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A Third-Order Fibonacci Sequence Associated to a Heptagonal Quasiperiodic Tiling of the Plane

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Since the experiment of Shechtman et al. [1], which shows the existence of a system with icosahedral symmetry, quasiperiodic systems have been intensively investigated. One of the quasiperiodic lattices is the so-called second-order Fibonacci chain [2], which can be seen as the projection along the icosahedral direction of a three-dimensional pattern with fivefold symmetry [3]. The third-order Fibonacci chain was considered recently by Terauchi et al. [4] who studied the diffraction pattern of this lattice grown by molecular beam epitaxy (MBE) of semiconducting layers A = AlAs, B = $Al_{0.5}Ga_{0.5}As$, and C = GaAs. Another work that used a third-order Fibonacci sequence was recently made [5]. In [5], hereafter named TI (Tiling I), we proposed a heptagonal quasiperiodic tiling of the plane associated to the third-order Fibonacci sequence $S_n = 2S_{n-1} + S_{n-2} - S_{n-3}$. Such tiling presents self-similarity and rotational symmetry but is not translationally invariant.

In this note we take a third-order Fibonacci sequence defined as [4]

$$S_n = S_{n-1} + 2S_{n-2} - S_{n-3} \tag{1}$$

which can be generated by

$$A \to B\,, \qquad B \to AC\,, \qquad C \to CB\,. \tag{2}$$

The corresponding transformation matrix is

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{3}$$

with the secular equation

$$x^3 - x^2 - 2x + 1 = 0\,. \tag{4}$$

The positive solutions of the above equation are α and $1/\beta$,

$$\alpha = \frac{2\sqrt{7}}{3}\cos\left(\frac{\theta}{3}\right) + \frac{1}{3} \tag{5}$$

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and

$$\beta = 2\frac{\sqrt{7}}{3}\cos\left(\frac{\psi}{3}\right) + \frac{2}{3}, \tag{6}$$

where

$$\cos\theta = -\cos\psi = \frac{-1}{2\sqrt{7}}.$$

α and β satisfy the relation

$$\alpha\beta = \alpha + \beta$$

with the following approximate values: $\alpha = 1.802$ and $\beta = 2.247$. As we said in TI, α and β are the distinct lengths of the diagonals of a heptagon (regular heptagon) of unitary side ($\alpha = 2\cos\Phi$ and $\beta = 2\cos^2\Phi + \cos 2\Phi$), where $\Phi = \pi/7$.

