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Third-order Fibonacci sequence associated to a heptagonal quasiperiodic tiling of the plane

B.J.O. Franco

Departamento de Física, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, C.P. 702, CEP 30161, Belo Horizonte, Brazil

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A heptagonal tiling of the plane is proposed. Using five basic forms a pattern which presents self-similarity is obtained. This pattern exhibits rotational symmetry, and no translational invariance.

In 1974 Penrose found a pair of "darts" and "kites" that tile the plane in a quasiperiodic way (pentagonal quasiperiodic tiling) and gave the corresponding inflation rule [1]: if we take such a tiling and bisect each dart symmetrically with a straight line segment, the resulting half-darts and kites can be collected together to make darts and kites on a slightly larger scale. Two half-darts and one kite make a large dart; two half-darts and two kites make a large kite. His scheme uses as side ratio the golden mean and angles which are multiples of $\pi/5$. A radial direction on the Penrose tiling is associated to a second order Fibonacci sequence defined by

$$S_n = S_{n-1} + S_{n-2} ,$$

$$A \to AB \text{ and } B \to A , \tag{1}$$

where S_n is the number of terms of the nth generation. A growth sequence after six steps would be ABAABABAABAAB.

The sequence of intervals A and B is characterized by the substitution law $r_i = M_{ij} r_j$, where the transformation matrix M is given by [2]

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

and its secular equation is $x^2 - x - 1 = 0$ which gives the golden mean τ. Recently, a power recurrence relation which generates numbers in base 10 that converted to base 2 gives the inflation rule for golden mean, silver mean, lead mean, etc., has been found [3] [#1].

Sometimes nature presents occurrence of heptagonal symmetries as can be seen in figs. 1a and 1b. In this Letter we report a heptagonal quasiperiodic tiling which is associated to the third-order Fibonacci sequence.

A third-order Fibonacci sequence is defined by [4]

$$S_n = 2S_{n-1} + S_{n-2} - S_{n-3} \tag{3}$$

or

$$S_n = S_{n-1} + 2S_{n-2} - S_{n-3} . \tag{4}$$

The sequence defined by eq. (3) can be generated by the inflation transformations A→C, B→CB and C→CBA. The corresponding transformation matrix is

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{5}$$

and its secular equation is

$$x^3 - 2x^2 - x + 1 = 0 , \tag{6}$$

[#1] In ref. [3] the mentioned power recurrence relation to generate a lattice of spins and to study the magnetic susceptibility as a function of wave vector and temperature for gold mean, silver mean and lead mean is used.

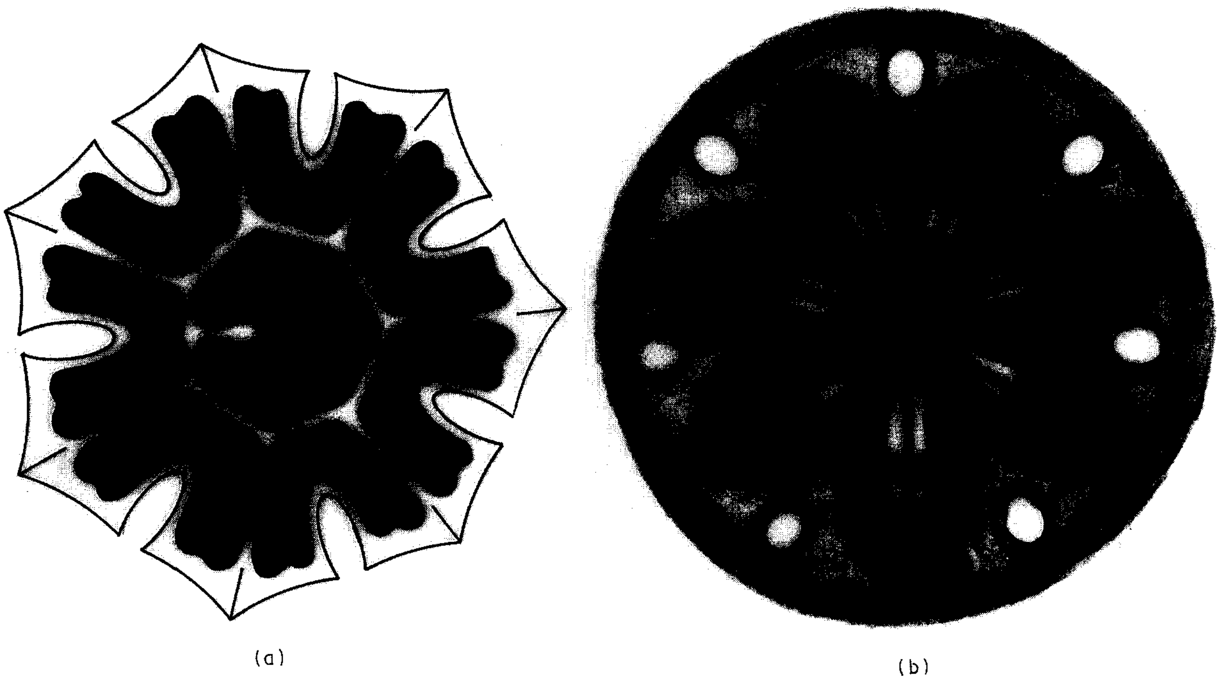

Fig. 1. (a) Seaweed: family Hidrodictyaceae, genera *Pediastrum*. (b) Sea animal: class Ascidiacea, category Stolidobranchiatra, genera *Botryllus*.

