INVERTIBLE SUBSTITUTIONS AND CONTINUED FRACTIONS

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Abstract.

1. Intro

xxx some introductory text

TODO [12, 25, 30, 37, 46] [1] [9] [34, Chapters 6 & 9]

Let $\mathbb{N}$ denote the positive integers, $\mathcal{A}_n = \{1, 2, \dots, n\}$ an alphabet with n letters, F_n the free group with n generators, and let $\mathcal{A}_n^*$ be the free monoid with n generators; i.e., $\mathcal{A}_n^* = \bigcup_{k=0}^{\infty} (\mathcal{A}_n)^k$ is the set of all words over $\mathcal{A}_n$, with concatenation as binary operator. In this paper we consider only the case $n = 2$. We denote the two letters by a and b, hence $\mathcal{A}_2 = \{a, b\}$, and $\mathcal{A}_2^*$ contains all words consisting of letters a and b, together with the empty word $\varnothing$ (of length 0).

A *word substitution* (or short simply *substitution*) over $\mathcal{A}_2^*$ is a map $\sigma : \mathcal{A}_2^* \to \mathcal{A}_2^*$ with $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in \mathcal{A}_2^*$. A substitution is uniquely defined by the image of the letters a and b. Therefore we denote a substitution σ by $\sigma = (\sigma(a), \sigma(b))$.

A substitution σ can be extended uniquely to an endomorphism of the free group F_2 by $\sigma(a^{-1}) = (\sigma(a))^{-1}$ and $\sigma(b^{-1}) = (\sigma(b))^{-1}$. A substitution is called *invertible*, if it is invertible as an endomorphism of F_2. In other words, σ is invertible if $\sigma \in \mathrm{Aut}(F_2)$ where $\mathrm{Aut}(F_2)$ denotes the automorphism group of F_2. We denote the set of invertible substitutions over $\mathcal{A}_2^*$ by $\mathrm{Aut}(\mathcal{A}_2^*)$.

It is useful to consider the *substitution matrix* M_σ of a substitution σ defined by

$$M_\sigma = \begin{pmatrix} |\sigma(a)|_a & |\sigma(b)|_a \\ |\sigma(a)|_b & |\sigma(b)|_b \end{pmatrix},$$

where $|w|_a$ denotes the number of a's in $w \in \mathcal{A}_2^*$, and $|w|_b$ denotes the number of b's in w. Moreover, we denote by $|w|$ the length of a word $w \in \mathcal{A}_2^*$, i.e., the number of letters of w.

Remark 1.1. If σ is invertible then $M_\sigma \in GL(2, \mathbb{Z})$. In particular, if σ is invertible then $\det(M_\sigma) = \pm 1$. Hence M_σ is also called the Abelianisation of σ.

Example 1.1. The silver mean substitution σ_1 is given by $\sigma_1(a) = aba$, $\sigma_1(b) = a$. So we write shortly $\sigma_1 = (aba, a)$. The substitution matrix is $M_{\sigma_1} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Applying σ_1

repeatedly on a single letter yields longer and longer words:

$$\sigma_1(a) = aba,\ \sigma_1^2(a) = abaaaba,\ \sigma_1^3(a) = abaaabaabaabaaaba,$$
$$\sigma_1^4(a) = abaaabaabaabaaabaabaaabaabaaabaabaabaaaba, \text{ etc.}$$

This substitution is invertible, one has $\sigma_1^{-1} = (b, b^{-1}ab^{-1})$. For this note that $\sigma_1^{-1}(a) = b$ is immediate from $\sigma_1(b) = a$, and we then have $a = \sigma_1^{-1}(aba) = \sigma_1^{-1}(a)\sigma_1^{-1}(b)\sigma_1^{-1}(a) = b\sigma_1^{-1}(b)b$.

A substitution σ is called *primitive* if the corresponding substitution matrix is primitive, i.e., if there is $k \in \mathbb{N}$ such that M_σ^k contains positive entries only. While a primitive substitution yields arbitrary long words if applied repeatedly on any single letter, non-primitive substitutions might produce only short words — or even empty words — in this way. Furthermore, a primitive substitution applied repeatedly on any single letter will eventually produce words that contain all letters of the alphabet. The substitution matrix M_σ of a primitive substitution σ yields information on the words generated by σ. This is made precise in the following lemma, which is essentially the Perron-Frobenius Theorem [32] applied to substitutions, see [4].

Lemma 1.1. *If σ is a primitive substitution over $\mathcal{A}_n^*$, then M_σ has a unique eigenvalue λ that is larger in modulus than the other eigenvalues of σ. Moreover, $\lambda \in \mathbb{R}$ and $\lambda > 1$. Furthermore λ is the average growth of the words $\sigma^n(x)$, i.e.,*

$$\lambda = \lim_{n\to\infty} \frac{|\sigma^{n+1}(x)|}{|\sigma^n(x)|}, \quad (x \in \mathcal{A}_n).$$

Hence, in our context, this eigenvalue λ is called the substitution factor *of M_σ. In a more general context λ is called* Perron-Frobenius eigenvalue *of M_σ.*

The normalised eigenvector of M_σ corresponding to λ contains the relative frequencies of letters. In particular, it contains positive entries only. I.e., if $(v_1, \dots, v_n)$ with $v_1 + \cdots + v_n = 1$ is an eigenvector of M_σ corresponding to λ then

$$v_k = \lim_{n\to\infty} \frac{|\sigma^n(x)|_{x_k}}{|\sigma^n(x)|} \quad (x_k \in \mathcal{A}_n),$$

i.e., v_k is the relative frequency of the k-th letter x_k.

Example 1.1 (continued). The eigenvalues of the substitution matrix $M_{\sigma_1} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of the silver mean substitution σ_1 are $1+\sqrt{2}$ and $1-\sqrt{2}$. So the substitution factor is $\lambda = 1+\sqrt{2}$. The normalised eigenvector for λ is $(\frac{1}{2}\sqrt{2}, 1-\frac{1}{2}\sqrt{2})^T$. Thus a word $\sigma^n(a)$ is of approximate length $c(\sqrt{2}+1)^n$ (for some $c \in \mathbb{R}$) and the ratio of the number of as and the number of bs in this word tends to $(1+\sqrt{2}) : 1$ for large n.

Lemma 1.2. *Let $\sigma, \tau \in \mathrm{Aut}(\mathcal{A}_2^*)$ be primitive. (In fact, σ and τ may be arbitrary primitve substitutions on two letters). If $\sigma \circ \tau$ and $\tau \circ \sigma$ are both primitive, then they have the same substitution factor.*

Proof. We have $\det(M_\sigma M_\tau) = \det(M_\sigma)\det(M_\tau) = \det(M_\tau M_\sigma)$ and $\operatorname{tr}(M_\sigma M_\tau) = \operatorname{tr}(M_\tau M_\sigma)$, hence the eigenvalues coincide. □

A common object of study in the context of substitutions is the collection of all biinfinite words generated by a substitution, the (symbolic) *hull* $\mathbb{X}_\sigma$ of a substitution σ. In the context of symbolical dynamical systems this yields a "shift space" (or "subshift") $(\mathbb{X}_\sigma, S)$, where S is the shift operator $S(u) = v$ where $v_n = u_{n+1}$. One way to define the hull is via legal words.

Definition 1.1. *The* hull *of a primitive substitution σ is the set*

$$\{u \in \mathcal{A}^{\mathbb{Z}} \mid \text{ each subword of } uissubwordof \sigma^k(x) \text{ for some } k \in \mathbb{N}, x \in \mathcal{A}\}$$

For primitive substitutions this definition coincides with others in the literature, see for instance [4, Remark 4.2]. In the sequel we will consider two substitutions as equivalent if they define the same hull.

This paper was motivated by the question about what happens if we consider compositions of substitutions. Let us illustrate this with some examples. It is easy to see that the square $\sigma^2 = \sigma \circ \sigma$ — or more generally any power σ^n for $n \in \mathbb{N}$ — yields the same hull as the substitution σ. Thus we will consider products of different substitutions only. For the composition $\sigma \circ \tau$ we write shortly $\sigma\tau$.

Example 1.2. Besides the silver mean substitution σ_1 from Example 1.1 there is another invertible substitution with the same substitution factor $1 + \sqrt{2}$: $\sigma_2 = (abb, ab)$. Let us consider products of σ_1, σ_2 and the well-known Fibonacci substitution $\tau = (ab, a)$.

	substitution matrix	eigenvalues
$\tau\sigma_1 = (abaab, ab)$	$\left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)$	$2 \pm \sqrt{3}$
$\tau\sigma_2 = (abaa, aba)$	$\left(\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right)$	$2 \pm \sqrt{3}$
$\sigma_1\tau = (abaa, aba)$	$\left(\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right)$	$2 \pm \sqrt{3}$
$\sigma_2\tau = (abbab, abb)$	$\left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right)$	$2 \pm \sqrt{3}$
$\sigma_2\sigma_1 = (abbababb, abb)$	$\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)$	$\frac{5}{2} \pm \frac{\sqrt{21}}{2}$
$\sigma_1\sigma_2 = (abaaa, abaa)$	$\left(\begin{smallmatrix} 4 & 3 \\ 1 & 1 \end{smallmatrix}\right)$	$\frac{5}{2} \pm \frac{\sqrt{21}}{2}$

The silver mean substitutions σ_1 and σ_2 combined with the Fibonacci (or golden mean) substitution yield three different substitutions, all with PF eigenvalue $2 + \sqrt{3}$. The two combinations of different silver mean substitutions yield two different subtitutions with PF eigenvalue $\frac{5}{2} + \frac{\sqrt{21}}{2}$.

These observations can be done systematically within the framework described in the next section. A key result is the following.

Theorem 1.1 ([45])**.** *Let $\varepsilon = (b, a)$, $\pi_1 = (ab, b)$, $\pi_2 = (ba, b)$. Then*

$$\operatorname{Aut}(A_2^*) = \langle \varepsilon, \pi_1, \pi_2 \rangle.$$

Note that $\varepsilon^2 = \varepsilon \circ \varepsilon$ is the identity, and that $b\pi_1(\cdot)b^{-1} = \pi_2(\cdot)$.

Corollary 1.1. *Let* $E = M_\varepsilon = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $P = M_{\pi_1} = M_{\pi_2} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. *If* $\sigma \in \mathrm{Aut}(\mathcal{A}_2^*)$ *then* $M_\sigma = P^{n_0} E P^{n_2} E P^{n_3} \cdots E P^{n_k}$, $n_0, n_k \geq 0, n_i \in \mathbb{N}$ $(1 \leq i \leq k-1)$.