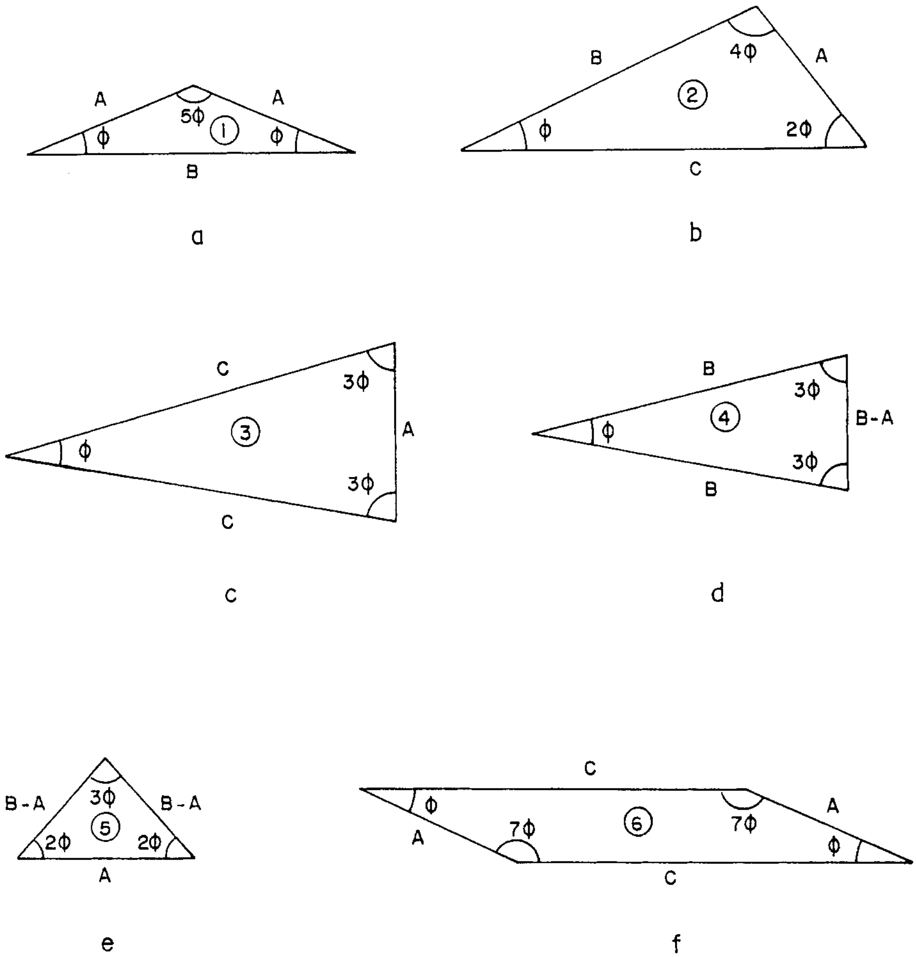

Fig. 1. Fundamental shapes of our tiling (A = 1.000, B = 1.802, C = 2.247, and $\Phi = \pi/7$)

In Fig. 1 we show six shapes which are used in the geometric constructions that appear in this note. Four of them were used in TI. They are four isosceles triangles, one scalene triangle, and one parallelogram. Angles are multiples of φ. We must note that the parallelogram (Fig. 1f) cannot be split in two of the other triangles because the angles of the resulting triangles are not multiples of Φ. We have the following correspondence for the shapes of Fig. 1:

$$\alpha \rightarrow \mathrm{B}, \qquad \beta \rightarrow \mathrm{C}, \qquad 1.000 \rightarrow \mathrm{A}. \tag{7}$$

Fig. 2 shows darts and kites that use shapes of Fig. 1. These are some possible combinations of the shapes of Fig. 1. The numbers idicated in Fig. 2 are the same as in Fig. 1. Internal parts of darts and kites of Fig. 2 will also be used in the infinite pattern of this note besides individual shapes of Fig. 1. Dart 2b was used in TI. Inflation transformations for (1) if we begin with C, are

1st generation: C
2nd generation: CB
3rd generation: CBAC
4th generation: CBACBCB (8)
5th generation: CBACBCBACCBAC
....