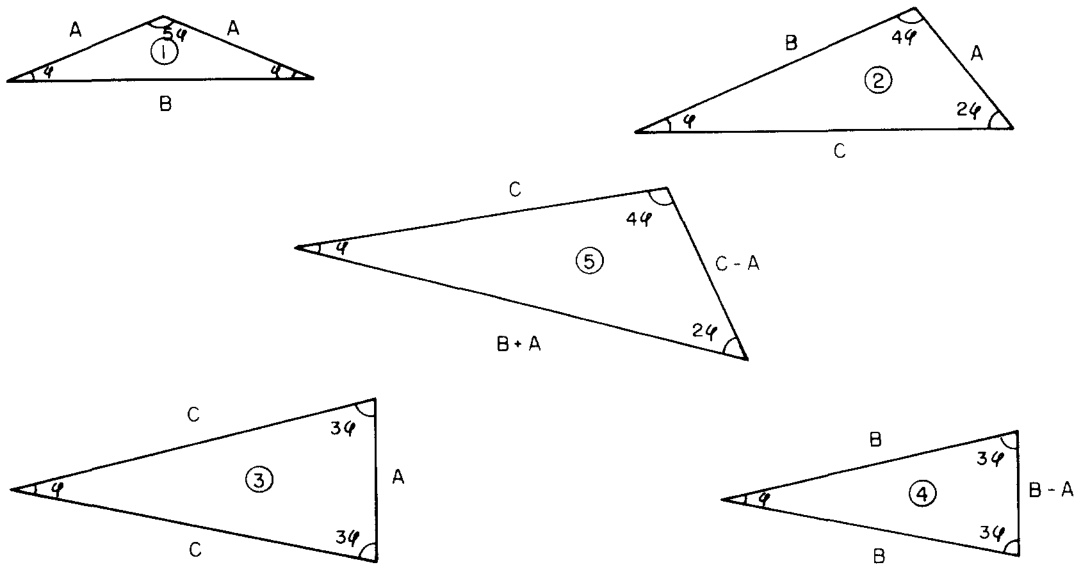

Fig. 2. Fundamental shapes of our tiling ($A=1.000$, $B\simeq 1.802$, $C\simeq 2.247$ and $\varphi=\pi/7$).

with the positive solutions $1/\alpha$ and β, where

$$\alpha \simeq 1.802, \quad \beta \simeq 2.247 . \tag{7}$$

Given a regular heptagon of side 1 we let α and β be the distinct lengths of the diagonals. ($\alpha = 2\cos\varphi$ and $\beta = 2\cos^2\varphi + \cos 2\varphi$, where $\varphi = \pi/7$). By using cyclotomic polynomials [5] we can easily show that α and β are roots of eq. (6).

In fig. 2 we show five shapes which are basic for the constructions that will appear in this paper. They are three isosceles triangles and two scalene triangles. Here we make the following correspondence for fig. 2: $\alpha \to$ B, $\beta \to$ C, $1 \to$ A. Angles are multiples of φ. These five shapes can be combined to form "darts" and "kites" as we show in fig. 3. They will be useful for the construction of the heptagonal quasi-periodic tiling. Those are some possible combinations of the five shapes considered. The numbers indicated in fig. 3 are the same as in fig. 2. If we exchange some internal pieces of the darts or kites a new dart or kite is obtained. In kite 3a we may exchange the adjacent pair 2–2 and the adjacent pair 3–3. For dart 3c the adjacent pair 1–1 may be exchanged by the adjacent pair 2–2 and for dart 3d the non-adjacent pair 2–2 may be exchanged by the adjacent pair 3–3. Besides the darts and kites of fig. 3 we will use parts of them, e.g., the two adjacent forms 2–2 of fig. 3d.

