A RESULT ON PERFECT SYMMETRIES FOR SUBSTITUTION TILINGS

D. FRETTLÖH, UNIV. BIELEFELD

Definition. If σ is a primitive tile substitution, $\mathbb{X}(\sigma)$ denotes the hull of σ (in the appropriate top, for instance local rubber top). $\mathbb{X}(\sigma)$ *aperiodic*, if each member of $\mathbb{X}(\sigma)$ is nonperiodic.

Supertile means a patch $\sigma(T)$, or more general, $\sigma^k(T)$, where T is some tile in $\mathcal{T} \in \mathbb{X}(\sigma)$. More precisely, the latter is called k-th order supertile. Edge or vertex of a supertile means some edge resp. vertex of the union of the tiles in $\sigma^k(T)$.

Here: substitution always selfsimilar. That is, the supertile $\sigma(T)$ is congruent to λT, where λ is the inflation factor.

Here: always tilings in $\mathbb{R}^2$. Let R denote the rotation through π about the origin.

Lemma 1. Let σ be a primitive tile substitution with the unique decomposition property, and let $\mathbb{X}(\sigma)$ be aperiodic. Let $\mathcal{T} \in \mathbb{X}(\sigma)$ such that $R(\mathcal{T}) = \mathcal{T}$. Then, for all $k \in \mathbb{Z}$ holds $R(\sigma^k(\mathcal{T})) = \sigma^k(\mathcal{T})$.

Proof. The claim is immediate for $k \geq 0$. (For $k = 0$ it is an assumption, and since two equal tiles/patches stay equal under k-th substitution, it is true for all $k \geq 0$.) So, let us assume the claim is wrong for $k = -1$. Then, $R(\mathcal{T}) = \mathcal{T}$, but the unique tiling $\sigma^{-1}(\mathcal{T})$ is not R-symmetric. But then some symmetric patch P in $\mathcal{T}$ corresponds to a non-symmetric patch in $\sigma^{-1}(\mathcal{T})$. Thus P can be desubstituted in two ways: $\sigma^{-1}(P) \neq \sigma^{-1}(R(P))$. Let P be the largest such patch. Either P is finite, then no local information can tell how to desubstitute P uniquely (since the rest $\mathcal{T} \setminus P$ is symmetric). Or P is infinite, then again, no local information tells how to desubstitute P. □

Lemma 2. If 0 is contained in the interior of some tile T in a primitive substitution tiling $\mathcal{T}$, and if $R(\mathcal{T}) = \mathcal{T}$, then $\mathcal{T}$ is determined uniquely by the sequence of types of the k-th order supertiles containing 0.

Proof. (Sketch) By selfsimilarity, 0 is not a vertex of any supertile. If the sequence of supertiles $\sigma(T), \sigma^2(T'), \ldots$ containing 0 fail to cover the entire plane, there is some point x which is not contained in the union U of the supertiles. By $R(\mathcal{T}) = \mathcal{T}$, and by Lemma 1, $-x$ is also not contained in U. This means that all k-th order supertiles having 0 as their symmetry centres, do not contain x and $-x$. Contradiction. □

Theorem. Let σ be a primitive tile substitution with the unique decomposition property, and let $\mathbb{X}(\sigma)$ be aperiodic and FLC. Then there are only finitely many elements of $\mathbb{X}(\sigma)$ which are invariant under a rotation by π about the origin.

Proof. Since R fixes the entire tiling $\mathcal{T}$, R fixes in particular the patch $P_0 = \{T \in \mathcal{T} \mid 0 \in T\}$. First, let us assume that 0 is contained in the interior of some tile T (hence $P_0 = \{T\}$). Then, since $R(T) = T$, 0 is exactly the (unique) centre of symmetry of T. By the Lemma 1, 0 is also the unique centre of each supertile containing 0. Let there be m different tile types, and let T be of type 1.

Case 1: There is more than one type of supertile containing a type 1 tile in its centre.

Case 1.1: Say, these are of type 1 and 2. Then there has to be a third supertile type containing a tile of type 2 in its centre, say, type 3. Thus we need a fourth supertile type, containing a tile of type 3 in its centre, and so on. Contradiction (at stage m, by the pigeon hole principle).

Case 1.2: Say, these are of type 2 and 3. Contradiction, analogously to the last case.

Case 2: There is only one type of supertile $\sigma(S)$ containing a type 1 tile in its centre.

Case 2.1: This supertile is of type 1. Then, the next order supertile $\sigma^2(S')$ has to be of type 1, too, and the same is true for all k-order supertiles: all are of type 1. By Lemma 2, this yields a unique tiling $\mathcal{T}$.

