BOUNDED DISTANCE AND BILIPSCHITZ EQUIVALENCE OF DELONE SETS

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Abstract. This paper surveys several old and new results on two equivalence relations between Delone sets, namely bilipschitz equivalence (i.e., there is a bijection between two Delone sets that is Lipschitz continuous in both directions) and bounded distance equivalence (i.e., there is a bijection between two Delone sets that is a small displacement). The most interesting cases arise in the realm between crystallographic Delone sets and chaotic Delone sets. Prominent examples being model sets (aka cut-and-project sets) or Delone sets arising from substitution tilings.

Contents

1. Introduction

Note on this draft: The two of us started to work on this topic in 2009. Since then we found out that several tracks we've been following have already been followed by other colleagues, either in the past or simultanuously, and that many of the results we've been proving were already known. Nevertheless, this fact also indicates growing interest in the subject. So we decided to make our material publicly available. This text can be read as a survey, or as a scientifc paper (there are a few original results, or at least some alternative proofs). Anyway, it is still under construction, so there are probably several errors, many gaps and some strange formulations.

In this paper we study geometric properties of Delone sets in d-dimensional Euclidean space $\mathbb{R}^d$. The notion of Delone set was introduced by B. Delone who called them (r, R)-*systems*.

In the following, let $C_r(x)$ denote the cube of side length r centered in x with edges parallel to the coordinate axes. I.e.,

$$C_r(x) = \{(y_1, \ldots, y_d) \,|\, x_i - \frac{r}{2} \le y_i \le x_i + \frac{r}{2} \;(i = 1, \ldots, d)\}.$$

Definition 1.1. A set $\Lambda \subset \mathbb{R}^d$ is a *Delone set*, if there are $R > r > 0$, such that

(1) For all $x \in \mathbb{R}^d$, $C_R(x)$ contains at least one element of Λ (Λ is *relatively dense*), and
(2) For all $x \in \mathbb{R}^d$, $C_r(x)$ contains at most one element of Λ (Λ is *uniformly discrete*).

In the initial definition of Delone d-dimensional cubes were replaced by Euclidean balls, but it is clear that these two definitions are equivalent and using metric with "cubical balls" could simplify our argumentation.

In particular, Delone sets are infinite countable sets. Sometimes Delone sets are also called *separated nets*. A Delone set Λ in $\mathbb{R}^d$ is called k-*periodic*, if it has k linearly independent periods. I.e., there are $t_1, \ldots t_k$ linearly independent vectors such that $\Lambda + t_i = \Lambda$ for $1 \le i \le k$.

Definition 1.2. A discrete point set $\Lambda \subset \mathbb{R}^d$ is *bilipschitz equivalent* to Λ' (short: $\Lambda \overset{\text{bil}}{\sim} \Lambda'$), if there is a bijective map $f : \Lambda \to \Lambda'$ which is Lipschitz in both directions. This is equivalent to

$$\exists C > c > 0 : \quad c|x - y| \le |f(x) - f(y)| \le C|x - y|$$

for all $x, y \in \Lambda$.

The following definition formulates a relation between Delone sets which is stronger than bilipschitz equivalence.

Definition 1.3. A discrete point set $\Lambda \subset \mathbb{R}^d$ is *bounded distance equivalent* to Λ' (short: $\Lambda \overset{\text{bd}}{\sim} \Lambda'$), if there is $C > 0$ and an invertible map $g : \Lambda \to \Lambda'$ such that $|x - g(x)| < C$ for all $x \in \Lambda$.
Such a map g is called a *bounded distance bijection*.

Lemma 1.4. *Bilipschitz equivalence and bounded distance equivalence are equivalence relations.*

The proof is straightforward. Trivially all lattices in $\mathbb{R}^d$ are bilipschitz equivalent to $\mathbb{Z}^d$, since they are images of $\mathbb{Z}^d$ under some linear bijection. Hence it is clear that a Delone set is bilipschitz equivalent to some lattice in $\mathbb{R}^d$, if and only if it is bilipschitz equivalent to $\mathbb{Z}^d$.

Lemma 1.5. *Let Λ, Λ' be Delone sets in $\mathbb{R}^d$. If $\Lambda \overset{\text{bd}}{\sim} \Lambda'$, then $\Lambda \overset{\text{bil}}{\sim} \Lambda'$.*

Proof. Let r (r') denote the constant of uniform discreteness of Λ (Λ'), compare Definition 1.1. We show that g^{-1} is a bilipschitz map from $g(\Lambda)$ to Λ. Let $x, y \in g(\Lambda)$. Then

$$\begin{aligned} |g^{-1}(x) - g^{-1}(y)| &= |g^{-1}(x) - x + x - y - g^{-1}(y) + y| \le |x - y| + 2C \\ &= r'(|x - y|/r' + 2C/r') \le r'(2C/r' + 1)|x - y|/r' = (2C/r' + 1)|x - y|, \end{aligned}$$

where we used the fact $t + C' \le (C' + 1)t$ for $t \ge 1$ and $C' > 0$. Furthermore,

$$\begin{aligned} |x - y| &\le |g^{-1}(x) - g^{-1}(y)| + 2C = r(|g^{-1}(x) - g^{-1}(y)|/r + 2C/r) \\ &\le r(2C/r + 1)|g^{-1}(x) - g^{-1}(y)|/r = (2C/r + 1)|g^{-1}(x) - g^{-1}(y)|, \end{aligned}$$

which proves the claim. □

In this paper we will review results on bounded distance and bilipschitz equivalence of arbitrary Delone sets but we will be mostly focused equivalence of model sets. The model sets are usual mathematical representatives for quasicrystals. See the Appendix A for more details. Similar questions on two mentioned equivalencies can be asked about Delone sets in arbitrary metric spaces. We restrict our attention to the Euclidean case apart from mentioning two results in Subsection 2.4.