Example 1.2 (continued). The Fibonacci substitution has the representation $\tau = \varepsilon\pi_2$. The silver mean substitutions have representations $\sigma_1 = \varepsilon\pi_2\pi_1$ and $\sigma_2 = \pi_1\varepsilon\pi_2$. Thus it is easy to see that

$$\tau\sigma_2 = (\varepsilon\pi_2)(\pi_1\varepsilon\pi_2) = (\varepsilon\pi_2\pi_1)(\varepsilon\pi_2) = \sigma_1\tau.$$

Note that a representation is not necessarily unique. TODO Is that a problem/issue? TODO For instance,

$$\sigma_1 = \varepsilon\pi_1\pi_2 = \varepsilon\pi_2\pi_1 = (aba, a).$$

Furthermore, in view of Lemma 1.2, it is clear that $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ have the same substitution factor.

2. Substitutions as automorphisms of F_2

In the sequel we assume that a substitution σ is always in reduced form with respect to the generators ε, π_1 and π_2; i.e., we assume that a representation of σ does not contain ε^n for $n \geq 2$.

Lemma 2.1. *The non-primitive substitutions in* $\mathrm{Aut}(A_2^*)$ *are exactly the ones of the form*

$$\varepsilon,\ \pi_1^{n_1}\pi_2^{n_2}\pi_1^{n_3}\cdots\pi_2^{n_k},\ \varepsilon\pi_1^{n_1}\pi_2^{n_2}\pi_1^{n_3}\cdots\pi_2^{n_k}\varepsilon,$$

where $k \geq 1$ *and* $n_i \geq 0$ *for* $1 \leq i \leq k$.

Proof. For the substitution matrices we have $M_{\pi_1} = M_{\pi_2} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = P$, thus we just need to show that

(a) that $E = M_\varepsilon$, P^n and EP^nE $(n \geq 0)$ are not primitive, and
(b) that all other products of E and P are primitive.

Regarding (a) we observe that $E^n = E$ for odd n and E^n is the identity matrix for even n. Moreover, $P^n = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ for all $n \in \mathbb{N}$, so no power of P (and of P^n) is primitive. Since $EP^nE = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ we have

$$(EP^nE)^k = EP^nE^2P^nE^2P^nE^2 \cdots P^nE = EP^{kn}E$$

and part (a) follows.

Regarding (b), note that any matrix not of the form above ist either of the form EP^n or P^nE, or it is a product of E and P containing PEP somewhere. Since

$$EP^n = \left(\begin{smallmatrix} n & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad P^nE = \left(\begin{smallmatrix} 0 & 1 \\ 1 & n \end{smallmatrix}\right),$$

we have

$$(EP^n)^2 = \left(\begin{smallmatrix} n^2+1 & n \\ n & 1 \end{smallmatrix}\right) > 0 \text{ and } (P^nE)^2 = \left(\begin{smallmatrix} 1 & n \\ n & n^2+1 \end{smallmatrix}\right) > 0,$$

where $A > 0$ means that all entries of the matrix A are positive. Thus EP^n or P^nE are primitive for all $n \in \mathbb{N}$. All entries of the matrix $PEP = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right)$ are positive, so

$PEP > 0$, in particular PEP is primitive. Multiplication with E interchanges rows (from left) respectively columns (from right), leaving all entries positive. Multiplication with $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ yields that $PA > 0$ and $AP > 0$ if $A > 0$. □

Different substitutions may generate the same hull. Firstly, it is easy to see that σ and σ^2 generate the same hull if σ is a primitive substitution. More generally, we have the following result.

TODO exact references?

Lemma 2.2 ([35, 34])**.** *For any primitive substitution σ and for all $n \in \mathbb{N}$ we have $\mathbb{X}_{\sigma^n} = \mathbb{X}_\sigma$.*

Secondly, there are also substitutions that are not powers of a common substitution that generate the same hulls.

Example 2.1. The substitutions $\tau = (ab, a)$ and $\tau' = (ba, a)$ both generate the same biinfinite (Fibonacci) words

$$\cdots abaababaabaababaababaabaababaabaababaab \cdots,$$

hence $\mathbb{X}_\tau = \mathbb{X}_{\tau'}$.

In order to make this precise we need the notion of conjugate substitutions. Recall that an automorphism $\gamma \in \mathrm{Aut}(F_2)$ is called *inner* automorphism, if there exists $w \in F_2$ such that $\gamma(x) = wxw^{-1}$ for every $x \in F_2$. We let γ_w denote this inner automorphism.

Definition 2.1. *We say that two given substitutions σ and τ are* conjugate*, if $\sigma = \gamma_w \tau$ for some $w \in F_2$. In this case, we will write $\sigma \sim \tau$.*

Remark 2.1. The use of the term 'conjugate' here is sloppy. A correct formulation would be "τ is obtained from σ by the action of an inner automorphism". For convenience, we take the freedom to use the short version. Note that if σ and τ are conjugate then w is a common prefix (if $\tau = \gamma_w \sigma$) or common suffix of $\tau(a)$ and $\tau(b)$ (if $\gamma_w \tau = \sigma$). Moreover, in this case we have $M_\sigma = M_\tau$.

Theorem 2.1 ([38, 8])**.** *Let σ and τ be two invertible primitive substitutions on two letters. Then $\mathbb{X}_\sigma = \mathbb{X}_\tau$ if and only if $\sigma^k \sim \tau^m$ for some $k, m \in \mathbb{N}$. In particular, if σ and τ have the same substitution factor, then $\mathbb{X}_\sigma = \mathbb{X}_\tau$ if and only if $\sigma \sim \tau$.*

Example 2.1 (continued). The two Fibonacci substitutions $\tau = (ab, a)$ and $\tau' = (ba, a)$ above are conjugate, since

$$\gamma_{a^{-1}}\tau(a) = a^{-1}\tau(a)a = a^{-1}aba = ba = \tau'(a), \text{ and } \gamma_{a^{-1}}\tau(b) = a^{-1}\tau(b)a = a^{-1}aa = a = \tau'(b),$$

thus their hulls are equal.

As a third — more general — possibility, two substitutions my generate the same biinfinite words up to switching the letters a and b.

Example 2.2. The substitutions $\varrho = (b, ba)$ and $\varrho' = (b, ab)$ are conjugate to each other by $\gamma_{b^{-1}} \varrho = \varrho'$. Thus they both generate the same biinfinite (Fibonacci) words

$$\cdots babbababbabbababbababbabbababbabbababba \cdots .$$

Replacing in these words each a by b and vice versa, one obtains the same words as in Example 2.1. This corresponds to conjugation by an outer automorphism, in this case $\varepsilon \varrho \varepsilon = \varrho'$.

Lemma 2.3. *Let σ and τ be two invertible primitive substitutions on two letters. If $\tau = \varepsilon\sigma\varepsilon$ then τ and σ have the same hull up to interchanging a and b.*

Because of Corollary 1.1, we obtain the following result.

TODO compare the following with results about continued fractions of eigenvectors

Lemma 2.4. *Any M_σ for $\sigma \in \mathrm{Aut}(\mathcal{A}_2^*)$ is of one of the four forms listed below. These forms correspond to the four cases whether a is the more frequent or the less frequent letter, and whether $\sigma(a)$ has more or less letters than $\sigma(b)$.*

(a) $M_\sigma = EP^k E \cdots P^s$: *$a$ is more frequent,* $|\sigma(a)| > |\sigma(b)|$.
(b) $M_\sigma = EP^k E \cdots P^s E$: *$a$ is more frequent,* $|\sigma(a)| < |\sigma(b)|$.
(c) $M_\sigma = P^k E \cdots P^s$: *$a$ is less frequent,* $|\sigma(a)| > |\sigma(b)|$.
(d) $M_\sigma = P^k E \cdots P^s E$: *$a$ is less frequent,* $|\sigma(a)| < |\sigma(b)|$.

This result is well-known, but we are not aware of a proof.

Proof. Since $EP^n = \left(\begin{smallmatrix} n & 1 \\ 1 & 0 \end{smallmatrix}\right)$ point (1) holds for $M_\sigma = EP^n$. Any compositions of such σs keep a even more frequent, and $|\sigma|$ even larger, so (a) holds in general.

Multiplication of the above M_σ by E from the left swaps the rows of M_σ. Hence in EM_σ th sum of the first column is greater thjan teh sum of the second column, but now a is rarer than b. This yields c.

Since conjugation by ε interchanges a and b, points (b) and (d) are true, too. □

Because of the two preceding results we will focus in the sequel on substitutions σ where a is always the more frequent letter; i.e., where $M_\sigma = EP^k E \cdots P^s$ or $M_\sigma = EP^k E \cdots P^s E$. Any result on such substitutions transfers immediately to the other two cases by Lemma 2.3.

[xxx wir brauchen noch: für $\sigma \in \mathrm{Aut}(\mathcal{A}_2^*)$ (mit obigem wlog) ist immer $|\sigma(a)|_a \geq |\sigma(b)|_a$ und $|\sigma(a)|_b \geq |\sigma(b)|_b$. Das ist wahr, denn in jeder Konjugationsklasse gibt's einen Vertreter mit $\sigma(b)$ ist Präfix von $\sigma(a)$.]

Recall the following theorem of Nielsen [29]. Here, the substitution matrix relies on the Abelianisation map from F_2 to $\mathbb{Z}^2$, i.e., the substitution matrix for a general element of $\mathrm{Aut}(F_2)$ counts the number of occurrences of each a (resp. b) in $\sigma(x)$ minus the number of occurrences of each a^{-1} (resp. b^{-1}) in $\sigma(x)$ ($x \in \{a, b\}$).

Theorem 2.2 ([29])**.** *Let $\sigma, \varrho \in \mathrm{Aut}(F_2)$. σ and ϱ are conjugate wrt an inner automorphism if and only if they have the same substitution matrix.*

Together with Theorem 2.1 this implies immediately the following result. For the last point one needs to recall that the normalised eigenvector to the substitution factor λ equals the frequencies of the letters; a 2×2 matrix with determinant ± 1 is uniquely determined by an eigenvalue λ and its eigenvector. [xxx A, B diagonalisable, same set of eigenvectors, then $A = P^{-1}BP$, P a permutation matrix.]