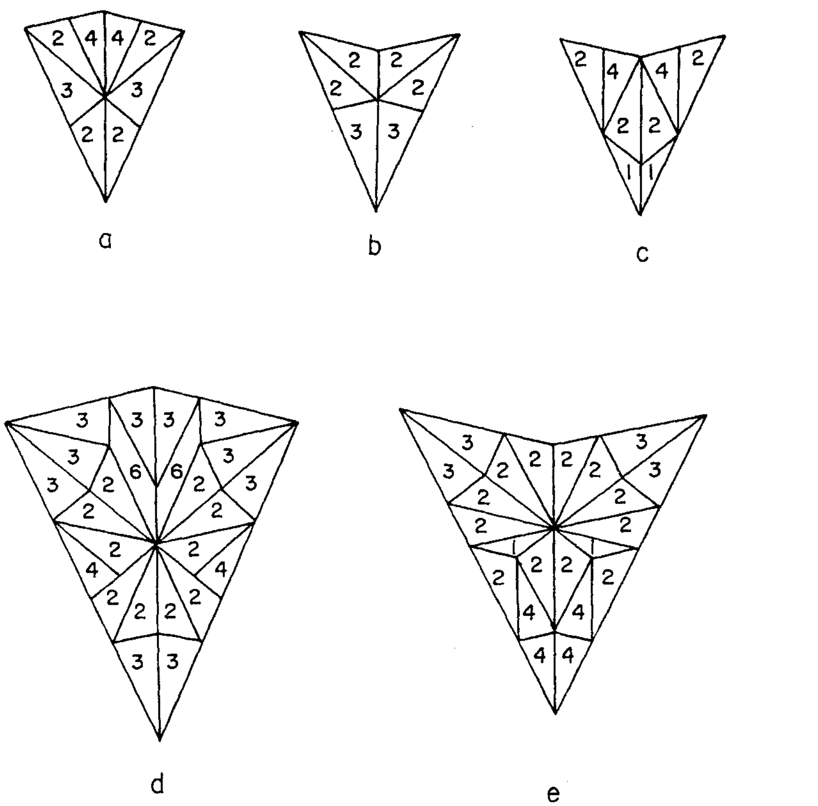

Fig. 2. Darts and kites obtained by convenient combination of the shapes of Fig. 1

Fig. 3 presents the infinite pattern corresponding to (1). It is a 14-gon with radius CBACBCBACCBAC (fifth generation in (8)). The central 14-gon of this pattern presents radius C (first generation in (8)) and uses fourteen times one of the shapes given by Fig. 1c. This is surrounded by seven equal 14-gons with centers located at the distance CABC(C + A + B + C) from the origin O (along OP). Along the heptagonal direction OQ kite 2d points to the origin O and along OP it points away from the origin O. Also along OP dart 2b points to the origin O and kite 2a points away from the origin O. Along OQ dart 2c appears pointing to the origin O and dart 2e pointing away from the origin O. If we observe Fig. 3, we notice that internal structure of the central 14-gon with radius equal to CBAC is contained in the whole figure. The internal structure of this last one, in its turn, will be contained in a larger 14-gon, and so on.

