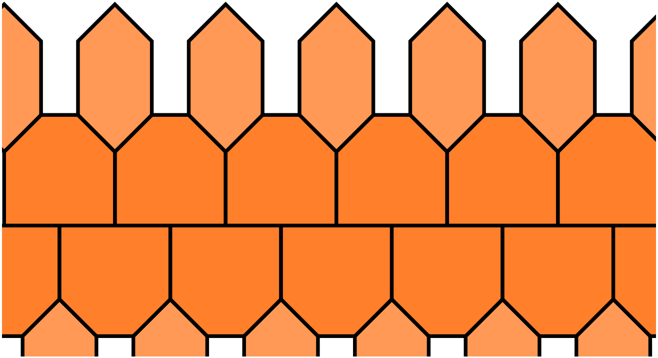

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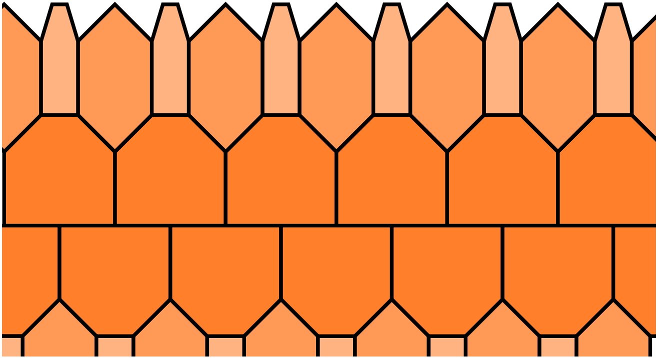

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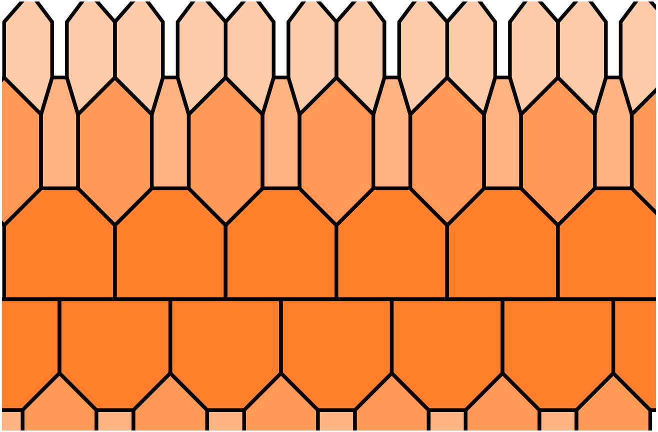

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Partial answer: at most finitely many.

Thomas Stehling: Über kombinatorische und graphentheoretische Eigenschaften normaler Pflasterungen, PhD thesis, Dortmund (1989)

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Akopyan provides an upper bound:

$$\#\text{ heptagons } \leq \frac{2\pi D}{A} - 6$$

D: maximal diameter, A: minimal area.

(so D/A is a measure for how "normal" the tiling is)

Answer: Arbitrarily many. (Even of unit area)

D.F., Alexey Glazyrin, Zsolt Lángi: Hexagon tilings of the plane that are not edge-to-edge, submitted, arxiv:1911:xxxxx