Corollary 2.1. *Let σ, ϱ be primitive invertible substitutions on the alphabet $\mathcal{A} = \{a, b\}$ with the same inflation factor. Then, the following are equivalent:*

(a) $\sigma \sim \varrho$
(b) $M_\sigma = M_\varrho$
(c) $\mathbb{X}_\sigma = \mathbb{X}_\varrho$
(d) *The relative frequency of a's and b's in $\mathbb{X}_\sigma$ equal the relative frequency of a's and b's in $\mathbb{X}_\tau$*

3. Substitutions with the same factor

Since conjugate matrices have identical eigenvalues the following result is immediate.

Lemma 3.1. *Let $\sigma, \tau \in \mathrm{Aut}(\mathcal{A}_2^*)$. If M_σ and M_τ are conjugate wrt E, P, i.e., if there are n_i such that*

$$M_\sigma = P^{n_1} E P^{n_2} E \cdots E P^{n_k} M_\tau P^{-n_k} E \cdots E P^{-n_1} \quad (n_i \geq 0)$$

then σ and τ have the same substitution factor. □

The following theorem (Theorem 4 in [16]) allows us to obtain the "if and only if" version of Theorem 3.2.

TODO include in next section??

TODO also need [16, Theorem 3]: Any matrix in $GL(2,\mathbb{Z})$ is integrally similar to a *standard* matrix (as defined in [16, Defininition 4.1]).

Theorem 3.1 ([16])**.** *Let $A \in GL(2,\mathbb{Z})$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$, where $a > c > 0$, $b \geq d \geq 0$ and $\mathrm{tr}(A) > 2$. Then A is uniquely defined by its first column and its determinant (1 or -1). Moreover, A then has the unique expansion*

$$A = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

in matrices of the form $\left(\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}\right)$ such that $\det(A) = (-1)^{n+1}$. The c_i are the entries in the (finite) continued fraction expansion $[c_0; c_1, \ldots, c_n]$ of $\frac{a}{c}$.

Note that there is an ambiguity in the continued fraction expansion since $[c_0; c_1, \ldots, c_m, 1] = [c_0; c_1, \ldots, c_m + 1]$.

TODO reformulate corollary?

Corollary 3.1. *Let $\sigma, \varrho \in \mathrm{Aut}(\mathcal{A}_2^*)$ be primitive. If $M_\sigma = M_\varrho$ then M_σ and M_ϱ have the same factorisation into E and P.*

The factorisations here are always to be understood to contain no E^2.

Proof. First, let $\mathrm{tr}(A) > 2$. If $A = M_\sigma = M_\varrho = EP^{k_0}E \cdots EP^{k_n}$ then by Lemma 2.4 a is the more frequent letter, and $|\sigma(a)| > |\sigma(b)|$, hence the conditions of Thm. 3.1 are fulfilled. Since $EP^m = \left(\begin{smallmatrix} m & 1 \\ 1 & 0 \end{smallmatrix}\right)$, any matrix A as above has a unique expansion of the form

$$A = EP^{c_0}EP^{c_1} \cdots EP^{c_n}, \tag{2}$$

since in this case $\det(A) = (-1)^{n+1}$. If on the other hand

$$A = M_\sigma = M_\varrho = EP^{k_0}E \cdots EP^{k_n}E, \tag{3}$$

then $\det(A) = (-1)^{n+2}$, and Theorem 3.1 does not apply. Thus we consider AE, it has a unique expansion of the form (2), hence A has a unique expansion of the form (3).

The two further cases in Lemma 2.4 have substitution matrices A with $a < c$ and *bled*. Considering EAE reduces these cases to the two cases above.

Now let $A = M_\sigma$ for some primitive $\sigma \in \mathrm{Aut}(\mathcal{A}_2^*)$, where $\mathrm{tr}(A) \leq 2$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a \geq c > 0$, $b \geq d \geq 0$. Such a matrix is of the form of one of the following matrices:

$$\left(\begin{smallmatrix} 2 & b_1 \\ 1 & 0 \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 1 & b_2 \\ 1 & 0 \end{smallmatrix}\right), \quad \text{or } \left(\begin{smallmatrix} 1 & b_3 \\ 1 & 1 \end{smallmatrix}\right) \quad (b_i \geq 0).$$

Because of $\det(A) = \pm 1$ we have $b_1 = 1$, $b_2 = 1$, and $b_3 = 0$. The third case is not primitive, hence impossible. The first case is EP^2, the second case is EP, in both cases is the expansion unique.

Altogether, if two substitution matrices M_σ and M_τ have different expansions in E, P then they have different expansions in the form (1). Thus they have different ratio a/c, thus $M_\sigma \neq M_\tau$. □

The conjugation class of a substitution σ can be read off explicitly from its representation in ε, π_1 and π_2, see [34, Exercise 9.2.5] for one implication ("if σ and ϱ have the same representation in ε, π_1 and π_2, then $\sigma \sim \varrho$"). Corollary 3.1 yields now the "if and only if" version of this result.

Theorem 3.2. *Let $\sigma = \varepsilon^{\alpha_1}\pi_{i_1}^{n_1}\varepsilon^{\alpha_2}\pi_{i_2}^{n_2} \cdots \varepsilon^{\alpha_s}\pi_{i_s}^{n_s}\varepsilon^{\alpha_{s+1}}$ and $\tau = \varepsilon^{\beta_1}\pi_{j_1}^{m_1}\varepsilon^{\beta_2}\pi_{j_2}^{m_2} \cdots \varepsilon^{\beta_s}\pi_{j_s}^{m_s}\varepsilon^{\beta_{s+1}}$, where $\alpha_k, \beta_k \in \{0,1\}$, $i_k, j_k \in \{1,2\}$ and $n_k, m_k \in \mathbb{N}$. We have $n_i = m_i$ for $1 \leq i \leq s$ and $\alpha_i = \beta_i$ for $1 \leq i \leq s+1$ if and only if $\sigma \sim \tau$.*

For instance, the conjugation class of the silver mean substitution $\sigma_1 = (aba, a) = \varepsilon\pi_2\pi_1$ consists of all substitutions of the form $\varepsilon\pi_{i_1}\pi_{i_2}$ $(i_1, i_2 \in \{1,2\})$. These are $\varepsilon\pi_1\pi_1 = (baa, a)$, $\varepsilon\pi_1\pi_2 = \varepsilon\pi_2\pi_1 = (aba, a)$, and $\varepsilon\pi_2\pi_2 = (aab, a)$.

The following result states that there are arbitrary many substitutions in $\mathrm{Aut}(\mathcal{A}_2^*)$ with different hulls and the same substitution factor.

Theorem 3.3. *For any $N \in \mathbb{N}$ there is a substitution factor λ such that there are more than N substitutions $\sigma_i \in \mathrm{Aut}(\mathcal{A}_2^*)$ with the same substitution factor λ that are not conjugate. I.e., all these substitutions have pairwise different hulls.*

Proof. Because of Lemma 3.1 two substitutions σ, τ with substitution matrices M_σ, M_τ that have different representations in E and P, but that are conjugate wrt E, P, have the same substitution factor.

Because of Corollary 2.1 substitutions with different matrices are in different conjugation classes and have different hulls.

Given $N \in \mathbb{N}$ it is now easy to find N different matrices with the same eigenvalues. For instance, the followign matrices are all conjugate, hence they have the same eigenvalues.

$$EP^N, PEP^{N-1}, P^2EP^{N-2}, \ldots, P^{N-1}EP.$$

□

Remark 3.1. In this context the number of different words in E, P containing exactly N P's and no EE, up to cyclic permutations, become of interest. They can be seen as cyclic ordered combinations of integers, or the number of necklaces of sets of beads. By Theorem 4.1 in [24] (see also sequence A008965 in OEIS) the number of cyclic ordered combinations, hence the number of different words in E, P containing exactly N P's and no EE is

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{n \mid \gcd(N,k)} \varphi(n) \binom{\frac{N}{n} - 1}{\frac{k}{n} - 1}$$

4. Mutual local derivability

The following results state that all the different substitutions in $\mathrm{Aut}^*(F_2)$ with the same substitution factor are strongly related. In particular, the different substitutions of Theorem 3.3 are. To be more precise: if two invertible substitutions have the same substitution factor then there is a local rule how to obtain one from the other (and vice versa). However, there are two such concepts, one on the symbolic level and one on a geometric level.

Definition 4.1. *A* sliding block code *is a map $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ (where $\mathcal{A}$ and $\mathcal{B}$ are two alphabets) such that there is $m \in N$ such that in $v = \Phi(u)$ each v_i depends only on $u_{i-m}, u_{i-m+1}, \ldots, u_{i+m}$.*

An important application of sliding block codes is that by the Curtis-Hedlund-Lyndon Theorem two shift spaces are topologically conjugate if and only if there is a sliding block code mapping one to the other (and vice versa).

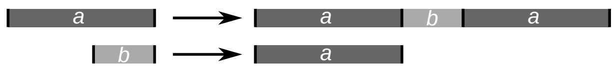
Example 4.1. By Theorem 3.3 the two silver mean substitutions $\sigma_1 = (aba, a)$ and $\sigma_2 = (abb, ab)$ have different hulls. In order to distinguish the letters of σ_1 and σ_2 we use capital letters for σ_2 in this example, so $\sigma_2 = (ABA, A)$. One can easily transform the biinfinite words from $\mathbb{X}_{\sigma_2}$ into the ones in $\mathbb{X}_{\sigma_1}$ in a unique way by a “local” rule: replace each AB by a, then replace all remaining As by b. This is indicated in the following diagram.

AB	B	AB	AB	AB	B	AB	AB	B	AB	AB	B	AB	AB	AB	B	AB	AB
a	b	a	a	a	b	a	a	b	a	a	b	a	a	a	b	a	a

In the other direction it is even simpler: replace each a by AB and each b by B. Nevertheless, this map is *not* a sliding block code. In order to describe this kind of relation we need a geometric picture.

A geometric realisation of a primitve symbolic substitution σ, an *inflation rule*, can be obtained as follows [4, Section 4]. The word $u = \ldots u_{-1}, u_0, u_1 \ldots$ is translated into a tiling of the real line by intervals. The left eigenvector $v = (v_1, v_2, \ldots)$ of M_σ corresponding to the PF-eigenvalue λ yields the lengths of tiles: each a is replaced by an interval of length v_1, each b by an interval of length v_2 (and so on). Hence any $u \in \mathbb{X}_\sigma$ can be translated into a tiling of the real line by replacing letters with the corresponding intervals (in the same order). Moreover, the resulting set can be described as the *geometric hull* of a *inflation rule*. An inflation rule consists of a set of different tiles $T_1, \ldots, T_m$ (here: intervals), an inflation factor λ and a rule how to dissect each λT_i into copies of the T_j.