2. Bounded distance equivalence

2.1. **Dimension one.** It is easy to see that all Delone sets in $\mathbb{R}$ are bilipschitzequivalent to $\mathbb{Z}$: the fact that two points do not come arbitrarily close corresponds to the slope of the bijection being not too steep, and the fact that there are no arbitrary large gaps corresponds to the slope of the bijection being not too small. Thus, by Lemma 1.4 follows:

Lemma 2.1. *Let Λ, Λ' be Delone sets in $\mathbb{R}$. Then $\Lambda \overset{\text{bil}}{\sim} \Lambda'$.*

Proof. We show that for any Delone set $\Lambda \in \mathbb{R}$ holds $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}$. Wlog, let $\Lambda = \dots \lambda_{-1}, \lambda_0 = 0, \lambda_1, \lambda_2 \dots$ be a Delone set with parameters r, R. Note that the smallest possible distance of two points in Λ is r, and the largest possible distance of two consecutive points is $2R$. Let $f : \Lambda \to \mathbb{Z}$, $f(\lambda_n) = n$. Then, $r|n-m| \le |\lambda_n - \lambda_m| \le 2R|n-m|$, thus f is bilipschitz. □

For bounded distance equivalence the situation is more complicated. As a first example, consider $\mathbb{Z}$ and $2\mathbb{Z}$. These two sets cannot be bounded distance equivalent. (An easy way to see this is to consider the $2n+1$ points in $A := \mathbb{Z} \cap [-n, n]$ and the $2n$ points in $B := 2\mathbb{Z} \cap [-2n, 2n-1]$. One of the points in A has to be mapped by g to some point not in B, thus to some point of distance larger than $n-1$ from A.)

The reason for $\mathbb{Z} \overset{\text{bd}}{\not\sim} 2\mathbb{Z}$ in the example above is obviously that the two sets have different densities. There are also examples of point sets with the same density being not bounded distance equivalent, as we will see later in this section.

In fact, the definition of the density of an infinite point set is a subtle point. It is clear how to define the density of a point set contained in some finite set, or the density of a periodic point set (e.g., a point lattice). For instance, if Λ is a periodic set in $\mathbb{R}$, then let

$$\text{dens}(\Lambda) = \lim_{r\to\infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]). \tag{1}$$

Lemma 2.2. *Let $\Lambda \subset \mathbb{R}$ be some periodic Delone set. Then $\Lambda \overset{\text{bd}}{\sim} \frac{1}{\text{dens}(\Lambda)}\mathbb{Z}$.*

Proof. Let t be the smallest period of Λ: $\Lambda = \Lambda + t$. Since Λ is uniformly discrete, any half-open interval $[a, a+t[$ contains the same number of points, say, m. Then $\text{dens}(\Lambda) = m/t$. Wlog, let $\Lambda = \dots \lambda_{-1}, \lambda_0 = 0, \lambda_1, \lambda_2 \dots$. The map $\lambda_n \mapsto \frac{t}{m} n$ gives the desired bounded distance equivalence. □

Consider the following nonperiodic example: Let Λ contain all integer points, together with those half integer points (n s.t. $n + \frac{1}{2} \in \mathbb{Z}$) that lie between 4^n and $2 \cdot 4^n$ and between -4^n and $-2 \cdot 4^n$. A simple computation of the terms in Equation (1) yields that the limit does not exist: the sequence oscillates between $\frac{4}{3}$ and $\frac{5}{3}$.

Later we want to give results for certain nonperiodic sets in $\mathbb{R}^d$. A relevant theorem for this is the following theorem by Kesten for point sets in $\mathbb{R}$.

Theorem 2.3 ([18])**.** *Let $\xi \in [0, 1]$, $0 \le a < b \le 1$ and define*

$$\Lambda := \{k \in \mathbb{Z} \mid a \le (k\xi \mod 1) < b\}.$$

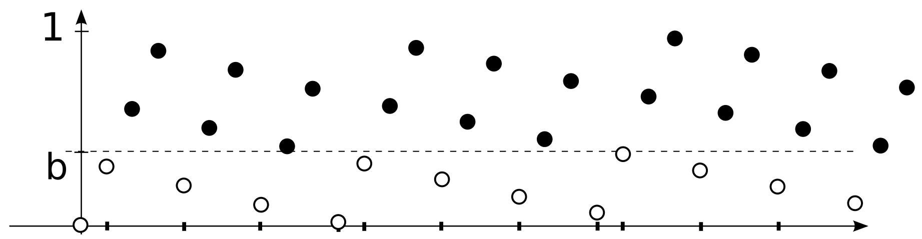

FIGURE 1. The points of L (circular dots) and of Λ (small black rectangles). White points in L correspond to points of Λ.

Then $D(n) := \#(\Lambda \cap [1,n]) - n(b-a)$ *is bounded, if and only if* $b - a = k\xi \mod 1$ *for some* $k \in \mathbb{Z}$.

