

# Tilings with transcendental inflation factor

Dirk Frettlöh

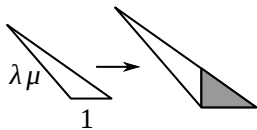
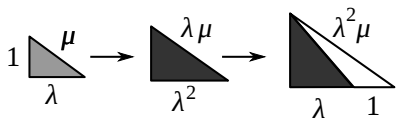
Joint work with Alexey Garber and Neil Mañibo

Technische Fakultät  
Universität Bielefeld

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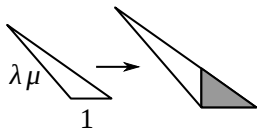
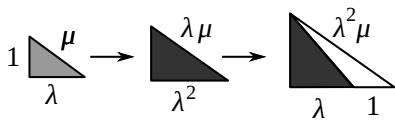
Amiens, 6 July 2023

In dimension  $d = 2$ :



- ▶ substitution matrix  $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ ,
- ▶ inflation factor  $\lambda = 1.3247\dots$  (the *plastic number*),
- ▶ minimal polynomial  $x^3 - x - 1$ .

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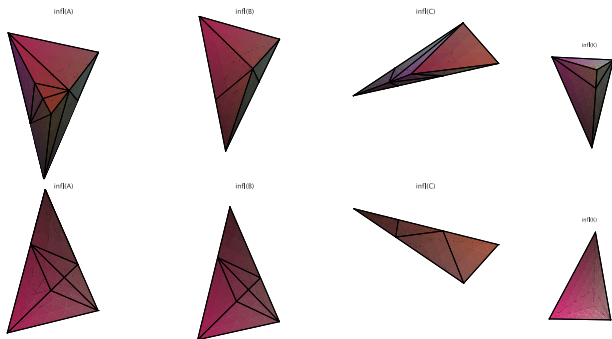
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Visit our "zoo": Tilings Encyclopa

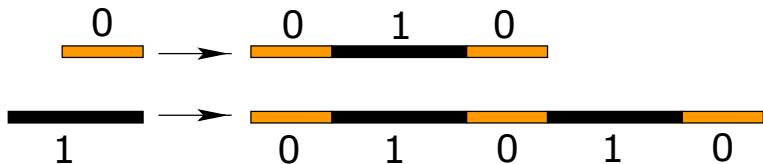
<https://tilings.math.uni-bielefeld.de>

In dimension  $d = 3$ :



- ▶ substitution matrix  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}$ ,
- ▶ inflation factor  $\lambda = \frac{1}{2}(\sqrt{5} + 1)$  (the *golden mean*),
- ▶ minimal polynomial  $x^2 - x - 1$ .

Substitution tiling in dimension  $d = 1$ :



- ▶ substitution matrix  $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ ,
- ▶ inflation factor  $\lambda = 2 + \sqrt{3}$ ,
- ▶ minimal polynomial  $x^2 - 4x + 1$ .

Since  $\lambda^d$  is an eigenvalue of an integer matrix, the inflation factor  $\lambda$  is always an algebraic integer.

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What if there are infinitely many prototiles?

In most examples with infinitely many prototiles studied so far (Ferenczi, Sadun, Durand-Ormes-Petite, Smilansky-Solomon...):

- ▶ constant length substitutions, infinitely many labels, or
- ▶ no proper inflation factor

Mañibo-Rust-Walton (preprint 2022): conditions for unique ergodicity of the dynamical systems arising from substitutions in dimension  $d = 1$  for infinitely many prototiles with distinct lengths.

Their Example 3.2: Prototiles  $0, 1, 2, 3, \dots$  and  $\infty$ .

$$0 \mapsto 0 \ 0 \ 0 \ 1$$

$$i \mapsto 0 \ i-1 \ i+1$$

$$\infty \mapsto 0 \ \infty \ \infty$$



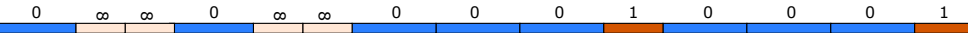
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$$0 \mapsto 0001$$

$$i \mapsto 0i-1i+1$$

$$\infty \mapsto 0\infty\infty$$



The tiles have indeed well-defined (distinct) lengths  $\ell_i$ :

$$\ell_i = 1 + \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\sqrt{2}}\right)^i,$$

and a proper inflation factor:  $\lambda = 3 + \frac{1}{\sqrt{2}}$



Their substitution "matrix":

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$



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When we saw this example we tried to find more.

But: unlike in the finite case one cannot just turn any "matrix" into a proper substitution  
(negative lengths, lengths  $\rightarrow \infty$ , all tile frequencies 0, ...)

0 0 0 1 0 0 2 0 0 0 0 1 0 0 0 1 0 1 3 0

There is also no simple analogue of Perron-Frobenius.

And in order to establish unique ergodicity they (Neil-Dan-Jamie) need to work a lot:

- ▶ The alphabet  $\{0, 1, 2, \dots, \} \cup \{\infty\}$  needs to be compact,
- ▶ the substitution "matrix" needs to be quasicompact,
- ▶ the symbolic substitution needs to be continuous,

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- ▶ the symbolic substitution needs to be continuous,
- ▶ and primitive,
- ▶ but what means primitive here?

