Empty polygons in Penrose tilings

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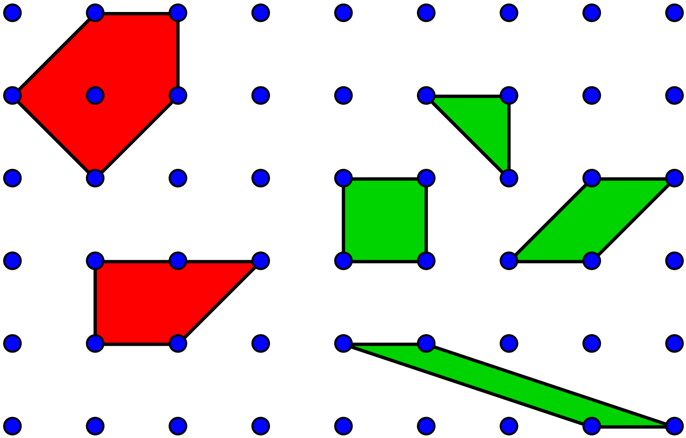
Empty lattice polygons:

("polygon" here always means convex polygon)



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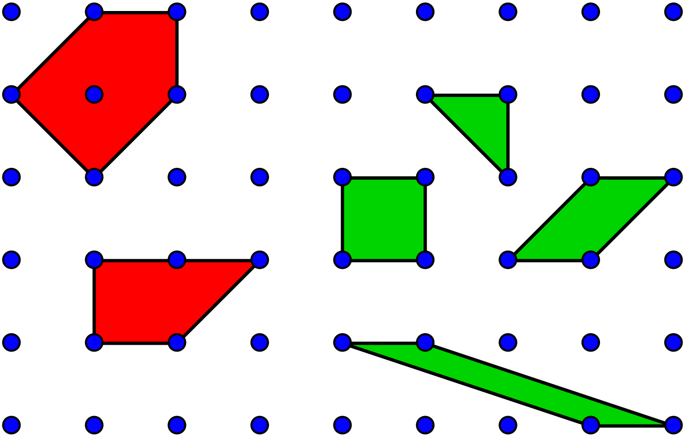


Several interesting results and applications.

One: Helly number of point lattices.

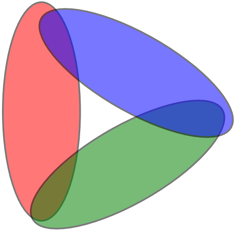
Helly numbers

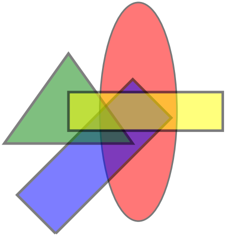
Theorem (Radon '21, König '22, Helly '23)

Let $K_1, \ldots, K_n$ be convex subsets of $\mathbb{R}^d$, with $n \geqslant d+1$. If any $d+1$ of these sets have a common point, then all have a common point; that is

$$\bigcap_{j=1}^{n} K_j \neq \varnothing.$$

Hence $d+1$ is the **Helly number** $h(\mathbb{R}^d)$ of $\mathbb{R}^d$.





For instance, $h(\mathbb{R}^2) = 3$.

Helly numbers

Let us generalize Helly numbers to other sets than $\mathbb{R}^d$.

Definition

Let $S \subseteq \mathbb{R}^d$ be nonempty. The S-Helly number $h(S)$ of S is the minimal number $n > d$ such that the following statement holds:

Let $K_1, ..., K_n$ be convex subsets of S. If any $h(S)$ of these sets have a common point, then all have a common point.

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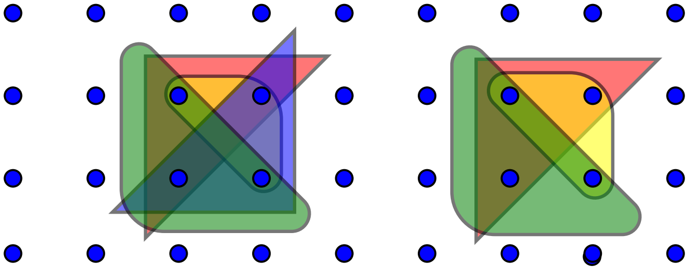
One result in this context:

Theorem (Doignon '73)

$h(\mathbb{Z}^d) = 2^d$.

Helly numbers

For instance, $h(\mathbb{Z}^2) = 4$.



Any three triangles have a common point, but not all four.

Helly numbers and empty polygons

Theorem (Averkov '13)

Let $S \subseteq \mathbb{R}^d$ be discrete. Then the Helly number $h(S)$ equals the maximal number of vertices of an empty S-polytope.

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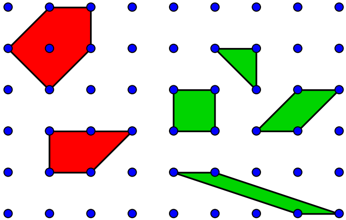