If ξ is rational, then Λ is periodic. This case is covered already by Lemma 2.2, thus we consider only irrational values of ξ.

Corollary 2.4. *Let* Λ *be as above,* ξ *irrational.* Λ *is bounded distance equivalent to some lattice, if and only if* $b - a = k\xi \mod 1$ *for some* $k \in \mathbb{Z}$. *In this case,* $\Lambda \overset{\text{bd}}{\sim} \frac{1}{(b-a)}\mathbb{Z}$.

The "if" part was known for a long time. It was shown by Hecke [17] (p 73) for $a = 0$, using analytic number theory (Dirichlet series of meromorphic functions). Ostrowski found a simple argument for generalizing this to arbitrary a [25]. Kesten settled the "only if" part by "heavy use of continued fraction expansions". Here we provide a simple proof of the "if" part, similar arguments were used by Duneau and Oguey [7] for more general results on Model sets.

Proof. (if part of Theorem 2.3) First we consider the case $a = 0$. We lift the sequence Λ of Theorem 2.3 to $\mathbb{R}^2$ as follows: Let $L := \{\dots \ell_{-1}, \ell_0 = 0, \ell_1, \ell_2, \dots\}$, where $\ell_k := (k, k\xi \mod 1)$ for $k \in \mathbb{Z}$. Let π denote the orthogonal projection to the first coordinate. Then Λ consists of all elements $\pi(\ell_k)$ of L with $a \le k\xi \mod 1 \le b$; compare Figure 1.

In other words, all points of L lying in the strip $\{(x,y) \,|\, 0 \le y \le b\}$ are projected to the line $\{(x,y) \,|\, y = 0\}$, yielding the set Λ. This is clearly the same set as the following: Erect on each point in L a vertical length segment of length b. Any point of L whose line segment is hit by the horizontal line through $(b,0)$ is projected orthogonally to the x axis. (Compare Figure 2, left.) Now we change the setting slightly in order to obtain a periodic point set Γ such that $\Gamma \overset{\text{bd}}{\sim} \Lambda$.

Note that b is of the form $k\xi \mod 1$ for some $k \neq 0$ in $\mathbb{Z}$. Since ξ is irrational, k is unique. (Otherwise: if there are $k \neq m$ in $\mathbb{Z} \setminus \{0\}$ with $k\xi = b = m\xi \mod 1$, then $(k-m)\xi = 0 \mod 1$, thus there is $N \in \mathbb{Z}$ with $(k-m)\xi = N$, hence $\xi = \frac{k-m}{N} \in \mathbb{Q}$, a contradiction.)

Let us first assume that the line segment ℓ from $(0,0)$ to $(k, k\xi \mod 1)$ contains no other point of L. Consider the $\mathbb{Z}$-span $\langle L \rangle_{\mathbb{Z}}$ of L, which is a point lattice $G = \langle (1,\xi), (0,1) \rangle_{\mathbb{Z}}$. Now the line segment ℓ is a diagonal of some fundamental domain of this lattice, because of the following fact: If $(p,q) \in \mathbb{Z}^2$ where p, q are coprime, then there is a fundamental domain F of $\mathbb{Z}^2$ such that the line segment from $(0,0)$ to (p,q) is the diagonal of F.

(This holds, because if p, q are coprime, there are $a, b \in \mathbb{Z}$ such that $1 = ap - bq = \det \begin{pmatrix} p & b \\ q & a \end{pmatrix}$. Thus (p,q) and (b,a) are sides of some fundamental domain of $\mathbb{Z}^2$. Hence (p,q) is the diagonal of the fundamental domain with sides $(p-b, q-a)$, (b,a).)

Now we erect on each point of L a line segment (a *flagstick*) of height b, but with the direction of the line S through $(0,0)$ and $(k, k\xi \mod 1)$. Instead of projecting the points of L whose flagstick is hit by the horizontal line through $(b,0)$ orthogonally to the

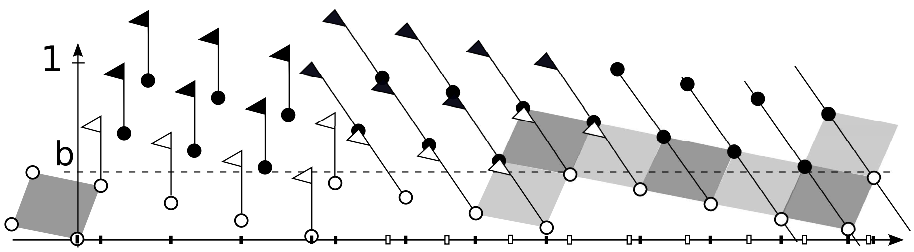

FIGURE 2. Projecting all points in the strip to the x-axis is the same as projecting each point whose flagstick is hit by the horizontal line H_b through $(0, b)$ (left). By slanting the flagsticks and projecting the intersection of the flagstick with H_b we obtain a periodic set Γ (white rectangles, right).

x-axis, we project the intersection of the flagstick with the horizontal line through $(b, 0)$ orthogonally to the x-axis (compare Figure 2, right). The resulting point set, regarded as a point set in $\mathbb{R}$, is called Γ.

Note that Γ is a periodic set in $\mathbb{R}$: It is obtained as a section through a set of equidistant parallel lines. Moreover, each of these parallel lines corresponds to exactly one point of Γ (with one exception): The fundamental cell F has height b.