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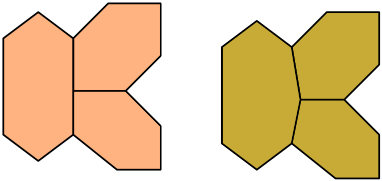
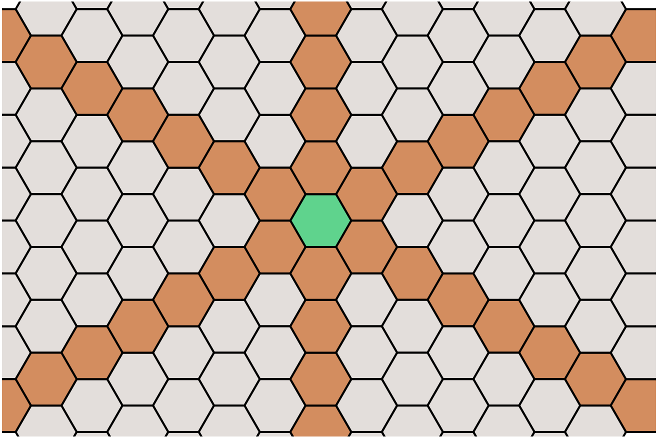
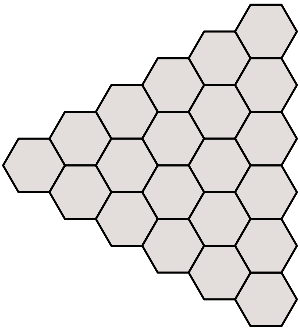
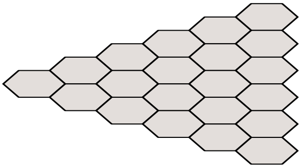
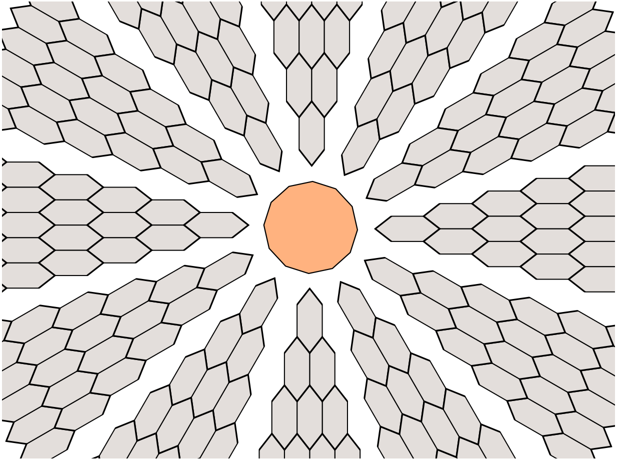
Corollary: A hexagon tiling can have arbitrarily many non-vertex-to-vertex situations (but not infinitely many)

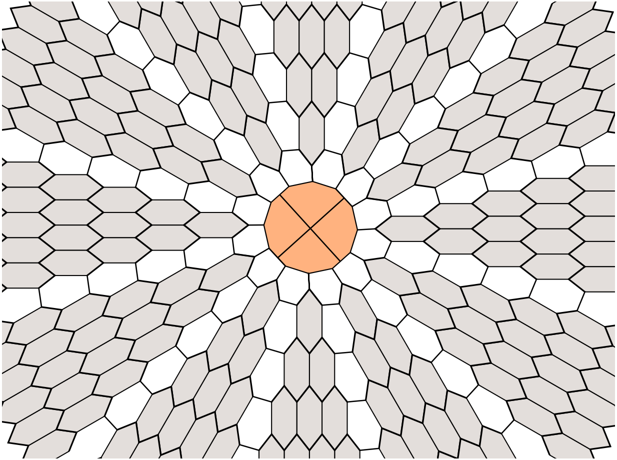


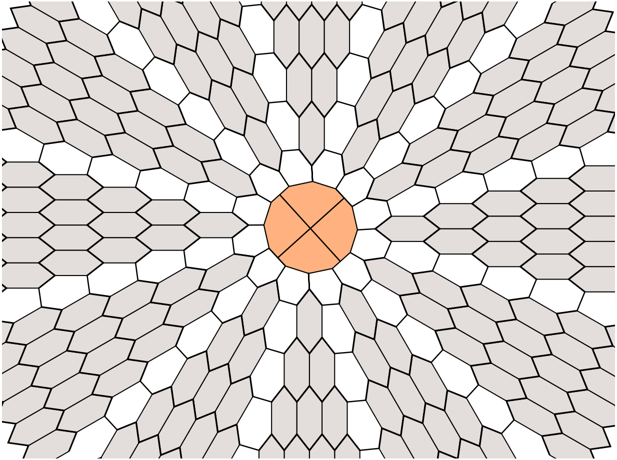


How to obtain "arbitrary many"

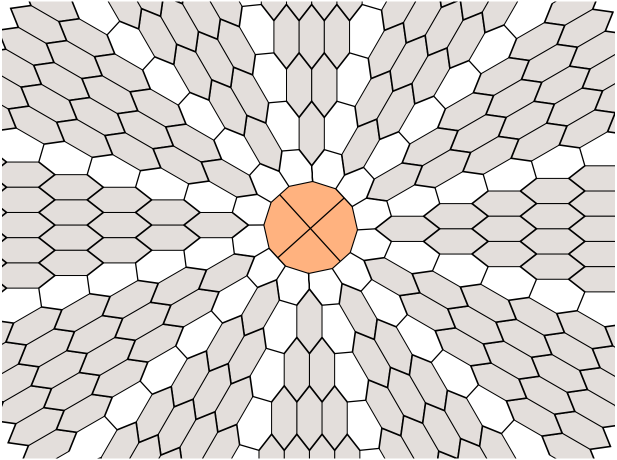









We can do the maths in order to compare with Akopyan's bound: This construction achieves 3/4 of his bound, hence his bound is asymptotically tight (linear in D/A).



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Thank you!