Example 4.2. For instance the silver mean substitution σ_1 of Example 1.1 yields the left eigenvector $v = (1 + \sqrt{2}, 1)$. The inflation rule corresponding to σ_1 is



where $\lambda = 1 + \sqrt{2}$, a is an interval of length $1 + \sqrt{2}$, and b is an interval of length 1.

Let us denote the inflation obtained from a substitution σ denote by σ,, too. The inflation can be iterated on the intervals in the same way as the substitution on the letters. The geometric hull $\mathcal{X}_\sigma$ of an inflation can be defined analogously to the symbolic hull of a substitution:

$$\mathcal{X}_\sigma = \{\mathcal{T} \text{ tiling of } \mathbb{R} \mid \text{ each finite subset of } \mathcal{T} \text{is a translate of some } \sigma^k(T_i)\}$$

Definition 4.2. *A tiling $\mathcal{T}$ of the real line is locally derivable from a tiling $\mathcal{S}$ of the real line if there is an interval K such that $\mathcal{S} \cap (x + K) = \mathcal{S} \cap (y + K)$ for some x, y implies $\mathcal{T} \cap \{x\} = \mathcal{T} \cap \{y\}$*

Here $\mathcal{T} \cap K$ means the collection of all tiles $T \in \mathcal{T}$ with $T \cap K \neq \varnothing$.

Definition 4.3. *Two tilings $\mathcal{S}, \mathcal{T}$ are* mutually locally derivable *(MLD) if $\mathcal{S}$ is locally derivable from $\mathcal{T}$ and vice versa.*

Two hulls that are mld share several topological and dynamical properties [4].

A very nice property of inflation rules derived from primitive invertible substitutions on two letters is stated in the following theorem. In order to make sense of it we need to introduce the concept of canonical projection tilings. We describe it for the simple case of

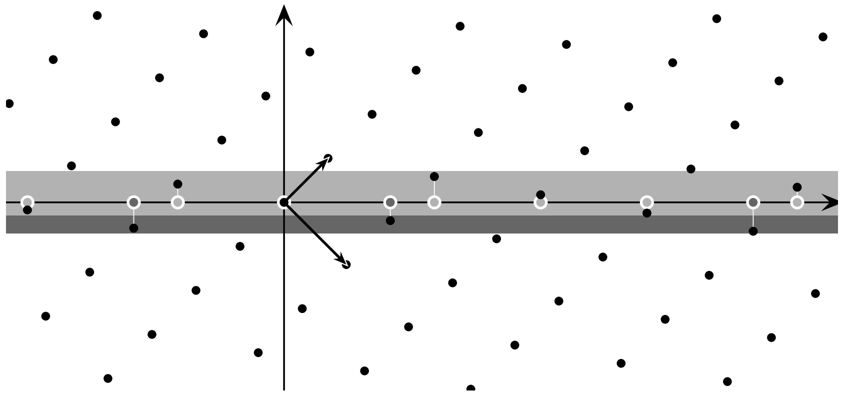


FIGURE 1. A cut-and-project scheme for the silver mean tilings in $\mathcal{X}_{\sigma_1}$. Image courtesy of [4].

dimension one, codimension one and two letters. For the general case see for instance [4, Chapter 7].

A cut-and-project sceme (CPS) is a way to obtain nonperiodic tilings of $\mathbb{R}^d$ (or more general in locally compact abelian groups) by projection from some higher dimensional point lattice. In general a CPS looks as follows:

$$\begin{array}{ccccc} G & \stackrel{\pi_1}{\leftarrow} & G \times H & \stackrel{\pi_2}{\rightarrow} & H \\ \cup & & \cup & & \cup \\ V & \leftarrow & L & \rightarrow & W \end{array}$$

where G, H are locally compact abelian groups, $W \subset H$ is compact (and usually W is the closure of its interior). Let π_1 and π_2 be the canonical projections from $G \times H$ to G respectively H. Then

$$V = \{\pi_1(x) \,|\, x \in L, \pi_2(x) \in W\}$$

is a *model set*, or a cut-and-project set. In the case of a model set on the real line the points of the model set partition the line into intervals. The resulting tiling by these intervals is a *cut-and-project tiling*. Figure 1 shows a CPS for the silver mean tilings. We have $G = H = \mathbb{R}$, $W = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

The following result connects invertible primitive substitutions over two letters with certain projection tilings. The result as such is stated in [21], which in turn uses results from [23, 38, 45]. A *canonical projection tiling* is one where W is the projection of some fundamental cell of the lattice L. [xxx not Voronoi, but fund cell = Delone cell]

Theorem 4.1 ([23, 38, 45, 21]). *Let σ be a primitive (nonperiodic) substitution on two letters. The elements in $\mathcal{X}_\sigma$ are canonical projection tilings if and only if $\sigma \in \mathrm{Aut}(\mathcal{A}_2^*)$.*

Theorem 4.2 ([5], see [4] Remark 7.6). *Let Λ_1, Λ_2 be two model sets with the same lattice L, but different windows W_1, W_2. Λ_1 and Λ_2 are MLD if and only if W_2 can be expressed as a finite union of sets each of which is a finite intersection of translates of W_1 (or its complement), with all translations from $\pi_2(L)$, and vice versa.*

Theorem 4.3. *Let $\sigma, \tau \in \mathrm{Aut}(\mathcal{A}_2^*)$. Let σ and τ have the same substitution factor λ, and the left eigenvectors $v = (\ell_\sigma, 1)$ and $w = (\ell_\tau, 1)$. If $\mathbb{Z}[\ell_\sigma] = \mathbb{Z}[\ell_\tau]$ then the geometric hulls $\mathcal{X}_\sigma$ and $\mathcal{X}_\tau$ are MLD.*

Proof. This is an application of Theorems 4.1 and 4.2.

For the special case considered here it is known [4] that the following construction (the Minkowski embedding) yields a correct CPS (if there is a CPS at all). The construction is quite general, here we describe it for the special case of two letters. The idea is that the vertex set V of an (appropriately scaled and translated) tiling is contained in the ring of quadratic integers $K[\omega] \subset \mathbb{Q}(\lambda)$. Then $K[\omega]$ (or a subring of $K[\omega]$) can be lifted to a lattice in $\mathbb{R}^2$.

Let $\lambda = a + b\sqrt{k}$ be the inflation factor of σ ($a, b, k \in \mathbb{Z}$, $k \geq 2$ square-free). Denote by $\mathbb{Z}[\omega]$ the ring of quadratic integers in $\mathbb{Q}(\lambda)$. This is,

$$\omega = \begin{cases} \sqrt{k} & \text{if } k \equiv 2,3 \pmod 4 \\ \frac{1+\sqrt{k}}{2} & \text{if } k \equiv 1 \pmod 4 \end{cases}$$

Consider the geometric hull of σ. Since the left PF-eigenvector $v = (\ell_\sigma, 1)$ yields the lengths of the tiles, and $\ell := \ell_\sigma \in \mathbb{Q}(\lambda)$, there is $q \in \mathbb{N}$ such that $q\ell \in \mathbb{Z}[\omega]$. Thus the lengths of the tiles can be chosen as $q\ell, q$.

xxx argh. Irgendwie brauchen wir doch den ganzen Z-Modul

Let $\mathcal{T} \in \mathcal{X}_\sigma$ such that 0 is a vertex of $\mathcal{T}$. Since $q\ell, q \in \mathbb{Z}[\omega]$, all vertices of $\mathcal{T}$ are contained in $\mathbb{Z}[\omega]$. Hence the integer span of the vertices of $\mathcal{T}$ is contained in $\mathbb{Z}[\omega]$. For $x = a + b\sqrt{k} \in \mathbb{Z}[\omega]$, let $x' = a - b\sqrt{k}$ denote the algebraic conjugate of x. Let $\Lambda = \langle (1,1)^T, (\ell, \ell') \rangle$.

By Theorem 4.1 the windows $W_1 = [x_1, x_2[$ and $W_2 = [y_1, y_2[$ are projections of fundamental cells of the lattice Λ. Hence $x_1, x_2, y_1, y_2 \in \mathbb{Z}[\omega] = \pi_2(\Lambda)$. It remains to show that Theorem 4.2 applies.

Wlog let $|x_2 - x_1| \leq |y_2 - y_1|$. Then

$$W_1 = [x_1, x_2] = (W_2 + x_1 - y_1) \cap (W_2 + x_2 - y_2).$$

For the other direction, take the union of $m = \lceil \frac{y_2 - y_1}{x_2 - x_1} \rceil$ translates of W_1 and intersect with a further translate $W_1 + y_2 - x_1$.

$$W_2 = [x_1 - x_1 + y_1, x_2 - x_1 + y_1] \cup [x_1 - x_1 + y_1 + (x_2 - x_1), x_2 - x_1 + y_1 + (x_2 - x_1)] \cup \cdots \cup [x_1 - x_1 + y_1 + m(x_2 - x_1),$$

All translations are elements of $\mathbb{Z}[\omega]$. Hence the geometric hulls $\mathcal{X}_\sigma$ and $\mathcal{X}_\tau$ are MLD. □

TODO: CPS (with intervals): [10, 7]???? Symplify after CF-treatment?

5. Results from the Theory of Continued Fractions

A *(regular) continued fraction* is a finite or infinite expression of the form

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \frac{1}{c_n}}}} \qquad \text{or} \qquad c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots}}}$$

where $c_0 \in \mathbb{Z}$ and $c_k \in \mathbb{N}$ for $k \geq 1$; in the first case we speak of a *finite continued fraction (of length n)* and abbreviate it by $[c_0; c_1, c_2, \ldots, c_n]$, in the latter case of an *infinite continued fraction* and we write $[c_0; c_1, c_2, \ldots]$.