The exception being that $(0, 0)$ and $(k, k\xi \mod 1)$ are mapped to the same point. In view of Theorem 2.3 we have to consider "half open" flagsticks: those whose lower endpoint belongs to it, its upper endpoint does not. (Anyway, even if we use closed flagsticks, we can repair it by moving all points left from 0 one step to the left. This leaves one empty position one step left from 0, which we use for $(k, k\xi \mod 1)$.)

Let $\Gamma = \{\dots \gamma_{-1}, \gamma_0 = 0, \gamma_1, \gamma_2, \dots\}$ and $\Lambda = \{\dots \lambda_{-1}, \lambda_0 = 0, \lambda_1, \lambda_2, \dots\}$. Clearly, $\Gamma \overset{\text{bd}}{\sim} \Lambda$, since for any $m \in \mathbb{Z}$ the maximal distance $|\gamma_m - \lambda_m|$ is bounded by the width of the fundamental cell we used. So the result is proved for the special case we have considered up to now.

We still have to deal with two restriction we have made: (1.) there are no further points of L in the line segment between $(0, 0)$ and $(k, k\xi \mod 1)$, and (2.) $a = 0$.

Regarding (1.): If there are $m > 1$ points of L contained on the line segment between $(0, 0)$ and $(k, k\xi \mod 1)$ we can use the same construction, using m stacked copies of the fundamental cell F. Then for each parallel line exactly m points are projected to the x-axis, still yielding a periodic Delone set Γ in $\mathbb{R}$.

Regarding (2.): If $a > 0$ we cannot longer assume that $\lambda_0 = \gamma_0 = 0$. But otherwise we can use the same construction. (If we like, we may define λ_0 (γ_0) to be that point in Λ (Γ) which are closest to 0. But this is not essential in the proof.) □

The sets Λ here are first basic examples of model sets, or Meyer sets (see Appendix). For instance, if we take $a = 0$, $\xi = \frac{1}{2}(\sqrt{5}+1)$, we obtain the so called Fibonacci sequence.

2.2. **Window is (a union of) fundamental domain(s) of sublattice(s).** We have seen already in dimension one that density of point sets plays some role, and that periodic point sets are simple to handle. This reflects in the following results proven in [7] (Lemma 4.3 and Theorem 5.2).

Theorem 2.5. *Let Λ, Λ' be two point lattices in $\mathbb{R}^d$ such that* $\text{dens}(\Lambda) = \text{dens}(\Lambda')$. *Then $\Lambda \overset{\text{bd}}{\sim} \Lambda'$. The union of any n lattices in $\mathbb{R}^d$ is bounded distance equivalent to some lattice in $\mathbb{R}^d$.*

The density of d-periodic sets can be defined in analogy to (1):

$$\text{dens}(\Lambda) = \lim_{r\to\infty} \frac{1}{r^d} \#(\Lambda \cap C_r(0)). \tag{2}$$

Theorem 2.6. *Let Λ and Λ' be Delone sets with well-defined density. If $\Lambda \overset{\text{bd}}{\sim} \Lambda'$, then* $\text{dens}(\Lambda) = \text{dens}(\Lambda')$.

Proof. The idea is the same as in the example after Lemma 2.1: Essentially the pigeon hole principle. Let $\text{dens}(\Lambda) > \text{dens}(\Lambda')$, $c := \big(\text{dens}(\Lambda)/\text{dens}(\Lambda')\big)^{1/d}$, and fix some ε such that $c > 1 + \varepsilon$. We show that some point of $\Lambda \cap C_{(1+\varepsilon)r}(0)$ has to be mapped by any bijection to a point in Λ' outside $C_{cr}(0)$.

By the definition of density, the number ℓ_r of points in $\Lambda \cap C_{(1+\varepsilon)r}(0)$ is

$$\ell = \big((1+\varepsilon)r\big)^d \text{dens}(\Lambda) + o(r^d) \geq r^d \text{dens}(\Lambda) + \varepsilon^d r^d \text{dens}(\Lambda) + o(r^d)$$

and the number ℓ'_r of points in $\Lambda' \cap C_{cr}(0)$ is $(cr)^d \text{dens}(\Lambda) + o(r^d)$. Thus

$$\ell_r - \ell'_r \geq \varepsilon^d r^d \text{dens}(\Lambda) + o(r^d).$$

In particular, $\ell_r - \ell'_r \geq 1$ for r large enough. Thus any bijection $g : \Lambda \to \Lambda'$ maps some point $x \in \Lambda \cap C_{(1+\varepsilon)r}(0)$ to the complement of $\Lambda' \cap C_{cr}(0)$. Hence $|g(x) - x| \geq (c - 1 - \varepsilon)r$ for any r large enough. Thus g cannot be a bounded distance bijection. □

Theorems 2.5 and 2.6 imply immediately the following result.

Theorem 2.7. *Let Λ, Λ' be d-periodic Delone sets in $\mathbb{R}^d$. Then* $\text{dens}(\Lambda) = \text{dens}(\Lambda')$ *iff* $\Lambda \overset{\text{bd}}{\sim} \Lambda'$.

