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A Non-Quadratic Irrationality Associated to an Enneagonal Quasiperiodic Tiling of the Plane

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A enneagonal tiling of the plane is proposed. A self-similar pattern is obtained by using eight basic shapes. This pattern presents rotational symmetry and no translational invariance.

1. Introduction

Quasiperiodic lattices have been intensively investigated after the discovery of a new phase of Al–Mn, a quasicrystalline phase, by Shechtman et al. [1]. Following this discovery other quasicrystalline systems were found: octogonal, decagonal, and dodecagonal phases [2 to 4]. Recently, one-dimensional quasicrystals (Fibonacci phases) were grown by MBE [5, 6], according to second and third-order Fibonacci sequences, that correspond to quadratic and cubic irrationalities, respectively. In 1974 Penrose [7] proposed a pentagonal tiling of the plane by using six basic shapes. After that he devised a new version for pentagonal tiling with two forms ('darts' and 'kites') and gave the corresponding inflation rule [8]. For darts or kites, the side ratio is the golden mean ($\tau = (1+\sqrt{5})/2$). Penrose tiling is associated to a second-order Fibonacci sequence (musical sequence) defined by

$$F_n = F_{n-1} + F_{n-2}\,, \qquad \mathrm{L} \rightarrow \mathrm{LS}\,, \quad \mathrm{S} \rightarrow \mathrm{L}\,. \tag{1}$$

In (1) F_n is the number of terms of the n-th generation. Numerical values for F_n are: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 ... A closed form for F_n is

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \tag{2}$$

which is known by Binet theorem [9]. A growth sequence for (1) after seven steps ($F_7 = 21$) would be LSLLSLSLLSLLSLSLLSLSL. The transformation given by (1) can be expressed by the so-called "divine proportion"

$$\frac{\mathrm{L}}{\mathrm{S}} = \frac{\mathrm{L}+\mathrm{S}}{\mathrm{L}}\,. \tag{3}$$

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If we put $x = \mathrm{L/S}$ we will have

$$x^2 - x - 1 = 0\,, \tag{4}$$

which has a positive solution τ. The sequence of intervals L and S can be also characterized by the substitution law $r_i = M_{ij} r_j$, where the transformation matrix M is given by [10]

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

and its secular equation is $x^2 - x - 1 = 0$ which gives the golden mean τ.

2. Fourth-Order Fibonacci-Like Sequences

In 1993, one of us proposed a third-order Fibonacci sequence associated to heptagonal tilings of the plane [11, 12]. The obtained pattern presents self-similarity and exhibits rotational symmetry. Heptagonal and enneagonal tilings were considered in 1988 by Whittaker and Whittaker [13] who used the projection method. The present work deals with a fourth-order Fibonacci-like sequence associated to an enneagonal quasiperiodic tiling of the plane, a quasiperiodic tiling of non-quadratic irrationality. Such tilings give rise to physically unstable systems and, indeed, ninefold symmetric quasicrystals have not been observed yet [14].