Obviously, a finite continued fraction yields a rational number. Conversely, every rational number can be written in a unique way as a finite continued fraction of either odd length or of even length (see [33, Satz 2.3]), noting that $[c_0; c_1, \ldots, c_{n-1}, c_n] = [c_0; c_1, \ldots, c_{n-1}, c_n - 1, 1]$ if $c_n \geq 2$. Given an infinite continued fraction $[c_0; c_1, c_2, \ldots]$, we call the rational number

$$\frac{p_k}{q_k} = [c_0; c_1, c_2, \ldots, c_k],$$

where $p_k, q_k \in \mathbb{Z}$, its *kth-order convergent*; for a finite continued fraction $[c_0; c_1, c_2, \ldots, c_n]$ this concept is defined in the same way, however there are only $(n+1)$ many convergents (of order $0, 1, 2, \ldots, n$). The numerator p_k and denominator q_k of the kth-order convergent can be obtained recursively by

$$\begin{aligned} &p_{-1} = 1, p_0 = c_0, p_j = c_j\, p_{j-1} + p_{j-2} \quad \text{for } j \geq 1, \\ &\qquad q_{-1} = 0, q_0 = 1, q_j = c_j\, q_{j-1} + q_{j-2} \quad \text{for } j \geq 1. \end{aligned} \tag{4}$$

For an infinite continued fraction $[c_0; c_1, c_2, \ldots]$, the sequence of its convergents $\frac{p_k}{q_k}$ converges to an irrational number; conversely, every irrational number has a unique infinite continued fraction (see [33, Satz 2.6]).

Using mathematical induction, see [43] and [16, Lemma 1], the recursions of Eq. (4) can be re-written in matrix form as

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}. \tag{5}$$

Note that the sequence of the matrices on the right can be easily obtained from the continued fraction expansion $\frac{p_k}{q_k} = [c_0; c_1, \ldots, c_k]$; also note that $\frac{p_{k-1}}{q_{k-1}} = [c_0; c_1, \ldots, c_{k-1}]$. We observe that $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = EP^c$, and that $\det(EP^c) = -1$. Thus, taking the determinant in Eq. (5) yields $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$ (compare [33, Eq. §6(1)]). Moreover, the four pair of integers (p_k, q_k), (p_{k-1}, q_{k-1}), (p_k, p_{k-1}) and (q_k, q_{k-1}) are coprime pairs (see [33, Satz 2.1]); in other words, the two integers in any column or any row of the matrix on the right in Eq. (5) are coprime. Taking the matrix transpose on both sides of Eq. (5) yields

(assuming $c_0 > 0$)

$$[c_k; c_{k-1}, \dots, c_0] = \frac{p_k}{p_{k-1}} \qquad \text{and} \qquad [c_k; c_{k-1}, \dots, c_1] = \frac{q_k}{q_{k-1}}.$$

One of the main results in the theory of continued fractions is Langrange's theorem stating that the (regular) continued fraction of a real number x is (eventually) periodic if and only if x is quadratic irrational (see [33, Satz 3.1 & 3.2]), i.e., $x = \frac{\sqrt{D}+b}{d}$ where $b \in \mathbb{Z}$, $D, d \in \mathbb{N}$ and D is not a complete square. For a (eventually) periodic continued fraction $[a_0; a_1, a_2, \dots]$, i.e., where there is an $n \geq 1$ and an $m \geq 1$ such that $a_k = a_{k+m}$ for all $k \geq n$, we write

$$\begin{aligned}[] [a_0; a_1, \dots, a_{n-1}, a_n, \dots, a_{n+m-1}, a_n, \dots, a_{n+m-1}, a_n, \dots, a_{n+m-1}, \dots] \\ = [a_0; a_1, \dots, a_{n-1}, \overline{a_n, \dots, a_{n+m-1}}], \end{aligned}$$

and call the part $a_0, \dots, a_{n-1}$ the *pre-period* and the part $a_n, \dots, a_{n+m-1}$ the *period (of length m)* of the continued fraction.

We now calculate eigenvalues and eigenvectors of the matrix in Eq. (5).

Lemma 5.1. *Let $c_0 > 0$ and denote by $A = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$ the matrix given in Eq. (5). Then A is primitve, has one real eigenvalue $\lambda > 1$ and one real eigenvalue λ' with $|\lambda'| = \frac{1}{\lambda} < 1$. Moreover, the Perron-Frobenius eigenvalue λ is given by*

$$\lambda = \begin{cases} \frac{1}{2}\left(\gamma + \sqrt{\gamma^2+4}\right) = [\gamma; \overline{\gamma}] & \text{if } \det A = -1 \text{ (i.e., if } k \text{ is even)}, \\ \frac{1}{2}\left(\gamma + \sqrt{\gamma^2-4}\right) = [\gamma - 1; \overline{1, \gamma-2}] & \text{if } \det A = 1 \text{ (i.e., if } k \text{ is odd)}, \end{cases}$$

where $\gamma = \operatorname{tr} A = p_k + q_{k-1}$ (note that $\gamma \geq 1$ in the first and $\gamma \geq 3$ in the second case), while for the second eigenvalue λ' we have

$$\lambda' = \begin{cases} -\frac{1}{\lambda} = \frac{1}{2}\left(1 - \sqrt{5}\right) = [-1; 2, \overline{1}] & \text{if } \det A = -1 \text{ and } \gamma = 1, \\ -\frac{1}{\lambda} = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2+4}\right) = [-1; 1, \gamma-1, \overline{\gamma}] & \text{if } \det A = -1 \text{ and } \gamma > 1, \\ \frac{1}{\lambda} = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2-4}\right) = [0; \gamma-1, \overline{1, \gamma-2}] & \text{if } \det A = 1. \end{cases}$$

Proof. If $c_0 > 0$, then the matrix $A = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$ is primitive: If $k = 0$, then $A = EP^{c_0} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}$ the square of which has positive entries only; if $k > 0$, then $A = EP^{c_0}EP^{c_1}\dots$ already has positive entries only, as can be seen immediately from the recursion in Eq. (4). Thus, the Perron-Frobenius theorem applies, compare Lemma 1.1, and A has one real eigenvalue, the Perron-Frobenius eigenvalue, $\lambda > 1$. Since $\det A = p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$, the second eigenvalue λ' is given by $\lambda' = \det A/\lambda$. and thus $|\lambda'| = \frac{1}{\lambda} < 1$.

The characteristic equation of A is given by

$$0 = x^2 - \operatorname{tr} A \cdot x + \det A = x^2 - (p_k + q_{k-1}) \cdot x + (-1)^{k+1} = x^2 - \gamma \cdot x + (-1)^{k+1}$$

with roots $\lambda = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 + 4 \cdot (-1)^k}\right)$ and $\lambda' = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2 + 4 \cdot (-1)^k}\right)$; the condition $\lambda > 1$ implies $\gamma \geq 1$ if k is even and $\gamma \geq 3$ if k is odd.

The continued fraction expansions for λ are obtained as follows: If k is even, we note that the above characteristic equation can be re-writen as $x = \gamma + \frac{1}{x}$ and thus iteration yields $\lambda = [\gamma; \overline{\gamma}]$. If k is odd, one can show that $\lambda = \gamma - 1 + \frac{1}{y}$ where y is the solution $y > 1$ of the quadratic equation $y = 1 + \frac{1}{\gamma - 2 + \frac{1}{y}}$, i.e., $y^2 - y - \frac{1}{\gamma - 2} = 0$; this establishes $\lambda = [\gamma - 1; \overline{1, \gamma - 2}] = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 - 4}\right)$. From these continued fractions, the continued fraction of λ' follows by the following two rules: If $x = [a_0; a_1, a_2, \ldots]$ with $a_0 > 0$, then $\frac{1}{x} = [0; a_0, a_1, a_2, \ldots]$; and if $x = [a_0; a_1, a_2, a_3, \ldots]$, then $-x = [-a_0 - 1; 1, a_1 - 1, a_2, a_3, \ldots]$ if $a_1 > 1$ and $-x = [-a_0 - 1; 1 + a_2, a_3, \ldots]$ if $a_1 = 1$, see [43, Section 4]. □

We note that the discriminantes $\gamma^2 \pm 4$ of λ can be found as sequence A087475 in OEIS respectively sequence A028347 in OEIS.

The value of $\gamma = \operatorname{tr} A = p_k + q_{k-1}$ can be calculated using the recursion in Eq. (4) from $c_0, c_1, c_2, \ldots$ and get – since $\det(EP^{c_0} EP^{c_1} \cdots EP^{c_k}) = \det(EP^{c_1} \cdots EP^{c_k} EP^{c_0})$ and $\operatorname{tr}(EP^{c_0} EP^{c_1} \cdots EP^{c_k}) = \operatorname{tr}(EP^{c_1} \cdots EP^{c_k} EP^{c_0})$ – the following "cyclic" expressions in the $c_i' s$:

k	0	1	2	3	...
p_k	c_0	$c_0 c_1 + 1$	$c_0 c_1 c_2 + c_0 + c_2$	$c_0 c_1 c_2 c_3 + c_0 c_1 + c_2 c_3 + c_3 c_0 + 1$	...
q_{k-1}	0	1	c_1	$c_1 c_2 + 1$	...
γ	c_0	$c_0 c_1 + 2$	$c_0 c_1 c_2 + c_0 + c_1 + c_2$	$c_0 c_1 c_2 c_3 + c_0 c_1 + c_1 c_2 + c_2 c_3 + c_3 c_0 + 2$	...

and for $k = 4$

$$\gamma = c_0 c_1 c_2 c_3 c_4 + c_0 c_1 c_2 + c_1 c_2 c_3 + c_2 c_3 c_4 + c_3 c_4 c_0 + c_4 c_0 c_1 + c_0 + c_1 + c_2 + c_3 + c_4.$$

Next we calculate a right and a left eigenvector of the matrix A for the Perron-Frobenius eigenvalue λ, also compare [16, Theorem 5] and [17, Lemma 8.12]. TODO Connection to other stuff in [17]?

Proposition 5.1. *Let $c_0 > 0$ and denote by $A = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$ the matrix given in Eq. (5). Then, a left and right eigenvector of A to its Perron-Frobenius eigenvalue λ are given by*

$$(1, \theta) \qquad \textit{respectively} \qquad \begin{pmatrix} 1 \\ \ell \end{pmatrix}$$

where

$$\theta = \frac{\lambda - p_k}{q_k} = [0; \overline{c_k, c_{k-1}, \ldots, c_1, c_0}] \qquad \textit{respectively} \qquad \ell = \frac{\lambda - p_k}{p_{k-1}} = [0; \overline{c_0, c_1, \ldots, c_{k-1}, c_k}].$$

Proof. The explicite expressions $\theta = \frac{\lambda - p_k}{q_k}$ respectively $\ell = \frac{\lambda - p_k}{p_{k-1}}$ are immediate from the eigenvector equation (also recall that $p_k, p_{k-1}, q_k > 0$ for $k \geq 0$). We note that $\theta, \ell > 0$ by the Perron-Frobenius theorem, compare Lemma 1.1.