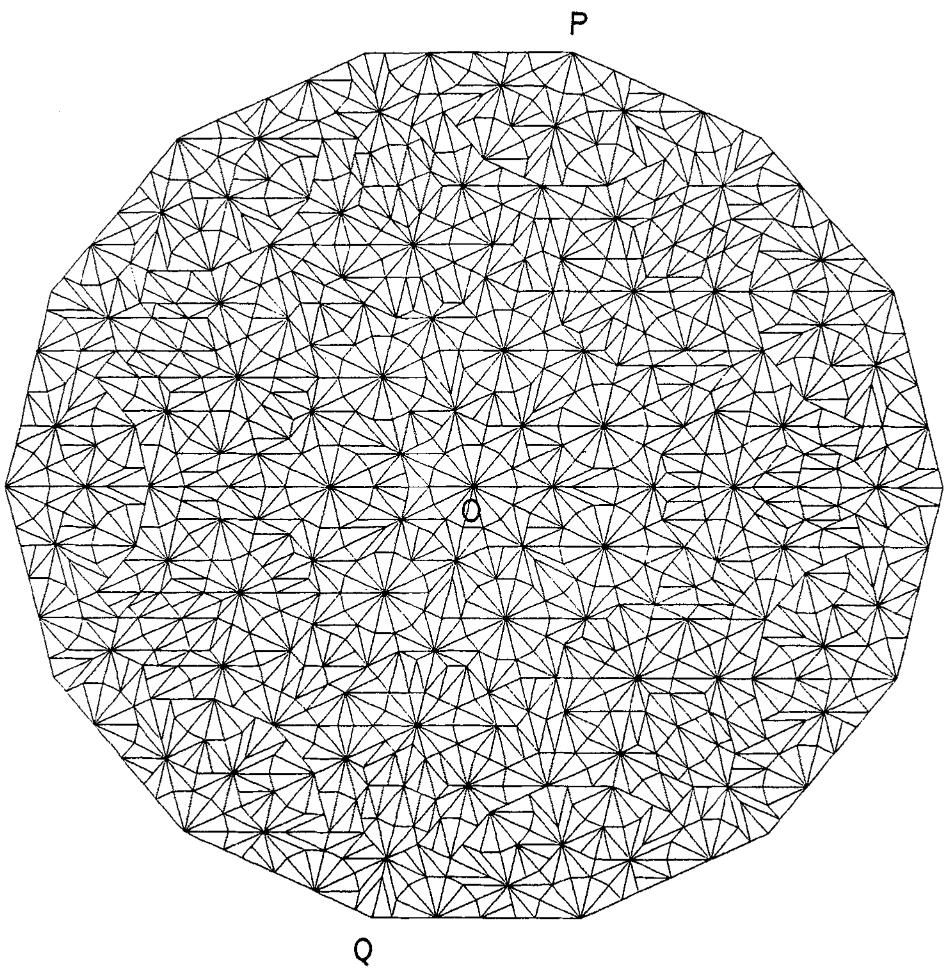

Fig. 3. The rule to inflate the infinite pattern

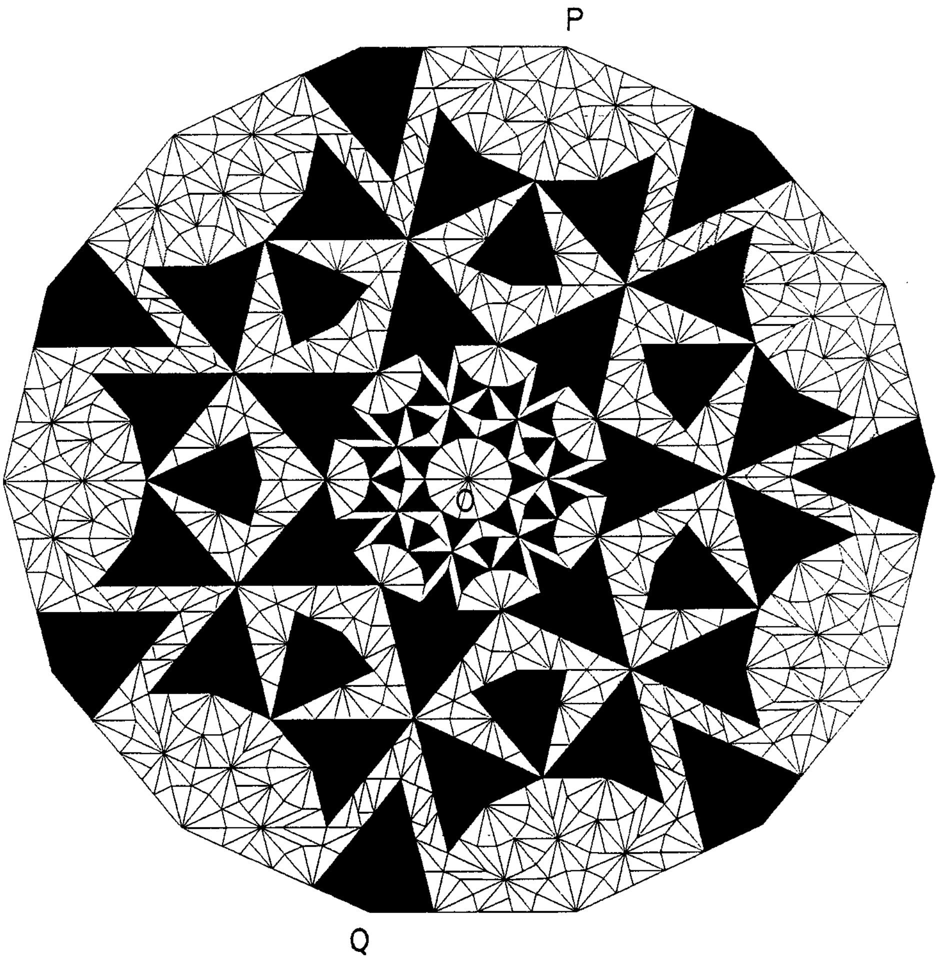

Fig. 4. The same as Fig. 3 pointing out the self-similarity

Fig. 4 is the infinite pattern of Fig. 3 pointing on the self-similarity. In this figure the central 14-gon (radius C) transforms into a concentric 14-gon with radius CBAC (third generation in (8)). The latter one, by its time, transforms into the whole pattern of Fig. 4 and so on. Thus the pattern of Fig. 4 presents self-similarity and the scaling factor is approximatly 3.247. We notice that Fig. 4 has rotational symmetry but is not translationally invariant.

The study of the structure given in Fig. 4 by light diffraction is in progress.

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