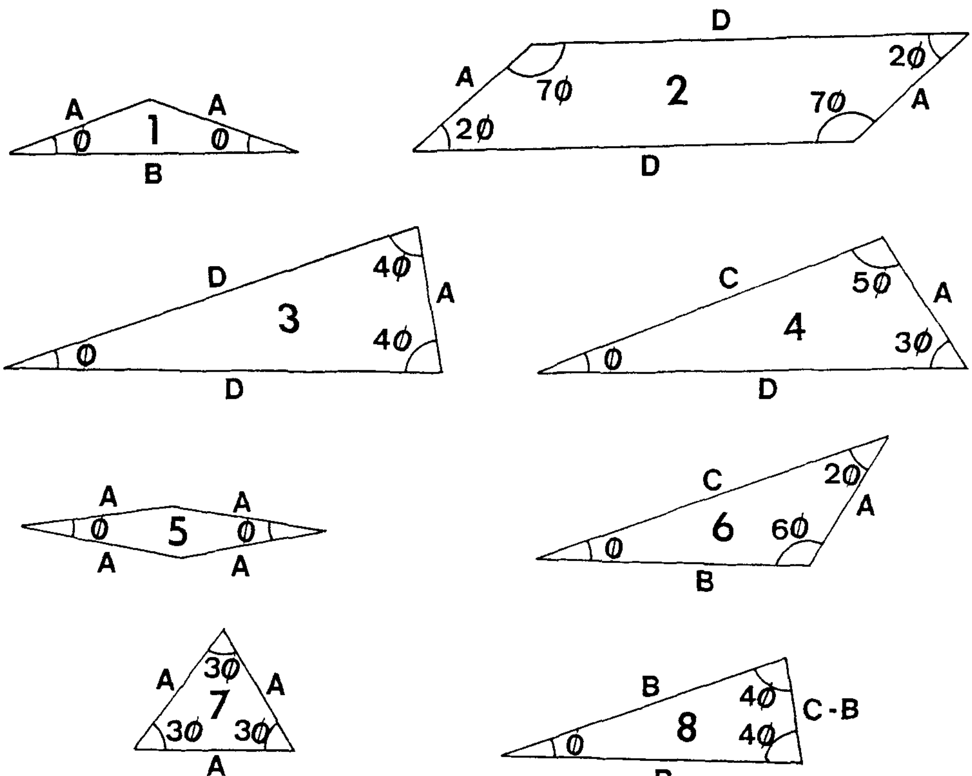

Fig. 1. Fundamental shapes of our tiling ($\mathrm{A} = 1.000$, $\mathrm{B} \approx 1.879$, $\mathrm{C} \approx 2.532$, $\mathrm{D} \approx 2.879$, and $\phi = \pi/9$)

Fourth- order Fibonacci-like sequences associated to an enneagonal tiling, are defined by

$$S_n = 2S_{n-1} + 3S_{n-2} - S_{n-3} - S_{n-4} \tag{6}$$

or

$$S_n = 4S_{n-1} - 3S_{n-2} - 3S_{n-3} + 3S_{n-4} \tag{7}$$

or

$$S_n = S_{n-1} + 3S_{n-2} - 2S_{n-3} - S_{n-4}\,. \tag{8}$$

In this work we are interested in a tiling associated to a non-quadratic irrationality defined by (6). The inflation rule for (6) is

$$\mathrm{A} \to \mathrm{D}, \quad \mathrm{B} \to \mathrm{DC}, \quad \mathrm{C} \to \mathrm{BCD}, \quad \mathrm{D} \to \mathrm{DABC} \tag{9}$$