The continued fraction expansion for ℓ follows immediately from [16, Theorem 5] which in our case states: If

$$A = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\frac{1}{\ell} = \frac{p_{k-1}}{\lambda - p_k} = [c_0; \overline{c_1, c_2, \ldots, c_{k-1}, c_k, c_0}].$$

Taking the inverse, yields the result for ℓ. Since a left eigenvector of A is a right eigenvector of its transpose, the result for θ follows. $\square$

We now have calculate eigenvalues and eigenvectors of the matrices of the form

$$A = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix} = EP^{c_0}\, EP^{c_1} \cdots EP^{c_k},$$

but we will also need eigenvalues and eigenvectors of matrices of the form

$$EA = P^{c_0}\, EP^{c_1} \cdots EP^{c_k}, AE = EP^{c_0}\, EP^{c_1} \cdots EP^{c_k}\, E \text{ and } EAE = P^{c_0}\, EP^{c_1} \cdots EP^{c_k}\, E$$

(recall that $EE = \mathrm{id}$). The corresponding results for EAE are immediate, this only permutes columns and rows of the matrix in Eq. (5).

Corollary 5.1. *Let $c_0 > 0$ and consider*

$$EAE = \begin{pmatrix} q_{k-1} & q_k \\ p_{k-1} & p_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & c_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & c_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & c_k \end{pmatrix} = P^{c_0} E\, P^{c_1} E \cdots P^{c_k} E.$$

Then EAE is a primitive matrix with eigenvalues $\lambda > 1$ and λ' as in Lemma 5.1. A left and right eigenvector of EAE to its Perron-Frobenius eigenvalue λ are given by

$$\left(1, \frac{1}{\theta}\right) \qquad \textit{respectively} \qquad \begin{pmatrix} 1 \\ \frac{1}{\ell} \end{pmatrix}$$

where θ and ℓ are given in Prop. 5.1, and thus we have

$$\frac{1}{\theta} = [c_k; \overline{c_{k-1}, \ldots, c_1, c_0, c_k}] \qquad \textit{respectively} \qquad \frac{1}{\ell} = [c_0; \overline{c_1, \ldots, c_{k-1}, c_k, c_0}].$$

$\square$

For EA and AE we need some slight modifications.

Proposition 5.2. *Let $c_0 > 0$, $k > 0$ and consider*

$$EA = \begin{pmatrix} q_k & q_{k-1} \\ p_k & p_{k-1} \end{pmatrix} = P^{c_0}\, E\, P^{c_1}\, E \cdots E\, P^{c_k}.$$

Define

$$\tilde{A} = EP^{c_1}\, EP^{c_2}\, \cdots EP^{c_{k-1}}\, EP^{c_0+c_k} = \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_0+c_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{p}_{k-1} & \tilde{p}_{k-2} \\ \tilde{q}_{k-1} & \tilde{q}_{k-2} \end{pmatrix}.$$

Then, EA and $\tilde{A}$ are primitive matrices, have the same eigenvalues, and we can use Lemma 5.1 with $\gamma = \tilde{p}_{k-1} + \tilde{q}_{k-2}$ noting that $\det(EA) = \det \tilde{A} = (-1)^k$. Moreover, a left and right eigenvector of EA to its Perron-Frobenius eigenvalue λ are given by

$$\left(1, \tilde{\theta}\right) \qquad \text{respectively} \qquad \begin{pmatrix} 1 \\ \tilde{\ell} \end{pmatrix}$$

where

$$\tilde{\theta} = [0; c_k, \overline{c_{k-1}, \ldots, c_1, c_0 + c_k}] \qquad \text{respectively} \qquad \tilde{\ell} = [c_0; \overline{c_1, \ldots, c_{k-1}, c_0 + c_k}].$$

Furthermore, the relationship of matrices of the form

$$AE = E(EA)E = \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix} = E\, P^{c_0}\, E\, P^{c_1}\, E \cdots E\, P^{c_k}\, E$$

to EA are as between EAE and A in Corollary 5.1; in particular, a left and right eigenvector of EA to its Perron-Frobenius eigenvalue λ are given by

$$\left(1, \frac{1}{\tilde{\theta}}\right) \qquad \text{respectively} \qquad \begin{pmatrix} 1 \\ \frac{1}{\tilde{\ell}} \end{pmatrix}$$

where

$$\frac{1}{\tilde{\theta}} = [c_k; \overline{c_{k-1}, \ldots, c_1, c_0 + c_k}] \qquad \text{respectively} \qquad \frac{1}{\tilde{\ell}} = [0; c_0, \overline{c_1, \ldots, c_{k-1}, c_0 + c_k}].$$

Proof. The statement about the eigenvalues follows from

$$\det(EA) = \det(P^{c_0} EP^{c_1} \cdots P^{c_{k-1}} EP^{c_k}) = \det(EP^{c_1} \cdots P^{c_{k-1}} EP^{c_0+c_k}) = \det \tilde{A} \quad \text{and}$$
$$\operatorname{tr}(EA) = \operatorname{tr}(P^{c_0} EP^{c_1} \cdots P^{c_{k-1}} EP^{c_k}) = \operatorname{tr}(EP^{c_1} \cdots P^{c_{k-1}} EP^{c_0+c_k}) = \operatorname{tr} \tilde{A},$$

and thus EA and $\tilde{A}$ have the same characteristic polynomial and consequently the same eigenvalues.

For a right eigenvector $(1, \tilde{\ell})^t$ of EA we note that $EA = P^{c_0}\, \tilde{A}\, (P^{c_0})^{-1} = P^{c_0}\, \tilde{A}\, P^{-c_0}$ and $\tilde{A}$ has the right eigenvector $(1, \ell)^t$ where $\ell = [0; \overline{c_1, \ldots, c_{k-1}, c_0 + c_k}]$ by Prop. 5.1; but this is just a transformation of the coordinate system and thus $(1, \tilde{\ell})^t = P^{c_0}(1, \ell)^t = (1, c_0 + \ell)$, i.e., $\tilde{\ell} = [c_0; \overline{c_1, \ldots, c_{k-1}, c_0 + c_k}]$. For a left eigenvector $(1, \tilde{\theta})$ of EA, a similar calculation using the right eigenvector $(1, \theta)$ of $\tilde{A}$ with $\theta = [0; \overline{c_0 + c_k, c_{k-1}, \ldots, c_1}]$ yields that $(1, \tilde{\theta})$ is a multiple of $(1 - c_0\theta, \theta)$ and thus of $\left(1, \frac{1}{\frac{1}{\theta} - c_0}\right)$ which yields the claim.

The statements about AE follow directly from Corollary 5.1. ∎

TODO

This shows that θ is unique, while λ is (usually) not!

TODO

Point out: power of substitution doesn't change eigenvectors!

TODO

Theorem 5.1. *TODO Summarize in the spirit of* [16, Theorem 5].
Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $\det A = (-1)^{n+1}$*. Then*

(a) *If* $\max\{a, b, c, d\} = a$*, then* $\frac{a}{c} = [c_0; c_1, \ldots, c_n]$ *we have* $A = EP^{c_0}EP^{c_1} \ldots EP^{c_n}$ *the eigenvalues and eigenvectors of which are given in Lemma 5.1 and Prop. 5.1.*
(b) *If* $\max\{a, b, c, d\} = d$*, then* $\frac{d}{b} = [c_0; c_1, \ldots, c_n]$ *we have* $A = P^{c_0}EP^{c_1} \ldots EP^{c_n}E$ *the eigenvalues and eigenvectors of which are given in Corollary 5.1.*
(c) *If* $\max\{a, b, c, d\} = c$*, then* $\frac{c}{a} = [c_0; c_1, \ldots, c_n]$ *we have* $A = P^{c_0}EP^{c_1} \ldots EP^{c_n}$ *the eigenvalues and eigenvectors of which are given in Prop. 5.2.*
(d) *If* $\max\{a, b, c, d\} = b$*, then* $\frac{b}{d} = [c_0; c_1, \ldots, c_n]$ *we have* $A = EP^{c_0}EP^{c_1} \ldots EP^{c_n}E$ *the eigenvalues and eigenvectors of which are given in Prop. 5.2.*

Proof. TODO only have to consider what application of E on the left/right does! □

TODO Corollary is Lemma 2.4.

TODO

[33]

[43] [16]

6. Further Considerations

M??gliche weitere Resultate/Bemerkungen:

xxx Wir k??nnen die Anzahl der (verschiedenen) Substitutionen in einer Konjugationsklasse an der $\varepsilon - \pi$-Darstellung ablesen: Wegen $\pi_1\pi_2 = \pi_2\pi_1$ k??nnen wir jede subst in kanonischer Form $\varepsilon\pi_1^{n_1}\pi_2^{n_1-j_1}\varepsilon\pi_1^{n_2}\pi_2^{n_2-j_2}\cdots$ schreiben. Die Konjugierten sind dann $\varepsilon\pi_1^0\pi_2^{n_1}\varepsilon\cdots, \varepsilon\pi_1^1\pi_2^{n_1-1}$ usw bis $\pi_1^{n_1}\pi_2^0\cdots$. Alle M??glichkeiten also dann doch wohl $(n_1+1)(n_2+1)\cdots$.