However, all this can be solved.

In an earlier paper on infinite alphabets they (Mañibo-Rust-Walton 2023) asked whether there are substitutions with transcendental (that is, not algebraic) inflation factor.

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**Theorem (F-Garber-Mañibo 2023+)**

*For any  $\lambda > 2$  there is a primitive substitution with infinitely many prototiles having  $\lambda$  as inflation factor.*



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### Theorem (F-Garber-Mañibo 2023+)

*For any  $\lambda > 2$  there is a primitive substitution with infinitely many prototiles having  $\lambda$  as inflation factor.*

### Corollary

*There are a lot of substitution tilings with transcendental inflation factor.*

**Proof:** (idea, simplified) Generalize the example above:

Let  $\mathbf{a} = (a_i)_i = a_0, a_1, a_2, \dots$  with  $a_i \in \{1, 2, \dots, N\}$  for some  $N \in \mathbb{Z}^+$ .

$$\text{Let } A = \begin{pmatrix} a_0 & 1 + a_1 & a_2 & a_3 & a_4 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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For instance,  $a_0 = 3$  and  $a_i = 1$  for  $i \geq 1$  is the example above.

$$\varrho_{\mathbf{a}} = \begin{cases} 0 \mapsto 0^{a_0} 1 \\ i \mapsto 0^{a_i} i-1 i+1 \\ \dots \dots \end{cases}$$

In order to show that this defines nice substitution tilings ("good" tile lengths and frequencies etc) we apply Mañibo-Rust-Walton:

We need to turn the set  $\{0, 1, 2, \dots\}$  (corr. to the prototiles) into a compact alphabet  $\mathcal{A}$ . (Amazingly sophisticated)

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...and show that

- ▶ The substitution  $\varrho_{\mathbf{a}}$  is a continuous map  $\varrho_{\mathbf{a}} : \mathcal{A} \rightarrow \mathcal{A}^+$ ,
- ▶  $\varrho_{\mathbf{a}}$  is primitive,
- ▶ the substitution operator (roughly, the "matrix") is quasicompact



It remains to realize all inflation factors  $\lambda > 2$ .

**Ansatz:**

Let  $(a_i)_i$  be fixed, and let  $\mu \in (0, \frac{1}{2}]$  be the unique number with

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} a_i \mu^i$$

**Claim:**

$\lambda = \mu + \frac{1}{\mu}$  is an eigenvalue with eigenvector  $\mathbf{v} = (1, \mu, \mu^2, \dots)^T$ .

$$A\mathbf{v} = \lambda\mathbf{v}.$$

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Row by row:

- ▶ 1<sup>st</sup> row:  $\mu + \sum_{i=0}^{\infty} a_i \mu^i = \mu + \frac{1}{\mu} = \lambda \cdot 1. \quad \checkmark$
- ▶  $i$ <sup>th</sup> row:  $\mu^{i-2} + \mu^i = (\mu^{-1} + \mu)\mu^{i-1} = \lambda\mu^{i-1}. \quad \checkmark$

It follows that  $\lambda$  is the inflation factor (by some infinite equivalent of Perron-Frobenius: eigenvector in the positive cone), and  $\mathbf{v}$  (normalized) is the vector of tile frequencies.

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- ▶ First, we don't. We need to allow  $a_i = 0$ .
- ▶ But to keep it simple, let us assume  $a_i \neq 0$ .
- ▶ Then we get all values  $\lambda > \frac{5}{2}$ .

Now we fix  $\mu \in (0, \frac{1}{2}]$ . We have to find  $(a_i)_i$  such that

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That's it!

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be the Thue-Morse sequence (with 1s and 2s).

Plugging it into  $\varrho_{\mathbf{a}}$  yields a transcendental inflation factor  $\lambda = \mu + \frac{1}{\mu}$  which we can compute (approximately).

Why "transcendental"?

Consider the classical Thue–Morse sequence  $t_n := (-1)^{s(n)}$ , where  $s(n)$  is the number of ones in the binary expansion of  $n$ .

1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, ...

### Theorem (Mahler 1929)

- ▶ Consider the generating function  $T(z) := \sum_{n \geq 0} t_n z^n$ .
- ▶ Let  $\alpha \neq 0$  be an algebraic number with  $|\alpha| < 1$ .

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The generating function of the 1-2-Thue-Morse sequence is

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{3}{2} \cdot \frac{1}{1-z} + \frac{1}{2} T(z)$$

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$$\frac{1}{\mu} = A(\mu) = \frac{3}{2} \cdot \frac{1}{1-\mu} + \frac{1}{2} T(\mu).$$

Now...

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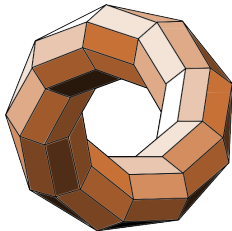


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Thank you!