Proof. The if-part is Theorem 2.6. For the other direction, note that any d-periodic Delone Λ set is the union of n translates of lattices. Thus, by Theorem 2.5, it is bounded distance equivalent to some lattice Γ. By Theorem 2.6 we have $\text{dens}(\Gamma) = \text{dens}(\Lambda)$. In the same way we find a lattice Γ' with $\Gamma' \overset{\text{bd}}{\sim} \Lambda'$ and $\text{dens}(\Gamma') = \text{dens}(\Lambda') = \text{dens}(\Lambda)$. Thus $\Lambda \overset{\text{bd}}{\sim} \Gamma \overset{\text{bd}}{\sim} \Gamma' \overset{\text{bd}}{\sim} \Lambda'$. □

Corollary 2.8. *Let Λ be a d-periodic Delone sets in $\mathbb{R}^d$. Then $\Lambda \overset{\text{bd}}{\sim} \frac{1}{\text{dens}(\Lambda)^{1/d}}\mathbb{Z}^d$.*

2.3. **A criterion by Laczkovich.** One kind of problems one can now formulate is illustrated by the following questions:

(1) Let $\Lambda_1 \overset{\text{bil}}{\sim} \Lambda_2$. Is $\Lambda_2 \overset{\text{bil}}{\sim} \Lambda_1 \cup \Lambda_2$? Under which conditions?
(2) Let $\Lambda \overset{\text{bd}}{\sim} \mathbb{Z}^d$, $\Lambda = \Lambda_1 \dot\cup \Lambda_2$, $\Lambda_1 \overset{\text{bd}}{\sim} \Lambda_2$. Is $\Lambda_1 \overset{\text{bd}}{\sim} 2^{-1/d}\mathbb{Z}^d$?

In the sequel we answer the second question, see Thm. 2.11 below. It can also been shown by a result of Laczkovich:

Theorem 2.9 ([19]). *Let $\Lambda \subset \mathbb{R}^d$ be a Delone set. Then $\Lambda \overset{\text{bd}}{\sim} \frac{1}{\text{dens}(\Lambda)^{1/d}}\mathbb{Z}^d$ if and only if for every bounded measurable $H \in \mathbb{R}^d$ there is $s > 0$ such that*

$$|\#(\Lambda \cap H) - \text{dens}(\Lambda)\lambda(H)| \leq s\lambda(\partial H + B_1)$$

Here λ denotes d-dimensional Lebesgue measure, and B_1 is the unit ball. Hence $\partial H + B_1$ is the thickened boundary of H. The left hand side of the inequation in the theorem above can be seen as a d-dimensional version of the deficiency $D(n)$ in Theorem 2.3. The criterion is fulfilled if the deficiency (the difference between the actual number of points and the expected value) does not deviate too much from the volume of the thickened boundary.

However we would like to provide an independent elementary proof of point (2) above without using Theorem 2.9. We will use the following "infinite" version of the Hall marriage theorem.

Lemma 2.10. *Consider a bipartite graph G with countable parts V_1 and V_2 such that any vertex has finite degree. Assume that for any finite subset X of one vertex part there are at least $\#X$ vertices from another vertex part that are connected with at least one vertex from X. Then there exists a set of disjoint edges of G that covers all vertices of G.*

Proof. Enumerate the vertices of V_1 as $a_1, a_2, \ldots$ and the vertices of V_2 as $b_1, b_2, \ldots$.

If there is finite $X \subset V_1$ such that $a_1 \in X$ and vertices of X connected with exactly $\#X$ vertices from V_2 (denote this subset of V_2 as X') then we can apply the usual finite version of Hall's theorem to the graph with vertex set $X \cup X'$ and forget about these vertices (that means that we choose edges between these vertices and we will never change these edges). It is easy to check that the conditions of this lemma remains true for the graph with the remaining vertex set.

If there is no such X, let v be some vertex that is connected with a_1. Now we will try to find a similar subset Y of V_2 such that $v \in Y$ and vertices with Y connected with exactly $\#Y$ vertices from V_1 (this set is denoted by Y'). If there is such an Y then we can apply Hall's theorem to the graph with vertex set $Y \cup Y'$ and forget about these vertices. Again the condition of this lemma remains true for the remaining graph.

If there is no such Y then we can delete the edge $a_1 v$ and the conditions of this lemma will remain true. We can not delete all edges from a_1, so at some step we will find a set X or Y. In both cases we will fix an edge from a_1 because in the first case a_1 is in X and in the second one a_1 is in Y'. So after this algorithm we will obtain a graph that satisfies the conditions of this lemma, but without vertex a_1 and some other vertices.

Now we choose the remaining vertex b_i with the smallest i and do the same steps and so on. Since there is a countable number of vertices we will divide all vertices into pairs. □

Theorem 2.11. *Consider a Delone set A in $\mathbb{R}^d$ such that $A \overset{\text{bd}}{\sim} \mathbb{Z}^d$. If A is represented as $A = M_1 \sqcup M_2 \sqcup \ldots \sqcup M_n$ in such a way that $M_i \overset{\text{bd}}{\sim} M_j$ then $M_i \overset{\text{bd}}{\sim} n^{1/d}\mathbb{Z}^d$.*

Proof. Without loss of generality we can assume that $A = \mathbb{Z}^d$. It is enough to construct a bounded distance bijection between M_1 and the set of all integer points in $T := \{(nk_1, k_2, \ldots, k_d) \,|\, k_i \in \mathbb{Z}\}$. By Theorem 2.7 we have $n^{1/d}\mathbb{Z}^d \overset{\text{bd}}{\sim} T$. Let c_{ij} be a distance in the bounded distance bijection between M_i and M_j and $c = \max c_{ij}$.

