

PARALLELOGRAM TILINGS, WORMS AND FINITE ORIENTATIONS

DIRK FRETTLÖH AND EDMUND HARRISS

ABSTRACT. This paper studies properties of tilings of the plane by parallelograms. In particular it is established that in parallelogram tilings using a finite number of shapes all tiles occur in only finitely many orientations.

1. INTRODUCTION

Parallelogram tilings, in particular rhomb tilings, are a fundamental family in the mathematical study of tilings, in both aperiodic order with its links to quasicrystals and in the study of random tilings. In addition they provide visual models for a wide range of mathematical structures.

The most famous aperiodic parallelogram tiling is certainly the Penrose tiling [Pen79, Gar77, HF] in the version with rhombic tiles. Many other aperiodic tilings use parallelograms as tiles, including the Ammann-Beenker tilings [AGS92, Bee82], or more generally, all canonical projection tilings [FHK02, HL04]. Further examples arise in the study of geometric representations of substitutive dynamical systems, see for instance [Pyt02]. Such models are of particular importance in the study of mathematical quasicrystals [SBGC84] and systems with long range aperiodic order [BM00, Moo97].

In random tilings parallelogram tilings are also of importance. Usually finite tilings are considered, lozenge tilings of polygons [KO07, BFRR11], or parallelotope tilings of zonotopes [DWMB05, CR06], but infinite random parallelogram tilings are also studied [BFR08]. In this paper we focus on infinite tilings of the plane, but some results are also valid and relevant for finite tilings, for instance Lemma 2.3 and Lemma 2.4.

Parallelogram tilings also turn up in the study of pseudo-line arrangements in combinatorics (for example [AS05, Goo80]), as a visualization of reduced words in Coxeter groups [Eln97] and, through random tilings and Dimer models, they link to a wide variety of physical models [Sti08]. The use of parallelogram tilings in a wide variety of mathematics is discussed further on Math Overflow [Spe].

Interestingly, there are aperiodic tilings of the plane with polygonal tiles where the tiles occur in an infinite number of orientations: The Pinwheel Tiling [Rad94] is an example of a tiling using only one kind of tile (a triangle with edge lengths 1, 2 and $\sqrt{5}$) where the tiles occur in infinitely many orientations throughout the tiling. A comprehensive study of this phenomenon is [Fre08]. There are also tilings using two kinds of quadrilaterals in which the tiles occur in infinitely many orientations [BFG07], see for instance the Kite Domino Tiling in the online resource [HF]. (Click “Kite-Domino” on the main page for information about the Kite Domino Tiling, and then “Infinite rotations” for a list of

other examples.) Our main result states that such a behavior is impossible if all tiles are parallelograms.

By definition, a parallelogram has two pairs of parallel edges. In a parallelogram tiling therefore there are natural lines of tiles linked each sharing a parallel edge with the next. These lines are called worms and were used by Conway in studying the Penrose tilings [GS87]. We will follow Conway to call these lines *worms*. Figure 3 shows three (finite pieces of) such worms. In this paper we consider the structure of these worms and use them to prove results on parallelogram tilings. In particular, our main result states that no parallelogram tiling using finitely many different parallelograms can have tiles in infinitely many orientations.

This paper is organized as follows: The first section defines the necessary terms and notations. The next section is dedicated to the proof of our main result Theorem 2.5. It contains several results which may be of interest on their own in the study of parallelogram tilings, for instance the Crossing Lemma 2.3 or the Travel Lemma 2.4. The last section contains some additional remarks.

A tiling \mathcal{T} of the plane \mathbb{R}^2 is a packing of \mathbb{R}^2 which is also a covering of \mathbb{R}^2 . Usually, the tiles are nice compact sets, and the tiling is a countable collection of tiles T_i : $\mathcal{T} = \{T_1, T_2, \dots\}$. The covering condition can then be formulated as $\bigcup_i T_i = \mathbb{R}^2$, and the packing condition as $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$ for $i \neq j$, where $\text{int}(A)$ denotes the interior of A .