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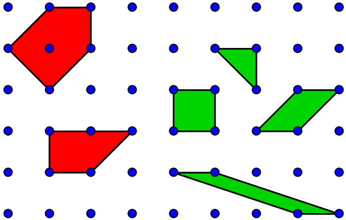
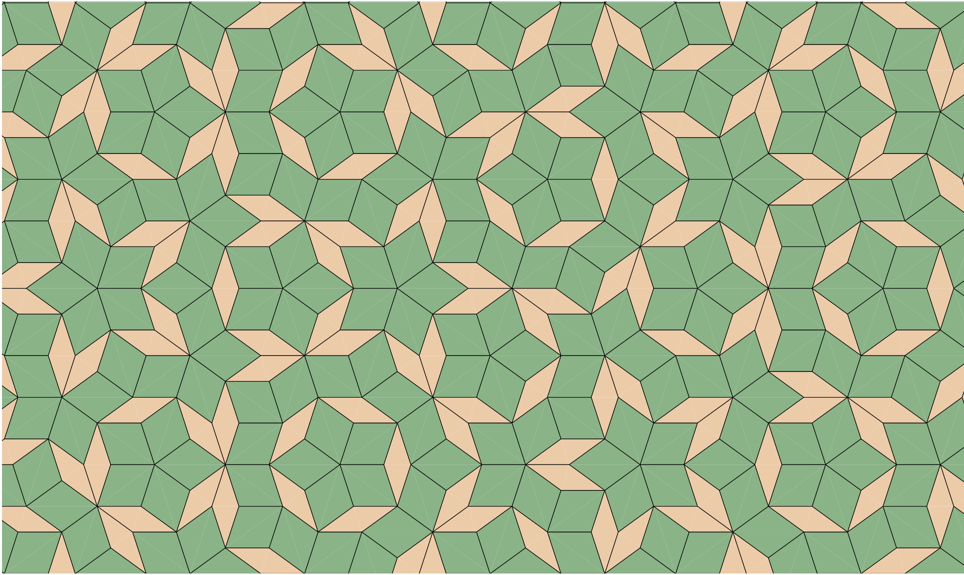


Which interesting discrete sets S we may study?

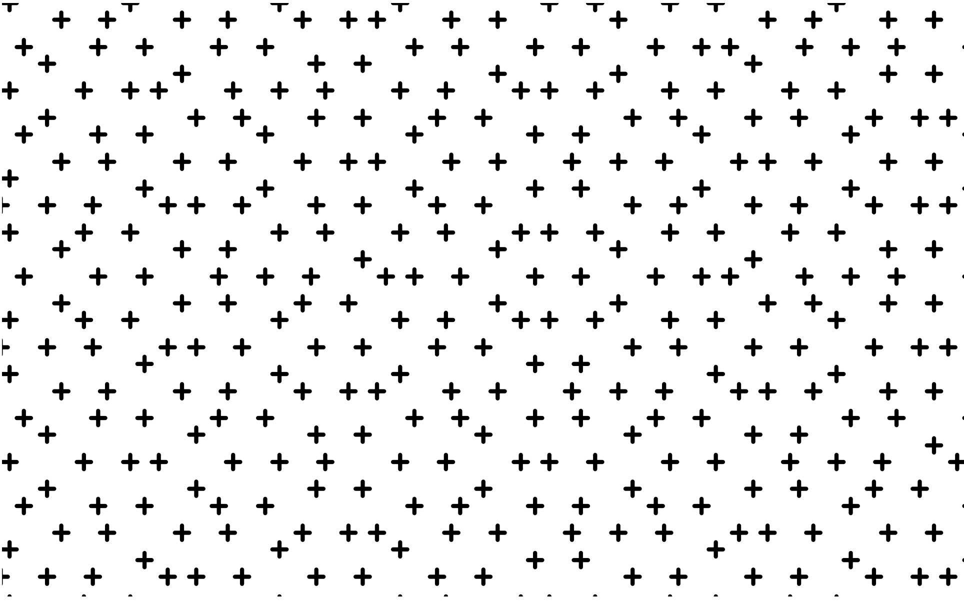
Penrose tilings

A famous aperiodic tiling: the Penrose Tiling



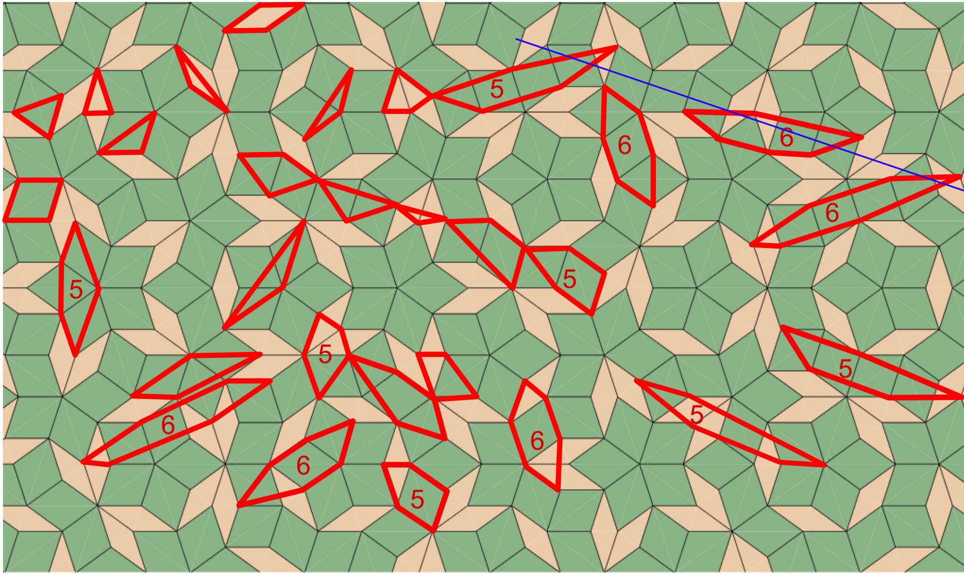
Penrose tilings

Consider its vertex set P:



Empty polygons in Penrose tilings

...and determine the empty polygons.



Empty polygons in Penrose tilings

By Averkov's result the maximal number of vertices yields the Helly number $h(P)$.

So far we have:

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Claim $h(P) = 6$.

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Thank you!