Consider a bipartite graph with vertex set $M_1 \cup T$. Two vertices are connected by an edge if and only if the distance between these vertices is at most $c + n$. We will show that this graph satisfies Lemma 2.10. For every k and for any k points from M_1 we have at least kn lattice points covered by c-balls in these points (we have k points from each M_i), and for every such lattice point we decrease the first coordinate to the closest integer divisible by n. In this way we will come to some point in T, and we moved for at most n, so totally we traveled for at most $c + n$. For a fixed point in T we came for at most n times (different remainders modulo n) so we obtained at least k different points from T in $(c + n)$-balls with centers in given k points of M_1.

The same is true for any k points from T: there are kn points in this $n \times 1 \times \ldots \times 1$ bricks and at least k of them are from one M_i. Thus at least k points from M_1 are at distance c from these bricks and at distance $c + n$ from these points from T. So the conditions of Lemma 2.10 are fulfilled and we get the desired bounded distance bijection determined by the edges from lemma 2.10. □

Example 2.12. As an example of application of previous theorem we can consider a half-Fibonacci sequence (see section 2.1). The Fibonacci sequence F is a one-dimensional model set with window $W = \left[0, \frac{\sqrt{5}+1}{2}\right)$. We can cut W into two equal interval $W_1 =$

$\left[0, \frac{\sqrt{5}+1}{4}\right)$ and $W_2 = \left[\frac{\sqrt{5}+1}{4}, \frac{\sqrt{5}+1}{2}\right)$ then we can construct two *half-Fibonacci* model sets F_1 and F_2 corresponding to these windows.

We will show that F_1 and F_2 are not bounded distance equivalent though they have the same density. By theorem 2.3 we know that $F_1 \dot\cup F_2$ is bounded distance equivalent to some lattice Λ and therefore if $F_1 \overset{\text{bd}}{\sim} F_2$ then by theorem 2.11 both F_1 and F_2 are equivalent to lattice $\frac{1}{2}\Lambda$. But this contradicts with Theorem 2.3 since windows W_1 and W_2 can not be represented in desired form.

2.4. **Other spaces.** In 1997 Bogopolskii proved the following result [4].

Theorem 2.13. *Every two Delone sets in hyperbolic space $\mathbb{H}^d$ ($d \geq 2$) are bounded distance equivalent.*

Consequently all Delone sets in $\mathbb{H}^d$ are bilipschitz equivalent. In view of Theorem 2.9 this is probably not too surprising: the deficiency (the expected number of points of Λ contained in some large ball minus the actual number of points of Λ inside this ball) should be bounded by the mass of the boundary (measured in some appropriate manner). Since in $\mathbb{H}^d$ the mass of a ball is concentrated on the boundary this is intuitively fulfilled easily.

In a similar flavour Papasoglu proved the following result in 1995 [26].

Theorem 2.14. *Any two infinite regular trees (of valency $k \geq n \geq 4$) are bilipschitz equivalent.*

In fact the author proved even bounded distance equivalence, see Lemma 1 in [26].

3. Bilipschitz equivalence

3.1. **Non-equivalence in Euclidean case and number of equivalence classes.** From Theorem 2.7 we have the following

Corollary 3.1. *Any two d-periodic Delone sets in $\mathbb{R}^d$ are bilipschitzequivalent.*

Proof. By Theorem 2.7 any d-periodic Delone set is bounded distance equivalent (thus b.l.e.) to some lattice. Clearly any two lattices in $\mathbb{R}^d$ are b.l.e., since they are bijective affine images of each other. □

Things became interesting when the following result was proven in 1998 by Burago and Kleiner and independently by C. McMullen..

Theorem 3.2 ([5, 22])**.** *There are Delone sets in $\mathbb{R}^d$ ($d \geq 2$) which are not bilipschitzequivalent to $\mathbb{Z}^d$.*

A natural question is now how the set of all Delone sets in $\mathbb{R}^d$ is partitioned into equivalence classe, w.r.t. $\overset{\text{bil}}{\sim}$ and $\overset{\text{bd}}{\sim}$. A first answer is about the number of these equivalence classes.

Theorem 3.3 ([21])**.** *For every integer $d \geq 2$ the set of biLipschitz equivalence classes in $\mathbb{R}^d$ has cardinality continuum.*

Theorem 3.4. *For every integer $d \geq 2$ the set of bounded distance equivalence classes in $\mathbb{R}^d$ has cardinality continuum; i.e., its cardinality is $\#\mathbb{R}$.*

Proof. By Theorem 2.6 the cardinality is at least $\#\mathbb{R}$, since for every value of density there is at least one equivalence class.

To see that the cardinality is at most $\#\mathbb{R}$ we index each equivalence class with two parameters $r \in \mathbb{R}^+, s \in \{A \,|\, A \subset \mathbb{Z}^d\}$. The value r is the minimal constant of uniform discreteness as in Definition 1.1. Now we divide $\mathbb{R}^d$ into boxes $C_r(x)$, $x \in r\mathbb{Z}^d$. By uniform discreteness, each such box contains at most on point of Λ. We map Λ to some subset of $r\mathbb{Z}^d$ by mapping any $\lambda \in \Lambda$ to the center x of the box containing λ. (In order to make rule this unique: If λ is contained in more than one box, it is mapped to the x with the lowest coordinates.) This is clearly a bounded distance bijection. The new set is the desired s. There are not more than r times s bounded distance equivalence classes, thus at most $\#\mathbb{R}$ many. □

The idea of choosing a representative $D \subset \mathbb{Z}^d$ for some bounded distance equivalence class may prove useful in the future. Thus we state a lemma about this concept here.