Case 2.2: This supertile is of type 2. Now, either there is more than one supertile containing a tile of type 2 in its centre, and we are in Case 1. Or there is exactly one supertile containing a tile of type 2 in its centre. It may be of type 1 (then we have a loop 1 2 1 2 ...), or of type 3. Proceeding in this manner, we will finally get into some loop 12...n of length at most m. By Lemma 2, this yields at most m different tilings $\mathcal{T}$ with $R(\mathcal{T}) = \mathcal{T}$.

Now, assume that 0 is not contained in the interior of some tile, but in the interior of some supertile. Then, by the same arguments, 0 has to be the centre of this supertile, and the centre of all higher order supertiles containing 0; and again, this yields only finitely many (at most m) different tilings $\mathcal{T}$ with $R(\mathcal{T}) = \mathcal{T}$.

The remaining possibilities we have to consider is when 0 lies on the boundary of kth order supertiles on any level k. This means that in 0 two or more supertiles are meeting. If more than two are meeting on each level, then 0 is a vertex on each level, and the substitution of the vertex constellation $P_0^{(-k)} = \{T \in \sigma^{-k}(\mathcal{T}) \,|\, 0 \in T\}$ is the vertex constellation $P_0^{(-k+1)} = \sigma(P_0^{(-k)}) = \{T \in \sigma^{-k+1}(\mathcal{T}) \,|\, 0 \in T\}$. Similar as in Lemma 2, the sequence of super-vertex constellations $P_0, P_0^{(1)}, P_0^{(2)}, \ldots$ determines the tiling uniquely. By selfsimilarity, vertices substitute to vertices. By FLC, there are only finitely many vertex constellations, say, n. Thus, this sequence is always ultimately periodic, with period at most n. This yields at most n different tilings $\mathcal{T}$ with $R(\mathcal{T}) = \mathcal{T}$.

The last possibility to consider is that 0 lies on the boundary of exactly two supertiles, from some level k on.

Consider again $P_0 = \{T \in \mathcal{T} \,|\, 0 \in T\}$. Now, P_0 contains exactly two tiles of the same type, and 0 is the centre of symmetry of P_0. By FLC, there are again only finitely many possibilities for P_0. Moreover, 0 is also the centre of symmetry for each constellation of the two supertiles $P_0^{(k)} = \{\sigma^k(T) \,|\, 0 \in \sigma^k(T)\}$. By considering all the centres of the P_0s as artificial vertices, we are in the situation of the last case: 0 is a vertex on each level, vertices substitute to vertices, there are only finitely many of them, say, n. Thus the possible sequence of super-vertices is ultimately periodic, yielding finitely many tilings.

□

Remarks: The proof generalises immediately to any rotation R about the origin. However, it does not work for mirror reflections: One can construct infinitely many pinwheel tilings which are mirror symmetric.

The proof indicates that all R-symmetric tilings are of the form $\bigcup_{k\geq 0} \sigma^{nk}(P)$ for some symmetric legal patch P in $\mathcal{T}$.

Let a be the number of R-symmetric tiles, b be the number of R-symmetric vertex constellations and c be the number of R-symmetric pairs of adjacent tiles, then $a+b+c$ is an upper bound for the number of R-symmetric tilings in $\mathbb{X}(\sigma)$.

A more detailed study how vertices substitute to vertices etc. yields the exact number of symmetric tilings. For instance, there are exactly four pinwheel tilings which are R-symmetric: One with vertex constellation V7 (Fig. 6 in [1]) in its centre, (the tiling being $\mathcal{T}_7 := \bigcup_{k\geq 0} \sigma^{2k}(V7)$, where $sigma^2(\mathcal{T}_7) = \mathcal{T}_7$) one with vertex constellation 11 in its centre ($\mathcal{T}_{11} := \sigma(\mathcal{T}_7)$, also fixed by σ^2), one with a domino D in its centre (see Figure 4 in [1], it is $\mathcal{T}_D = \bigcup_{k\geq 0} \sigma^{2k}(D)$), and its substitution $\mathcal{T}_s := \sigma(\mathcal{T}_D)$. Again, we have $\mathcal{T}_D = \sigma(\mathcal{T}_s)$ and vice versa, thus $\sigma^2(T_D) = \mathcal{T}_D$ and $\sigma^2(T_s) = \mathcal{T}_s$.

References

[1] M. Baake, D. Frettlh, U. Grimm: A radial analogue of Poisson's summation formula with applications to powder diffraction and pinwheel patterns, *J. Geom. Phys.* 57 (2007) 1331-1343.