The set of congruence classes of tiles in a tiling \mathcal{T} is called *protoset* of \mathcal{T} . Its elements are called *prototiles* of \mathcal{T} . A tiling is called *locally finite*, if each ball $B \subset \mathbb{R}^2$ intersects only a finite number of tiles of \mathcal{T} . A tiling \mathcal{T} is called *vertex-to-vertex*, if all prototiles are polygons, and for all tiles in \mathcal{T} holds: If $x \in \mathbb{R}^2$ is vertex of some tile $T \in \mathcal{T}$, then $x \in T' \in \mathcal{T}$ implies x is a vertex of T' .

General Assumption: Throughout the paper, only locally finite vertex-to-vertex tilings are considered. Moreover, the the prototiles will always be parallelograms. In particular, all tiles have nonempty interior.

Definition 1.1. Let T be a tile in a parallelogram tiling, and let e be one of its edges. There is a unique tile T_1 sharing the edge e with T . Let $e_1 \neq e$ denote the edge of T_1 parallel to e . There is a unique tile T_2 sharing this edge with T_1 . Repeating this yields a (one-sided) infinite sequence of tiles T, T_1, T_2, \dots uniquely given by T and e . Repeating this on the edge of T which is opposite of e yields a further (one-sided) infinite sequence of tiles T_{-1}, T_{-2}, \dots . In this way, T and e define a unique biinfinite sequence $W := \dots T_{-2}, T_{-1}, T, T_1, T_2, \dots$. We will call this sequence the *worm given by T and e* , and any such sequence just a *worm*. The intersection of two worms W, W' is defined as $W \cap W' = \{T \mid T \in W, T \in W'\}$.

Worms are also called *de Bruijn lines* [DWMB05] or *ribbons* [BFRR11]. Worms correspond to pseudo-line configurations [AS05, Goo80], and to grid lines used in the construction of canonical projection tilings by the multigrid method [GR86].

Definition 1.2. Let e be some edge in a parallelogram tiling. The set of all worms which are defined by edges parallel to e is called the *worm family (given by e)*.

2. PROOF OF THE MAIN RESULT

In this section we collect some basic results on worms, which will be used in the next section. The first result states that a worm cannot bend “too much”. In order to state this precisely, we define the open cone $C(x, \alpha)$ with axis $x \in \mathbb{R}^2$ and angle $\alpha \in [0, \pi[$ by $C(x, \alpha) := \{z \in \mathbb{R}^2 \mid |\angle(x, z)| < \alpha\}$, where $\angle(x, z) = \arccos \frac{\langle x, z \rangle}{\|x\| \|z\|}$ denotes the angle between x and z .

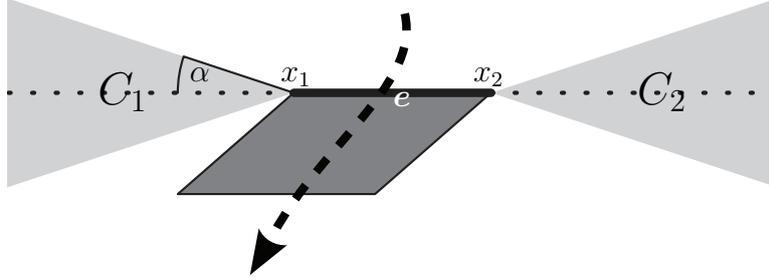


Figure 1. The worm defined by the edge e cannot intersect the open cones C_1 or C_2 .

Lemma 2.1 (Cone Lemma). *Let \mathcal{T} be some parallelogram tiling, with finite or infinite protoset. Denote the infimum of all interior angles of the prototiles by α . Let W be a worm given by some tile T and some edge e . Denote the two endpoints of e by x_1 and x_2 . If $\alpha > 0$ then W is contained in*

$$\mathbb{R}^2 \setminus (C_1 \cup C_2), \quad \text{where } C_1 := x_1 + C(x_1 - x_2, \alpha), C_2 := x_2 + C(x_2 - x_1, \alpha).$$

If $\alpha = 0$ then W is contained in $e \cup (\mathbb{R}^2 \setminus \text{aff}(e))$, where $\text{aff}(e)$ denotes the affine span of e .