Lemma 3.5. [12] *Let $\Lambda \subset \mathbb{R}^d$ be a Delone set. Then there exists a Delone set $D \subset \mathbb{Z}^d$ such that Λ and D are biLipschitz equivalent.*

3.2. **Sufficient condition for the two-dimensional case.** The following theorem was proved by Burago and Kleiner ([6, Theorem 1.3]), which reads in our framework as follows.

Theorem 3.6. *Let $\Lambda \subset \mathbb{R}^2$ be a Delone set. For $\varrho > 0$, define $e_\varrho(x,r)$ as*

$$e_\varrho(x,r) = \max\left(\frac{\varrho r^2}{N(x,r)}, \frac{N(x,r)}{\varrho r^2}\right),$$

and let $E_\varrho(r) = \sup_{x\in\mathbb{R}^2} e_\varrho(x,r)$. If there is $\varrho > 0$ such that the infinite product $\prod_m E_\varrho(2^m)$ converges, then Λ is bilipschitzequivalent to $\mathbb{Z}^2$.

3.3. **Arbitrary two-dimensional quasicrystals.** The first main theorem of this section states that each linearly repetitive Delone set in $\mathbb{R}^2$ is bilipschitzequivalent to $\mathbb{Z}^2$. It is a consequence of Theorems and 3.6 and 3.8. During preparing this manuscript we learned that there is a preprint obtaining our main theorem for arbitrary dimension [1]. First we need a few definitions. A *patch* in Λ is just a finite subset of Λ. An *r-patch* in Λ is a patch in Λ which is contained in some ball of radius $r > 0$.

Definition 3.7. A Delone set Λ is *repetitive*, if for each $r > 0$ there is $R > r > 0$ such that each R-patch in Λ contains a translate of each r-patch in Λ.
A Delone set Λ is *linearly repetitive*, if there are $c, a > 0$ such that $(cr + a)$-patch in Λ contains a translate of each r-patch in Λ.
A Delone set has *finite local complexity* (short FLC) if for any $r > 0$ there are only finitely many pairwise noncongruent r-patches in Λ.

Let $N_P(x,r)$ denote the patch counting function

$$N_P(x,r) = \#\{P' \in \Lambda \cap C_r(x) \,|\, P' = P + t, t \in \mathbb{R}^d\}.$$

The following theorem [20, Theorem 5.1] states that for all linearly repetitive Delone sets, the frequency f_P of each patch P are well-defined and exist uniformly in x. Moreover, it gives an estimate on the rate of convergence of these frequencies.

Theorem 3.8. *Let Λ be a linearly repetitive Delone set in $\mathbb{R}^d$. Then Λ has uniform patch frequencies. Moreover, the frequency $f_P = \lim_{r\to\infty} \frac{1}{r^d} N_P(x,r)$ of any patch $P \subset \Lambda$ does not depend on x, and it holds:*

$$\left|\frac{1}{r^d} N_P(x,r) - f_P\right| \le C r^{-\delta}$$

for some $C, \delta > 0$.

In particular, this result holds for all 1-point patches. Thus the above estimate holds for the *point frequency* $f = f(\Lambda)$ of Λ. In the sequel, denote the point counting function $\#\{y \in (\Lambda \cap C_r(x)\}$ just by $N(x,r)$.

Now we can proof the main result of this section.

Theorem 3.9. *Any linearly repetitive Delone set in $\mathbb{R}^2$ is bilipschitzequivalent to $\mathbb{Z}^2$.*

Proof. By setting $\varrho = f$ we obtain by Theorem 3.8 and Theorem 3.6:

$$E_f(r) = \sup_{x\in\mathbb{R}^2} \max\left(\frac{fr^2}{N(x,r)}, \frac{N(x,r)}{fr^2}\right).$$

From Theorem 3.8 follows $|\frac{N(x,r)}{fr^2} - 1| \leq \frac{C}{f} r^{-\delta}$. From this inequality we obtain for r large enough

$$1 - \frac{C}{f} r^{-\delta} \leq \frac{N(x,r)}{fr^2} \Rightarrow \frac{fr^2}{N(x,r)} \leq \frac{1}{1-\frac{C}{f}r^{-\delta}} = 1 + \frac{\frac{C}{f}r^{-\delta}}{1-\frac{C}{f}r^{-\delta}}$$
$$\Rightarrow \frac{fr^2}{N(x,r)} - 1 \leq \frac{C}{f} r^{-\delta} \frac{1}{1-\frac{C}{f}r^{-\delta}} \leq \frac{2C}{f} r^{-\delta}.$$

This yields $0 \leq E_f(r) - 1 \leq \frac{2C}{f} r^{-\delta}$ for r large enough. The product $\prod_m E_f(2^m)$ converges iff $\sum_m (E_f(2^m) - 1)$ converges. Since $\delta > 0$, the sum

$$0 \leq \sum_m (E_f(2^m) - 1) \leq \sum_m \frac{2C}{f} (2^m)^{-\delta} = \frac{2C}{f} \sum_m (2^{-\delta})^m$$

converges as sum of geometric sequence, and the claim follows. □

Now we can apply this result to the zoo of nonperiodic tilings. There are essentially two main methods to construct nonperiodic tilings of high local and global order: Tile substitutions and cut and project methods. See for instance [3] for explanations, examples and details. Theorems 3.11 and 3.13 give conditions on bounded distance equivalence and b.l.e. of tilings generated by these two methods.