and the corresponding transformation matrix is

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{10}$$

with the secular equation

$$x^4 - 2x^3 - 3x^2 + x + 1 = 0\,. \tag{11}$$

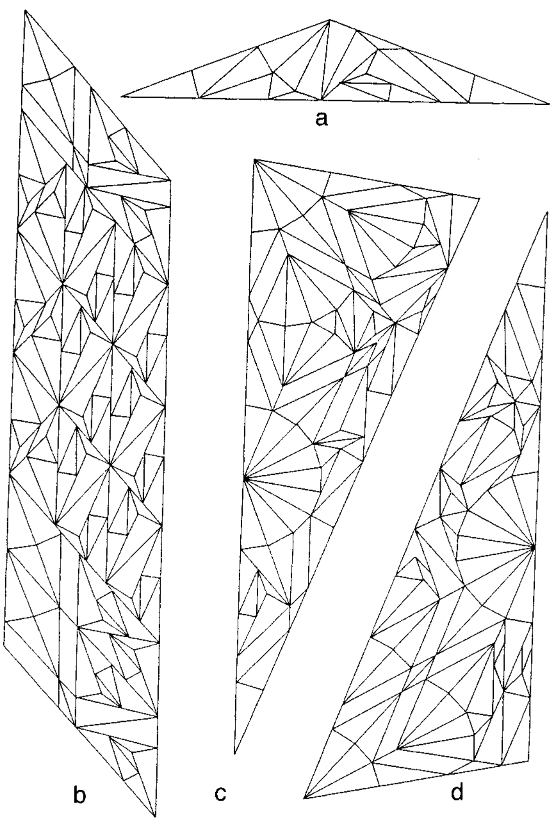

Fig. 2. Transformations of the shapes a) 1, b) 2, c) 3, and d) 4 of Fig. 1

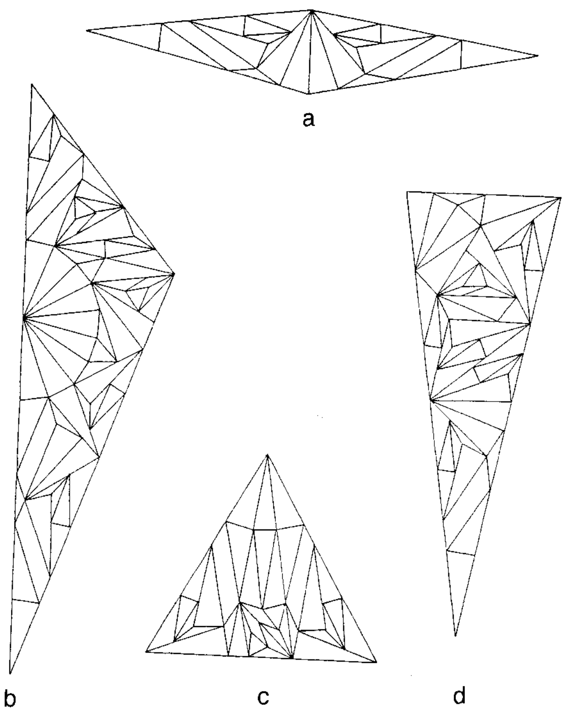

Fig. 3. Transformations of the shapes a) 5, b) 6, c) 7, and d) 8 of Fig. 1

The secular equation for (7) is

$$x^4 - 4x^3 + 3x^2 + 3x - 3 = 0 \tag{12}$$

and the characteristic equation for (8) is

$$x^4 - x^3 - 3x^2 + 2x + 1 = 0\,. \tag{13}$$

Positive solutions for (11), (12), and (13) are, respectively ξ, ν, and μ with the following approximate values:

$$\xi = 2.879, \;\; \nu = 2.532, \;\; \text{and} \;\; \mu = 1.879\,. \tag{14}$$

μ, ν, and ξ are the distinct lengths of a 9-gon (regular enneagon) of unitary side ($\mu = 2\,\cos\,\phi$, $\nu = 2\,\cos^2\,\phi + \cos\,2\phi$, and $\xi = 2\,\cos^3\,\phi + \cos\,\phi\,\cos\,2\phi + \cos\,3\phi$), where $\phi = \pi/9$. By using cyclotomic polynomials [15] it can be proved that ξ, ν, and μ are, respectively, roots of equations (11), (12), and (13).

Next, we display some generations of the inflation transformation given by (9),

1st generation: D
2nd generation: DABC
3rd generation: DABCDDCBCD
4th generation: DABCDDCBCDDABCDABCBCDDCBCDDABC
5th generation: DABCDDCBCDDABCDABCBCDDCBCDDABC
DABCDDCBCDDABCDDCBCDDCBCDDABCD
ABCBCDDCBCDDABCDABCDDCBCD
. . . .
(15)

