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Exercises to“Average Lattices for Quasicrystals” and “Quasicrystals and Symmetry”

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For unknown terms or notations, or useful facts, see the slides of the talks.
Exercises marked with (⋆) might be hard to solve, or even new results.

Exercise 1:
Consider the following symbolic substitutions:

1. $a \to ab,\ b \to a$
2. $a \to ab,\ b \to ba$
3. $a \to abb,\ b \to a$
4. $a \to aaba,\ b \to a$
5. $a \to c,\ b \to bc,\ c \to abc$
6. $a \to c,\ b \to abc,\ c \to cbc$

These can be turned into tile substitutions in $\mathbb{R}$. For each of them, compute the substitution factor, the lengths of the prototiles (such that the shortest prototile has length 1) and the relative frequencies of the prototiles.

(⋆) What is the density of the resulting tilings, i.e. the average number of tiles per unit area?

Exercise 2:
Given that the substitution factor of some primitive tile substitution σ is $\lambda = \eta^{1/d}$, where d denotes the dimension, and η is the largest eigenvalue of the substitution matrix M_σ. Prove that the left eigenvector corresponding to η contains the length, resp. areas, resp. volume, ... hence the d-dimensional volume, of the prototiles. Prove that the (normed) right eigenvector corresponding to η contains the relative frequencies of the prototiles.

Exercise 3:
(a) Find a tile substitution σ_1 in one dimension such that the substitution factor equals $2+\sqrt{2}$.

(b) Find two different tile substitutions σ_1, σ_2 in one dimension such that the substitution factor of both equals $1+\sqrt{2}$, and such that $M_{\sigma_1} \neq M_{\sigma_2}$ (and also $PM_{\sigma_1}P^{-1} \neq M_{\sigma_2}$ for $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$).

(c) Find three different tile substitutions $\sigma_1, \sigma_2, \sigma_3$ in one dimension such that the substitution factor of all three of them equals $2+\sqrt{3}$, and such that $M_{\sigma_i} \neq M_{\sigma_j}$ for $i \neq j$ (and also $PM_{\sigma_i}P^{-1} \neq M_{\sigma_j}$ for $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$).

Exercise 4:
Find a tile substitution with substitution matrix $M_\sigma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right)$

(a) in dimension one

(b) in dimension two (⋆)

(c) in dimension three (⋆)

Exercise 5:
Given a CPS Λ with $d = e = 1$, lattice $\Gamma = \langle \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}\sqrt{2}\\-\sqrt{2}\end{smallmatrix}\right)\rangle_{\mathbb{Z}}$, window $W = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, and π_1 and π_2 the orthogonal projections to the x-axis, resp. the y-axis.

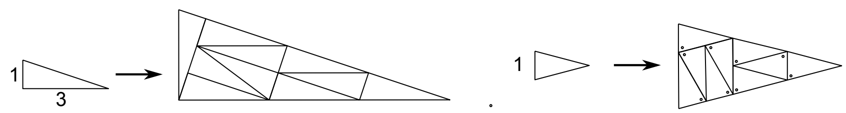
(a) Construct the points of Λ that are contained in the interval $[-10, 10]$. *(lengthy)*

(b) Find a tile substitution that generates Λ, too.

(c) Find a simple algebraic expression for the star map $x^\star$ of x.

Exercise 6:
Prove that no power M^k of $M = \frac{1}{5}\left(\begin{smallmatrix}3 & 4\\-4 & 3\end{smallmatrix}\right)$ equals $\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)$ for $k \in \mathbb{Z}, k > 0$.

Exercise 7:
Prove that the following tile substitutions generate tilings with DTO: $(\star)$



Exercise 8:
Construct a tiling with DTO in $\mathbb{R}^3$ out of Conway's pinwheel's substitution σ_P, by considering $(\sigma_P)^2$ (that has substitution factor 5) and stacking 5 layers of the thickened pinwheel. (At least one $1 \times 1 \times 2$ block of two tiles has to be rotated appropriately).

(a) Show that the resulting tilings have DTO, i.e., the orientations of the tiles are dense on the sphere. $(\star)$

(b) Show that the orientations of the tiles in the resulting tilings are equidistributed on the sphere. $(\star)$