Theorem 3.10. *Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^d$ having FLC. Then $\mathcal{T}$ is linearly repetitive.*

This result is proven for the Penrose tiling in [11]. A general proof for substitution tilings can be found in [27], see also [3]. Together with Theorem 3.9 this immediately implies the following result.

Theorem 3.11. *Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^d$ with FLC. Then $\mathcal{T}$ is bounded distance equivalent to some lattice. Thus $\mathcal{T}$ is b.l.e. to $\mathbb{Z}^d$.*

Clearly any model set is FLC. Therefore we have the following result as a special case of 3.11.

Corollary 3.12. *Any model set which can also be generated by a primitive substitution is linearly repetitive.*

The following result generalises the if-part of Kesten's Theorem (Thm. 2.3 above) to arbitrary dimensions. It can be proven by a d-dimensional construction due to Duneau and Oguey [7] generalising the construction in the proof of the if-part of Thm. 2.3 above. In particular, one uses that the set of the fundamental parallelepipedes (of which the window is a projection) belonging to the projected points has the shape of a stepped layer of exactly the appropriate width. This is the d-dimensional equivalent of the fact that each fundamental parallelepipede has exactly one predecessor and exactly one successor; compare Figure 2 right. See also [15] for an alternative proof of this result.

Theorem 3.13. *Any canonical projection tiling is bounded distance equivalent to some lattice; thus b.l.e. to $\mathbb{Z}^d$.*

3.4. **Arbitrary quasicrystals in any dimension.** Aliste-Prieto, Coronel and Gambaudo proved the generalisation of Theorem 3.9 above in 2013.

Theorem 3.14 ([1]). *Any linearly repetitive Delone set in $\mathbb{R}^d$ is bilipschitzequivalent to $\mathbb{Z}^d$.*

Appendix A. Model sets, aka cut and project sets

Model sets (or cut-and-project sets) are generalisations of lattices. They were introduced by Meyer in the seventies [23] and became well studied objects in the context of quasiperiodic structures, see for instance [3], [24]. Any model set Λ shares the following properties with lattices:

- Λ is a Delone set.
- $\Lambda - \Lambda$ is uniformly discrete.
- $\Lambda - \Lambda \subseteq \Lambda + F$, where F is a finite set.

Model sets are defined as follows.

Definition A.1. Let G, H be locally compact Abelian groups, Γ be a lattice in $G \times H$ (that is, Γ is a cocompact discrete subgroup of $G \times H$), $\pi_1 : G \times H \to G$, $\pi_2 : G \times H \to H$ be projections, such that $\pi_1|_\Gamma$ is injective, and $\pi_2(\Gamma)$ is dense in H. Let $W \subset H$ be a compact set — the *window* — such that the closure of the interior of W equals W. This is summarised in the following diagram, which is called *cut-and-project scheme*.

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ V & & \Gamma & & W \end{array} \tag{3}$$

Then

$$V := V(G, H, \Gamma, W) = \{\pi_1(x) \,|\, x \in \Gamma,\ \pi_2(x) \in W\}$$

is called a *model set*.

If $\mu(\partial(W)) = 0$, then V is called *regular* model set.

If $\partial(W) \cap \pi_2(\Gamma) = \varnothing$, then V is called *generic* model set.

If $G = \mathbb{R}^d$, $H = \mathbb{R}^e$ and if W is the projection of a fundamental parallelepipede of Γ, then V is called a *canonical projection tiling.*

In fact, many of the prominent examples of model sets in the literature are canonical projection tilings. In a certain sense these are the nicest model sets, compare for instance [8, 13]. Examples are the Fibonacci tilings, the Penrose tilings, the Ammann-Beenker tilings (see for instance [10]), and the icosahedral tilings in $\mathbb{R}^3$ of Kramer, Danzer, Ammann and Socolar. To be precise: The tilings themselves are clearly not model sets, since they are packings rather than Delone sets. But the vertex set (or more general, the set of some control points coding the tiling) of these tilings form canonical projection tilings, from which the tilings can be constructed in a unique way. In a similar way, many substitution tilings living on a integer lattice or hexagonal lattice (like chair tiling, sphinx tiling...) can be obtained as models sets if we choose H to be the p-adic numbers $\mathbb{Q}_p$ (more precisely: $(\mathbb{Q}_p)^e$ for $e \geq 1$).

Furthermore, the point sets in Theorem 2.3 are very close to being models sets (compare Figure 1). Here $G = H = \mathbb{R}^1$, the integer span of the lifted points $(k, k\xi \mod 1)$ is the lattice Γ. The interval $[0, b)$ is the window W. The orthogonal projection $\pi_1 : G \times H \to G$ is not injective, hence the definition is not fulfilled. Anyway, tilting π_1 slightly will make it injective. And this is a bounded displacement wrt the points projected, so the sets are very close to being model sets wrt bounded distance equivalence.

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