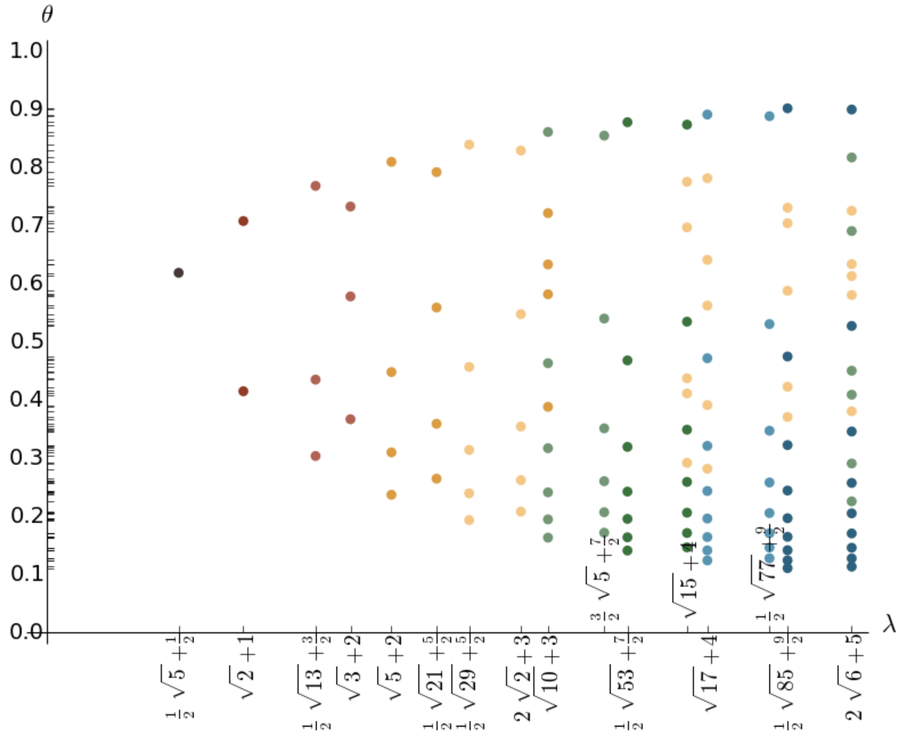
(Dazu m??ssen wir noch $\pi_1^{n-k}\pi_2^k \neq \pi_1^{n-j}\pi_2^j$ zeigen, und damit evtl sogar noch $\pi_1^{n-k}\pi_2^k\varepsilon\pi_1^{n-j}\pi_2^j \neq \pi_1^{n-i}\pi_2^i\varepsilon\pi_1^{n-l}\pi_2^l$ zeigen)

TODO Eindeutigkeit der EP-Darstellung der Matrizen: [2, Section 2.2] and [13, Ex. 9.4.4]

xxx Fenster bzw θ, siehe [34, Section 6.4], seine Kettenbruchentwicklung aus der PE-Darstellung ablesen. TODO Jetzt klar(?), Fenster direkt, Bedeutung des rot-angles $\alpha =$

$\frac{\theta}{1+\theta}$ in [34, Section 6.4.1]? (relative Häufigkeit? α als Kettenbruch ausdr??cken?). Experimentell: $\theta = [0; \overline{c_k, c_{k-1}, c_{k-2}, \ldots, c_1, c_0}]$ gibt $\alpha = [0; c_k + 1, \overline{c_{k-1}, c_{k-2}, \ldots, c_1, c_0, c_k}]$, Beweis wie in [43, Section 4]?

xxx Wie klein/gro?? kann λ sein bzgl der Zahl der Ps in M_σ?



Semi-experimentally:

- lower envelope by $P^n E$ which yields $\theta = [0; \overline{n}] = \frac{1}{2}(-n + \sqrt{n^2 + 4})$ (where $n \geq 1$; for discriminantes see sequence A087475 in OEIS) and $\lambda = [n; \overline{n}] = \frac{1}{2}(n + \sqrt{n^2 + 4})$
- upper envelope by $P^n EP$ which yields $\theta = [0; 1, \overline{n+1}] = \frac{1}{2(n+1)}(n - 1 + \sqrt{(n+1)^2 + 4})$ (where $n \geq 1$) and $\lambda = [n+1; \overline{n+1}] = \frac{1}{2}(n + 1 + \sqrt{(n+1)^2 + 4})$
- middle upper envelope (i.e., approaching $\frac{1}{2}$ from below) by $P^n EP^2$ which yields $\theta = [0; 2, \overline{n+2}] = \frac{1}{2(n+3)}(n - 2 + \sqrt{(n+2)^2 + 4})$ (where $n \geq 1$) and $\lambda = [n + 2; \overline{n+2}] = \frac{1}{2}(n + 2 + \sqrt{(n+2)^2 + 4})$
- middle lower envelope (i.e., approaching $\frac{1}{2}$ from above) by $P^n EPEP$ which yields $\theta = [0; 1, \overline{1, n+1}] = \frac{1}{2(2n+1)}(n - 1 + \sqrt{(n+3)^2 - 4})$ (where $n \geq 1$; discriminante given by sequence A028347 in OEIS) and $\lambda = [n + 2; \overline{1, n+1}] = \frac{1}{2}(n + 3 + \sqrt{(n+3)^2 - 4})$

TODO

Upper bound for λ given by τ^n, i.e., $(PE)^n$ (just a power of Fibonacci, so $\theta = \frac{1}{\tau}$ for all n)! Lower bound for λ given by the above cases, i.e., $\frac{1}{2}(n + \sqrt{n^2+4})$.

Continued fraction expansion for λ lines up with observation about conjugate matrices earlier (Remark 3.1) – but things are a bit more complicated, cyclic permutations are just one possibility to get same λ (however: see next section, cyclic permutations yields matrices of the same integral similarity class and thus with the same "limit translation module"). For a number N of Ps, the set of substitutions of the form $P^{n_1}EP^{n_2}E\ldots EP^{n_{k-1}}EP^{n_k}E$ with $N = n_1+n_2+\ldots+n_k$ yield how many different substitution factors?

N	1	2	3	4	5	6	7	8	9	10
A008965 in OEIS	1	2	3	5	7	13	19	35	59	107
A091696 in OEIS	1	2	3	5	7	12	17	29	45	77
# distinct λs	1	2	3	5	7	12	17	27	42	70

First difference found at $N = 6$: PEP^2EP^3E and PEP^3EP^2E. Similarly for $N = 7$: PEP^2EP^4E and PEP^4EP^2E, as well as $PEPEP^2EP^3E$ and $PEPEP^3EP^2E$ — so is this just A008965 modulo matrix transpose? No, this would be A091696! Minimal "different" examples for $N = 8$: $PEPEPEPEP^4E$ and $P^2EP^3EP^3E$ have $\lambda = 13+\sqrt{170} = [26;\overline{26}]$; $PEPEPEPEPEP^3E$ and $P^2EP^2EP^2EP^2E = (P^2E)^4$ have $\lambda = 17+12\sqrt{2} = [33;\overline{1,32}]$. Two $N = 10$ examples: $PEPEPEP^2EP^5E$ and $PEPEP^3EPEP^4E$ have the same $\lambda = 24+\sqrt{577} = [48;\overline{48}]$; or $PEPEP^2EP^2EP^4E$ and $PEP^2EPEP^3EP^3E$ with $\lambda = 30+\sqrt{901} = [60;\overline{60}]$. Above sequence not found in OEIS! Also clear: might get the same λ for different N (see figures, e.g., at $\lambda = 3+\sqrt{10}$, i.e.,P^6E vs. $PEPEP^2E$; note that difference doesn't have to be even, e.g., $PEPEPEPEP^2E$ vs. PEP^3EP^3E yield $\lambda = 8+\sqrt{65}$).

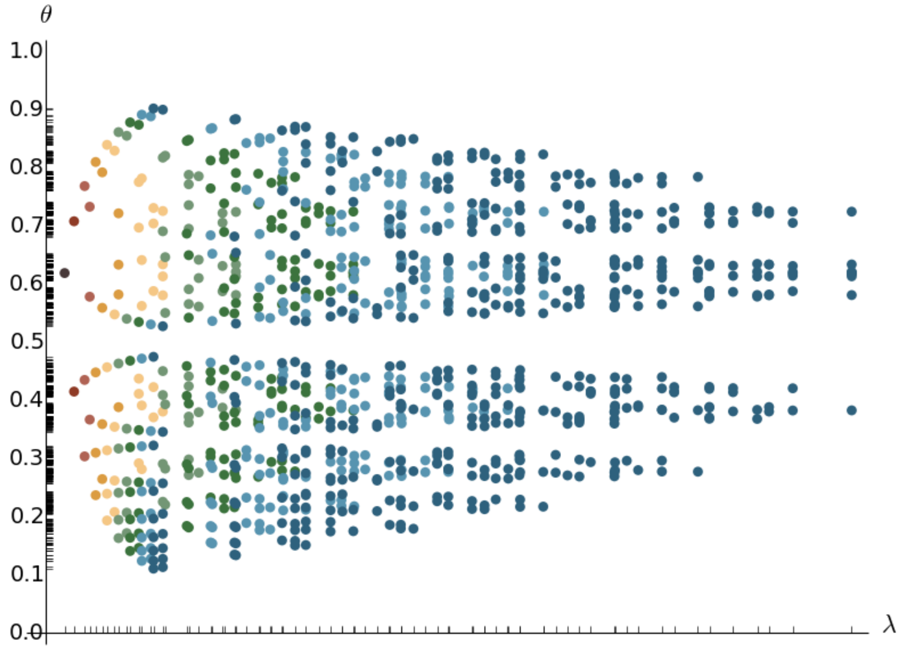
Note: In the above table, powers of simpler "substitutions" are contained. What if we make table removing these? So remove powers of substitutions and make a new table (TODO use results earlier to interpret this table – also, in view of next section, use only one representative per integral similarity class):

N	1	2	3	4	5	6	7	8	9	10
# overall substitutions up to N times Ps $2^{N+1}-N-2 =$ A000295 in OEIS	1	4	11	26	57	120	247	502	1013	2036
# distinct θs A119917 in OEIS	1	3	9	21	51	105	231	471	975	1965
# distinct θs without preperiod $\frac{1}{2}$(A119917 +1) = A085945 in OEIS	1	2	5	11	26	53	116	236	488	983
# distinct λs in all substitutions	1	3	6	10	16	25	37	58	86	133
# distinct λs in distinct substitutions	1	2	4	7	13	19	32	51	80	125

Note: For $N = 2$, substitutions include $PEPE = (PE)^2$ which has the same θ as PE – however, their λ will be different (one being the square of the other). However, the two silver mean substitutions PEP and P^2E will have the same λ, but different θ (first one with preperiod).

Observe self-similar structure of continued fractions here!

Next step: Investigate combinations, i.e., composition of substitutions! (Probably then/now trivial!) TODO JA, nun klar von Kettenbruchentwicklung!



7. Module Generated by the PF-Eigenvector

TODO reformulate the following – Limit Translation Module(?), compare [4, Section 5.1.2 & Example 5.3] and references therein (same argument because inflation factor is unit). So, are we calculating "equivalence classes" of the limit translation module (all members of the cycle of a PE-representation belong to the same class), or something like that? What does this mean for relationship between LTM and MLD ([4, Corollary 5.1]) and Theorem 4.3?

TODO [16, Theorem 1], also found in [40], says that this module generated by the components of the PF-Eigenvector is an ideal in the integral domain $\mathcal{O}_{\mathbb{Q}(\lambda)}$ (notation for integral domain vs. $\mathbb{Z}[\lambda]$!?!)