Proof. By inspection of Figure 1 we observe that the worm defined by the edge e cannot intersect the cones (shaded) C_1 and C_2 , where $\alpha > 0$ is the smallest interior angle of the prototiles. Using the prototile with the smallest angle α alone, the worm can line up along the boundary line of one of the cones at best.

If $\alpha = 0$, the open cones are empty. Nevertheless, since any tile has nonempty interior, the particular tile in the worm W containing e as an edge has some positive interior angle. Then, by the same reasoning as above, W cannot touch the line $\text{aff}(e)$ apart from e . \square

The last lemma shows also that a worm can have no loop. We state this result as an additional lemma.

Lemma 2.2 (Loop Lemma). *A worm has no loop.*

Lemma 2.3 (Crossing Lemma). *Two worms cross at most once. Worms in the same family never cross.*

In other words: The intersection of any two worms is empty or consists of a single tile. The intersection of two different worms defined by parallel edges is empty.

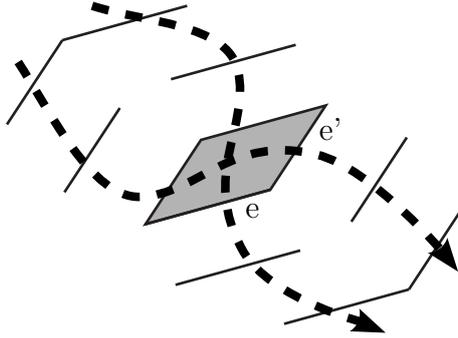


Figure 2. Two worms can cross at most once.

Proof. Consider two worms in distinct families, and put their defining edges together at a point (see Figure 2). Since the tiles are parallelograms, and the interior angles of parallelograms are less than π , the two worms can only cross in the direction where the edges form an angle less than π . After crossing this direction is reversed. Thus no further crossing is possible.

If the two worms are in the same family the crossing tile would have to have all four edges parallel, but no tile (with positive area) can have this property. \square

Lemma 2.4 (Travel Lemma). *Let \mathcal{T} be some parallelogram tiling, with finite or infinite protoiset. Denote the infimum of all interior angles of the prototiles by α . If $\alpha > 0$, then any two tiles S, T in \mathcal{T} can be connected by a finite sequence of tiles which are contained in a finite number of worms. Moreover, less than $\lceil 2\pi/\alpha \rceil$ different worms are needed. (Here $\lceil x \rceil$ denotes the smallest $k \in \mathbb{Z}$ such that $k \geq x$.)*

In plain words: If the prototiles are not arbitrarily thin, then we can always travel from tile S to tile T by walking on worms, with finitely “turns”; that is, with finitely many changes from one worm to another one. The proof stresses the fact that tiles cannot be arbitrarily thin.

Proof. We start from tile S , and choose W_0 , one of the two worms passing through S . As, by Lemma 2.2 W_0 is not a loop, an infinite number of worms cross W_0 . The tile T will either lie on one of these worms or between two of them. (The latter situation is indicated in Figure 3.)

If T lies on one of the worms we are done. Otherwise there are two worms, defined by adjacent tiles in W_0 , such that T lies between these two worms. Let W_1 be the worm further from S and S_1 be the tile given by the intersection of W_0 and W_1 . Again T either lies on a worm crossing W_1 or between two worms. If T does lie on such a worm we are done. Otherwise call the worm further from S_1 W_2 .

Using this method we obtain a sequence of worms W_0, W_1, \dots . Note that we cannot visit infinitely many tiles of the same worm: In this case we would have found infinitely many worms passing between S and T , which is impossible, since worms cannot be arbitrarily thin.

Thus we obtain a sequence of finite pieces of worms by our method. We proceed to show that we have to arrive at T in this way after finitely many turns.