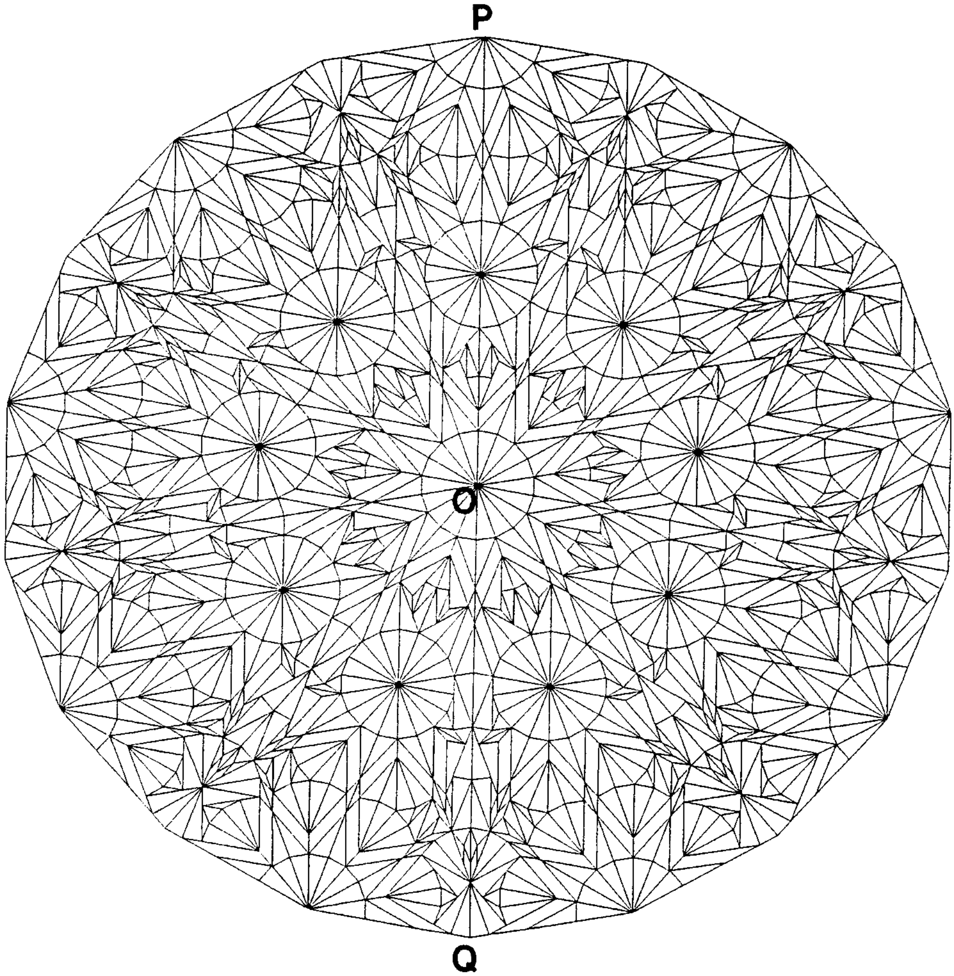

Fig. 4. The infinite pattern for enneagonal symmetry

In the above transformations the number of intervals in each generation is given by $S_1 = 1$, $S_2 = 4$, $S_3 = 10$, $S_4 = 30$, $S_5 = 85$, ... according to (6).

3. Results and Conclusions

In the following we will describe how to tile the plane with an enneagonal quasiperiodic symmetry. In Fig. 1 we show eight forms which are basic for the constructions that will appear in this paper. They are three isosceles triangles, two scalene triangles, one equilateral triangle, one parallelogram, and one lozenge. Angles are multiples of ϕ and sides are according to the correspondence: $1 \to$ A, $\mu \to$ B, $\nu \to$ C, $\xi \to$ D.

Fig. 2 and 3 are the transformations of Fig. 1 by the approximate scaling factor 8.29. Fig. 2 (a, b, c, and d) corresponds, respectively, to forms 1, 2, 3, and 4 of Fig. 1. Fig. 3 (a, b, c, and d) corresponds, respectively, to forms 5, 6, 7, and 8 of Fig. 1. Forms of Fig. 2 and 3 have sides according to the Fibonacci-like sequence defined by (6).