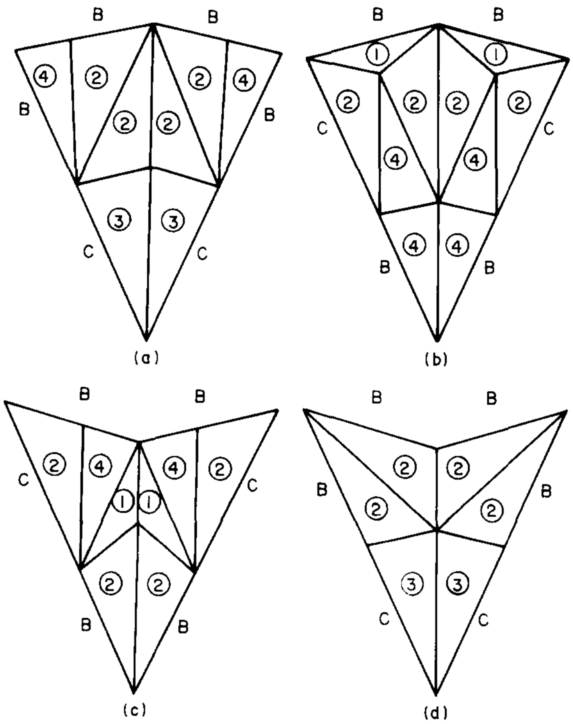

Fig. 3. "Kites" and "darts" obtained by a convenient combination of the shapes of fig. 2.

Next we display four generations of the inflation transformation given by eq. (3). They will be used in figs. 4 and 5. Beginning with CB, we get
1st generation:

CB ,

2nd generation:

CBACB ,

3rd generation:

CBACBCCBACB ,

4th generation:

CBACBCCBACBCBACBACBCCBACB .

The central fourteen sides regular polygon of fig. 4 has radius CB(C+B) and uses seven kites as in fig. 3a. This figure is surrounded by other seven equal fourteen sides regular polygons whose centers are located at the distance CBACBC from the origin O. The

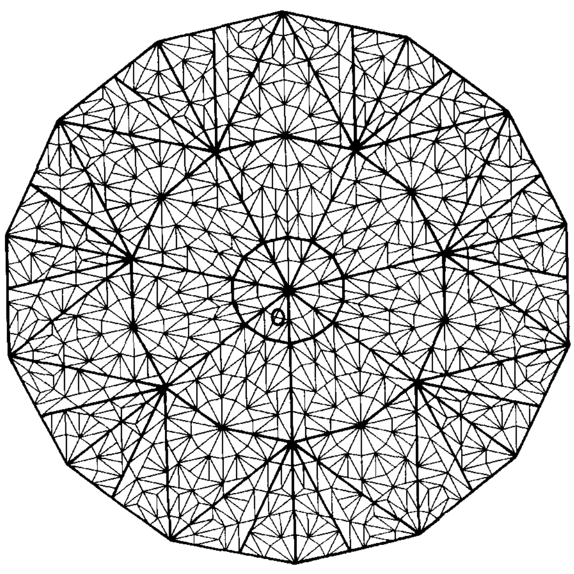

Fig. 4. The rule to inflate the infinite pattern.

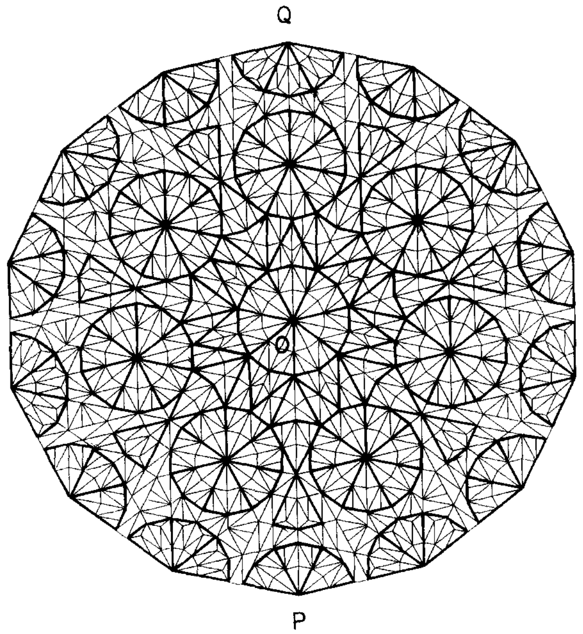

Fig. 5. The same as fig. 4 with details of constructions.

whole figure is a new fourteen sides regular polygon with radius equal to CBACBCCBACB (third generation). This fourteen sides regular polygon can be considered as a new central fourteen sides regular polygon surrounded by a seven equal fourteen sides regular polygon, and so on. If we observe fig. 4 we notice that the internal structure of the central fourteen sides regular polygon is contained in the whole figure. The internal structure of this last one in its turn will be contained in a larger fourteen sides regular polygon in which fig. 4 will be the central fourteen sides regular polygon, and so on. Thus, the pattern of fig. 4 presents self-similarity and the scaling factor is 5.049. We notice that it has rotational symmetry but it is not translationally invariant. If we consider the heptagonal direction PQ in fig. 5, dart 3c points off the origin and dart 3d to the origin. Kites 3b point to the origin if they are along PO and off the origin when they are along OP. Kite 3a besides forming a fourteen sides regular polygon of radius CB appears to be pointing off the origin and to lie along OP. We must note that the radial directions separated by φ do not correspond (in the inflation transformation such radial directions correspond to walks in opposite directions starting at the central fourteen sides regular polygon). So, we have only seven-fold symmetry.

The study of the structure given by fig. 4 by light diffraction is in progress.

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