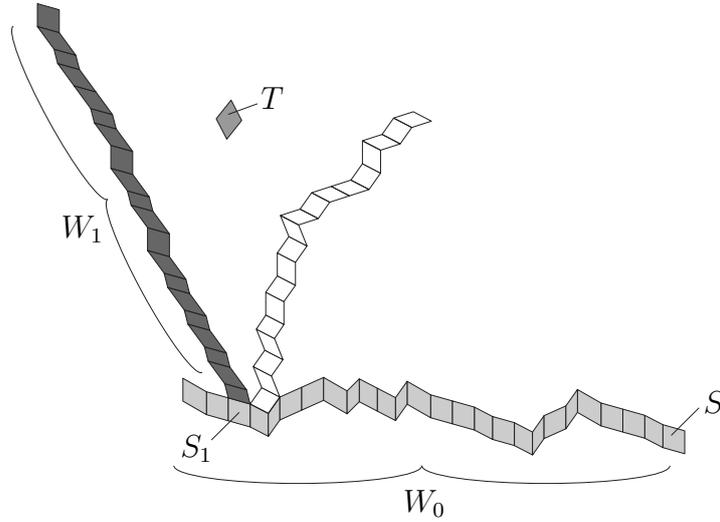


Figure 3. Starting from a tile S and worm W_0 , there are a set of worms crossing W_0 . A tile T that does not lie on one of these worms must lie between two. We label the worm further from S with W_1 .

By assumption we have $\alpha > 0$. Thus after at most $k := \lceil 2\pi/\alpha \rceil$ turns we completed a full turn by 2π or more. Note that, by construction, the tile T is always on the right hand side of the current worm, or always on the left hand side. Without loss of generality, let T be always on the right hand side of the current worm. Now let us travel further on the current worm W_k , say. Either S is contained in W_k (Case 1), or on the left hand side of it (Case 2), or on the right hand side of it (Case 3). In the first two cases we have completed a loop (Case 2 is indicated in Figure 4).

Assume we have completed a loop. By construction, T lies inside this loop. Since, by construction, our sequence of worms is not crossed by any worm defined by T , the two worms defined by T must be contained entirely inside this loop. Since worms are infinite, and tiles are not arbitrarily small, this is impossible.

It remains to consider Case 3: S is on the right hand side of our current worm W_k . There are two subcases: Either W_k reaches the first worm W_0 , defined by S . In this case we have closed a loop again, yielding a contradiction. Or W_k avoids the first worm W_0 . Now we utilize the Cone Lemma 2.1 to show that this case is impossible, too. Let us denote the tile where we entered W_k by S_k . By Lemma 2.1, the cone C_0 pointing outward our sequence (to the left) defined by S and the edge defining W_0 cannot contain tiles of W_0 . Similarly, the cone pointing outward defined by S_k and the edge defining W_k cannot contain tiles of W_k .

Either C_k intersects W_0 , then W_k intersects our worm sequence, closing a loop. Otherwise, since $k \geq 2\pi/\alpha$, the axis of C_k is tilted towards S , thus C_0 and C_k have non-empty intersection. (This situation is indicated in Figure 5.) Neither W_0 nor W_k can intersect

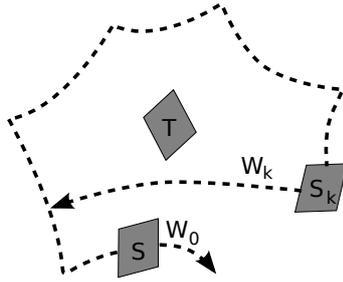


Figure 4. Case 2: The constructed worm sequence encircles T .

$C_0 \cap C_k$. W_0 cannot intersect C_0 , and W_k cannot intersect C_k , thus W_0 and W_k have to intersect each other. This closes a loop, which yields a contradiction.