TODO also need [16, Theorem 2] – part A) (C)) $\Leftrightarrow$ B) tells us that integrally similar matrices generate modules/ideals belonging to the same ideal class

Two matrices A and A', say in $GL_2(\mathbb{Z})$ (although this holds more generally for matrices $M_n(\mathbb{Z})$ for any n), with the same characteristic polynomial and thus the same eigenvalues λ, λ', are *similar* to each other, i.e., there is a matrix $U \in GL_2(\mathbb{R})$, in fact, we can even choose $U \in GL_2(\mathbb{Q})$, such that $UAU^{-1} = A'$, compare [11, Theorem 1]. However, they might not be integrally similar, where we say that two matrices A and A' are *integrally similar* if there is a matrix $U \in GL_2(\mathbb{Z})$ such that $UAU^{-1} = A'$. Since this is an equivalence relation, one can define the *integral similarity class* of matrices in $M_2(\mathbb{Z})$ that are integrally similar to each other.

We look at an example: The matrices $A = EP^6 = \begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix}$ and $A' = EPEPEP^2 = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$ both have characteristic polynomial $x^2 - 6x - 1$ and thus eigenvalues $\lambda = 3 + \sqrt{10}$ and $\lambda' = 3 - \sqrt{10}$. Thus, they are similar, e.g., by the matrix $U = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in GL_2(\mathbb{Q})$. However, they are not integrally similar: Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $UA - A'U = 0$ leads to $c = \frac{a}{2} + \frac{b}{2}$ and $d = \frac{a}{2} - \frac{5b}{2}$. Then, the condition $\det U = \pm 1$, leads to the equation $(a-3b)^2 - 10b^2 = \pm 2$, i.e., a *Pell's equation* $x^2 - 10y^2 = \pm 2$ which has no integral solutions (since there is no solution of $x^2 \equiv \pm 2 \bmod 5$). So the two matrices A and A' belong to different similarity classes (two other representatives of these two classes can be found in [11, Example 11], where the intersting observation is made that, however, these two matrices are similar w.r.t a matrix $U \in GL_2(\mathbb{Z}[\sqrt{2}])$).

This example agrees with [16, Theorem 6] which states that two matrices in $GL_2(\mathbb{Z})$ are integrally similar iff their factorisation into elementary matrices E and P^c is just a cyclic reordering of each other. It also agrees with the *Latimer-MacDuffee-Taussky theorem*, see [22, 40], [28, Theorem III.13], [42, Theorem 5], [20, Lecture 2], [11, Theorem 3], which states that there is a one-to-one correspondence between the integral similarity classes of matrices $A \in M_n(\mathbb{Z})$ with irreducible monic polynomial $p(x) \in \mathbb{Z}[x]$ of degree n, and the ideal classes in $\mathbb{Z}[\lambda]$ (where $\lambda \in \mathbb{C}$ is a root of $p(x) = 0$). Indeed, the class number of $\mathbb{Q}(\sqrt{10})$ is 2, and two matrices are the representatives of the corresponding two integral similarity class.

Let us consider another example: Let $A = EPEP^2EP^3 = \begin{pmatrix} 10 & 3 \\ 7 & 2 \end{pmatrix}$ and $A' = EPEP^3EP^2 = \begin{pmatrix} 9 & 4 \\ 7 & 3 \end{pmatrix}$, both with eigenvalues $\lambda = 6 + \sqrt{37}$ and $\lambda' = 6 - \sqrt{37}$. While, for example, $U = \begin{pmatrix} 1 & -\frac{1}{7} \\ 0 & 1 \end{pmatrix}$ (noting that $\det U = 1$) shows that these two matrices are similar, again it follows from [16, Theorem 6] that these two matrices are not integrally similar. However, the class number of $\mathbb{Q}(\sqrt{37})$ is 1, compare sequence A003172 in OEIS (even its *narrow class number* is 1, see sequence A003655 in OEIS), thus there should be only one integral similarity class (respectively,even one *narrow integral similarity class* for which we require $U \in SL_2(\mathbb{Z})$, compare [11, Remark 9] TODO narrow class number [14, p. 180]). Note that A is integrally similar to the transpose $(A')^t$ of A', e.g., using $U = (EP)^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ in agreement with [16, Theorem 6]. However, a matrix needn't be integrally similar to its transpose as [11, Example 15] shows.

TODO comparing [6], [16] and our considerations: $\Delta = \gamma^2 \pm 4$, in particular, always $\Delta > 0$ in our case. Iteration in [6] has form $A \mapsto (EP^{-n})A(EP^{-n})^{-1} = (EP^{-n})A(P^nE)$ to create matrices of the form $(P^aE)(P^bE)(P^c \cdots E)(P^mE)$ and its cyclic permutations – at most first or first two steps not in the cycle depending on form $EP \cdots PE$, $EP \cdots EP$, $PE \cdots EP$ or $PE \cdots PE$. Compare with *standard* matrices, [16, Definition 4.1], which have the form $(EP^a)(EP^b)(E \cdots P^k)(EP^m)$.

As another test for this example, we look at [6]: The discriminate $\Delta = \mathrm{tr}^2 - 4 \cdot \det$ of the characteristic polynomial of A and A' is $\Delta = 148 = 4 \cdot 37 > 0$. Thus, [6, Theorem 4.3] can be used to check whether the two matrices fall into the same integral similarity class by applying a certain "reduction operator" successively to these matrices that will lead a unique cycle of *reduced* matrices in each integral similarity class. In this case with discriminate $\Delta > 0$ and not a complete square, a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called *reduced* if $c > 0$ and $|\sqrt{\Delta} - 2c| < (d - a) < \sqrt{\Delta}$. In fact, both matrices A and A' lead to a cycle of length 3 of reduced matrices, one cycle containing the transposed matrices in the other cycle (as one might expect from the calculations before), but not the same cycle (contrary to the expectation from the class number).

starting matrix	cycle of length 3		
$A = \begin{pmatrix} 10 & 3 \\ 7 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 7 \\ 3 & 10 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 4 & 11 \end{pmatrix}$
$A' = \begin{pmatrix} 9 & 4 \\ 7 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 7 \\ 4 & 9 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 3 & 11 \end{pmatrix}$

We note that the cycles correspond to the matrices PEP^2EP^3E, P^2EP^3EPE, P^3EPEP^2E respectively PEP^3EP^2E, P^3EP^2EPE, P^2EPEP^3E. With this trace and determinant, the algorithm following [6, Theorem 4.3] finds two more cycles, namely a cycle of length 1 consisting of the matrix $P^{12}E$, and another cycle of length 3 consisting of $PEPEP^5E$, PEP^5EPE, P^5EPEPE. Note that this is in accordance with [16, Theorem 6] and our results earlier (we got all possibilities with k even, i.e., determinant 1, and $\gamma = 12$). TODO for this example, also re-visit [11, Example 13]

TODO Something is not quite right with the Latimer-MacDuffy-Tausky theorem (or I mis-interpret it): Given that $\mathbb{Q}(\sqrt{37})$ has class number 1, there should only be one integral similarity class of matrices of that characteristic polynomial, but we got 4! Why??? TODO clear now: $\mathbb{Z}[\lambda]$ vs. integral domain $\mathcal{O}_{\mathbb{Q}(\lambda)}$ (notation?) – only if $\mathbb{Z}[\lambda] = \mathbb{Z} \oplus \lambda\mathbb{Z}$ is integral domain of number field, and thus $1, \lambda$ an integral basis (observe cases $D \equiv 1 \bmod 4$ vs. $D \equiv 2, 3 \bmod 4$ for quadratic number fields $\mathbb{Q}(\sqrt{D})$ for D square-free), is the number of matrix classes equal to ideal class number, otherwise additional cases (?related to discriminante of the number field?). Possible references: [28, Section III.16] **and** [14, 26, 27]**.**

TODO in $\mathbb{Q}(\sqrt{37})$ splitting/inert primes given via Legendre Symbol, note $\left(\frac{37}{3}\right) = \left(\frac{3}{37}\right)$. E.g., inert are $2, 5, 13, 17, 19, 23, 29, 31, \ldots$, splitting are $3, 7, 11, \ldots$. Will we get something out of this?

TODO look all examples in [11]

TODO maybe also consider lattices ("geometry of numbers") related to different module classes [27]

TODO Vergleich mit [36, Theorem 7 & Sections 8 & 9] (dies ist der $SL(2,\mathbb{Z})$ Fall? Narrow Ideal Class?); Note: $U = EPEP^{-1}$, $T = UEP$

TODO Further $SL(2,\mathbb{Z})$ results: See [17, Section 7.2, Theorems 7.14 & 7.18] (note: In $SL(2,\mathbb{Z})$ case if both eigenvalues are positive, det $= 1$). Also see [3] (or only Baake-Roberts?), [1], [41, Setion 5] and references therein, [42] and references therein

TODO Great reference: [44]!! (Can we also cite Gauss now??)

det $= -1$ case, h class number of $\mathbb{Q}([\gamma; \overline{\gamma}])$, H number of conjugacy classes in $GL(2,\mathbb{Z})$, compare [44, Corollary to Theorem 2] & [16, Theorem 6]

γ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h	1	1	1	1	1	2	1	1	2	2	1	1	1	1	3
H	1	1	1	2	1	2	1	2	2	2	2	4	1	3	3

e.g., $\{EP^4, EPEPEP\}$, $\{EP^8, EP^3EPEP\}$, $\{EP^{11}, EPEPEPEP\}$, $\{EP^{14}, EP^6EPEP, EP^2EP^2EP^2\}$

det $= 1$ case, h class number of $\mathbb{Q}([\gamma - 1; \overline{1, \gamma - 2}])$, H number of conj. classes in $GL(2,\mathbb{Z})$

γ	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
h	1	1	1	1	1	2	1	1	1	2	2	1	2	1	2
H	1	1	1	2	2	2	1	3	2	2	2	3	2	3	2

e.g., $\{EP^4EP, EP^2EP^2\}$, $\{EP^5EP, EPEPEPEP\}$, $\{EP^8EP, EP^4EP^2, EP^2EPEPEP\}$, $\{EP^9EP, EP^3EP^3\}$, $\{EP^{12}, EP^4EP^3, EP^6EP^2\}$, $\{EP^{14}EP, EP^7EP^2, EP^4EPEPEP\}$

Russian school?: [18, 19]

Appendix A. Sage code

TODO

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