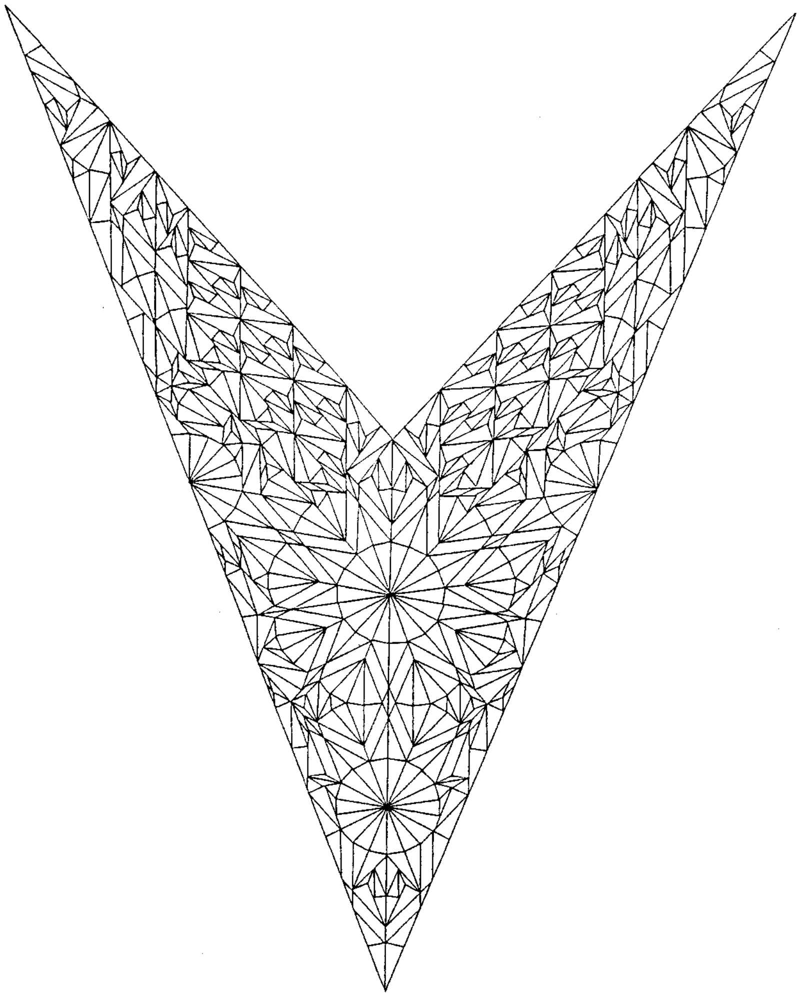

Fig. 5. One ninth of a enneagonal star on the plane

Fig. 4 presents the infinite pattern corresponding to (6). It is a 18-gon with radius DABCDDCBCD (third generation in (15)). The central 18-gon of Fig. 4 has radius D and uses eighteen forms given by form 3 of Fig. 1. This is surrounded by other nine equal 18-gons with centers located at the distance DABCD (D + A + B + C + D) from the origin O (along OP). The whole figure can be considered as a new central 18-gon surrounded by nine equal 18-gons, and so on. Thus the pattern of Fig. 4 presents self-similarity and the approximate scaling factor is 8.29. We must note that radial directions separated by ϕ do not correspond (in the inflation transformations such radial directions correspond to walks in opposite directions starting at the central 18-gon). So, we have only ninefold symmetry.

Fig. 5 is formed by Fig. 2a, b, c, and d and represents a part of one star on the plane (one ninth of a star). It furnishes more details about the pattern of Fig. 4. If we observe

Fig. 4 and 5, we see that the central star of Fig. 4 (radius DCB) transforms into a large star that contains the same internal structure of the central star of Fig. 4. This large star, in turn, would transform in a bigger star with the same internal structure of the star generated Fig. 5 nine times, and so on. We notice that Fig. 4 has rotational symmetry, but it is not translationally invariant.

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