Thus our constructed worm sequence cannot avoid T . Thus T is reached from S by using (finitely many) tiles contained in finitely many (k or less) worms. \square

Theorem 2.5. *Let \mathcal{T} be a parallelogram tiling with a finite protoset. Then each prototile occurs in a finite number of orientations in \mathcal{T} . Moreover, the number of orientations of each prototile is bounded by a common constant.*

Proof. This is now an immediate consequence of Lemma 2.4: Let α denote the minimal interior angle in the prototiles. Fix some tile S in the tiling. Any other tile T of the same shape can be reached from S by a sequence of worms with at most $k = \lceil 2\pi/\alpha \rceil$ turns. Let there be m prototiles altogether. If we can connect S with T without any turn at all (i.e., S and T are contained in the same worm) there are 2 possibilities how T is oriented with respect to S . (T can be a translated copy of S or a reflected copy of S , yielding two different orientations unless S and T are squares.) If we can connect S with T using one turn, there are $2m$ possibilities how T is oriented with respect to S : m possibilities arising from the choice of the prototile defining the turn, and two possibilities for any such choice (left-handed or right-handed). Analogously, for any turn there are $2m$ possibilities of changing the orientation. Thus there are $(2m)^\ell$ possibilities if we need exactly ℓ turns. Altogether, this leaves

$$N := 2 + 2m + (2m)^2 + \cdots + (2m)^k = 1 + \sum_{n=0}^k (2m)^n = \frac{1}{2m-1} ((2m)^{k+1} + 2m - 2)$$

as an upper bound for the number of different orientations of T with respect to S . This is true for any tile T congruent to S . Thus there are at most N^2 possibilities for the orientation of any two congruent tiles T, T' to each other. \square

At this point one may ask how the result generalizes to prototiles being centrally symmetric convex polygons.

Corollary 2.6. *Let \mathcal{T} be a tiling with a finite protoset, and let all prototiles be centrally symmetric convex polygons. Then each prototile occurs in a finite number of orientations in \mathcal{T} .*

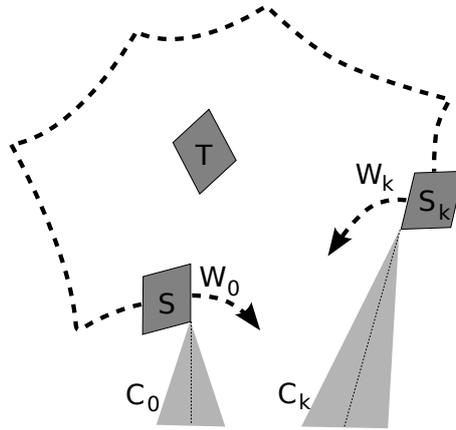


Figure 5. Case 3: The constructed worm sequence does not entirely encircle T , but nevertheless W_0 and W_k have to cross (since the cones C_0 and C_k are tilted towards each other, W_0 cannot intersect C_0 , and W_k cannot intersect C_k).

Proof. It is known that each centrally symmetric convex polygon can be tiled by parallelograms, see [KaS92], Theorem 1. Any tiling \mathcal{T} by centrally symmetric convex polygons with tiles in infinitely many orientations could be refined into a tiling by parallelograms in infinitely many orientations, in contradiction to Theorem 2.5. \square

3. REMARKS

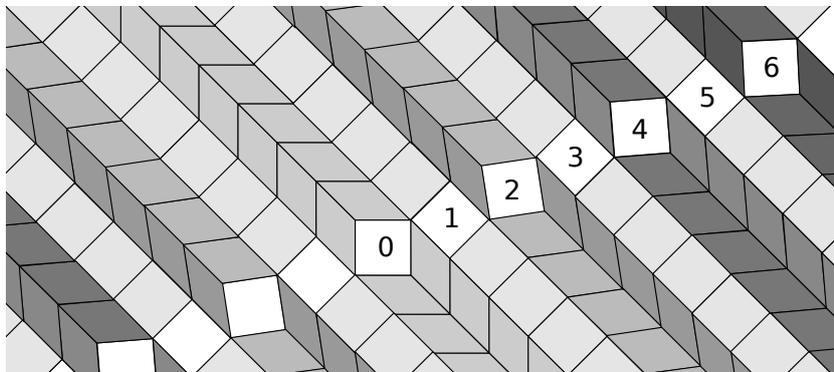


Figure 6. This patch illustrates how a tile (in this case a square) can lie in infinitely many orientations, if we allow infinitely many prototiles.

The tiling indicated in Figure 6 demonstrates that Theorem 2.5 is sharp, in the following sense: it becomes false if we allow an infinite number of prototiles, even if the minimal interior angle α is bounded away from 0. In this example, the squares labelled $1, 3, 5, \dots$ are rotated through $\frac{\pi}{4}$ with respect to the square labelled 0 (compare Figure 6). The squares with even label $i \neq 0$ are rotated through $\frac{\pi}{10i}$ with respect to the square labelled 0. Thus these squares occur in infinitely many orientations in the tiling.

Nevertheless, the Travel Lemma still holds in this example: The minimum interior angle is $\alpha = \frac{\pi}{4} - \frac{\pi}{10}$ (occurring in the thinner rhomb that touches square number 2).

Thus any tile can be connected to the square 0 by a (finite) sequence of tiles which is contained in a finite number of worms.

The upper bound obtained in the proof of Theorem 2.5 is not optimal. For instance, the angle changes induced by turns are not always independent: the pair of turns induced by a tile of type 1 (right-handed, say) followed by a turn induced by a tile of type 2 (right-handed) yields the same relative orientation as the pair of turns induced by a tile of type 2 (right-handed) followed by a turn induced by a tile of type 1 (right-handed). However, we don't see how to bring this upper bound below $O((2m)^k)$.

Throughout the paper we made the general assumption that all tilings are vertex-to-vertex, and that all tilings are locally finite. The requirement of local finiteness is essential, otherwise the Travel Lemma would become false: We can no longer guarantee that any two tiles can be connected by a finite sequence of tiles at all.

The requirement that the tilings are vertex-to-vertex allows for the definition of worms. However, if the tilings are locally finite, but not vertex-to-vertex, then we can dissect each parallelogram into finitely many parallelograms such that the new tiling is vertex-to-vertex. Thus all results apply also to locally finite parallelogram tilings which are not vertex-to-vertex.

One may ask whether Theorem 2.5 can be generalized to arbitrary centrally symmetric polygons. By the results in [KaS92] and [K93], a centrally symmetric polygon can be tiled by parallelograms if and only if it satisfies certain technical conditions: the “balance condition” and the “convex crossing condition”, see Theorem 2 in [KaS92]. Thus Theorem 2.5 generalizes to any tiling by prototiles fulfilling these conditions, along the lines of the proof of Corollary 2.6. A simple example of a centrally symmetric polygon which does not fulfil these conditions — thus can not be tiled by parallelograms — is a bow-tie (or hour-glass) shaped centrally symmetric non-convex hexagon. For such tiles the methods of this paper break down: The Cone Lemma is not longer true in this case, thus the Loop Lemma cannot be derived from it. Also the Crossing Lemma is false for such tiles: it is easy to construct counterexamples to the Crossing Lemma using the mentioned bow-tie shaped hexagons plus some parallelograms. This leads us to the formulation of the following problem.

Problem. Is there a locally finite tiling by centrally symmetric polygons where the tiles occur in an infinite number of orientations?

Equivalent results to those proved in this paper should apply to tilings of \mathbb{R}^d ($d \geq 2$) by parallelotopes. In order to generalize Lemma 2.4, however, we need to enclose some tile T with a finite collection of tiles. This would require further objects. In addition to the pseudo-lines for worms, one must also include pseudo-planes (and pseudo-hyperplanes) made up of linked parallelotopes sharing a parallel edge in common.

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INSTITUT FÜR MATHEMATIK, FU BERLIN

E-mail address: frettloe@mi.fu-berlin.de

URL: <http://page.mi.fu-berlin.de/frettloe/>

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS

E-mail address: edmund.harriss@mathematicians.org.uk

URL: <http://www.mathematicians.org.uk/eoh>