Aperiodic Frieze Patterns

Dirk Frettlöh

Joint work with Jan Mazáč

Technische Fakultät
Universität Bielefeld

Open University
Milton Keynes, 18^{th} Nov 2026

Coxeter Frieze Patterns

Row 0 is all zeroes.

Row 1 is all 1s.

Diamond rule: for every $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ holds $bc - ad = 1$.

Example:

$$\begin{array}{ccccccccccccccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots \\
& \cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots & \\
\cdots & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & \cdots \\
& \cdots & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & \cdots & \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
& \cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots &
\end{array}$$

Coxeter Frieze Patterns

Row 0 is all zeroes.

Row 1 is all 1s.

Diamond rule: for every $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ holds $bc - ad = 1$.

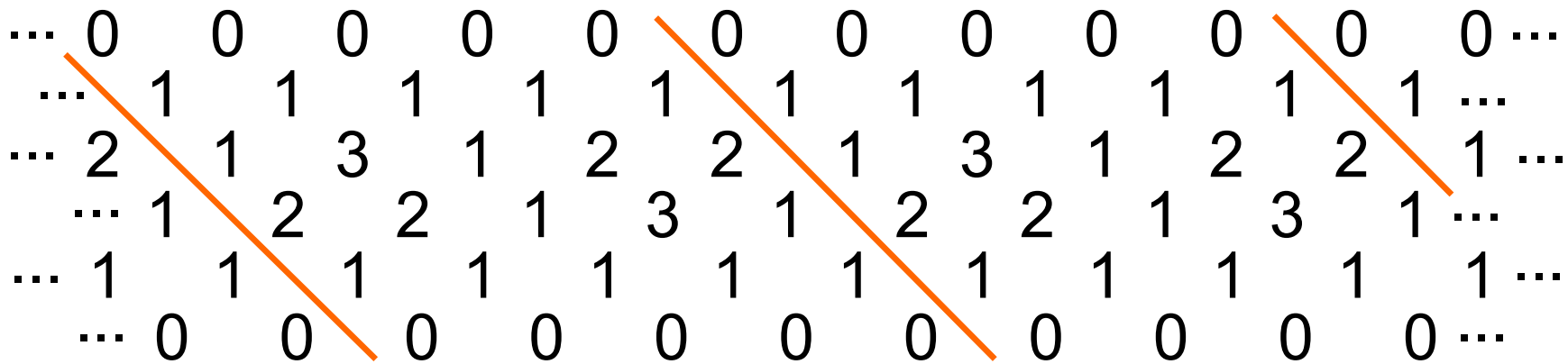
Example:

$$\begin{array}{ccccccccccccccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots \\
 & \cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots & \\
\cdots & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & \cdots \\
 & \cdots & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & \cdots & \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
 & \cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots &
\end{array}$$

Question: Which numbers can we write in the second row?
(such that we get finitely many rows, last one all zeroes)

Coxeter Frieze Patterns

Diamond rule: for every $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ holds $bc - ad = 1$.



Coxeter Frieze Patterns

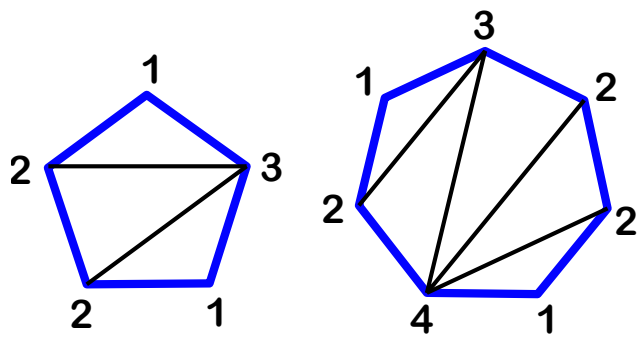
Diamond rule: for every $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ holds $bc - ad = 1$.

⋯	0		0		0		0		0		0		0	⋯
	⋯	1		1		1		1		1		1	⋯	
⋯	2		1		3		1		2		2		1	⋯
	⋯	1		2		2		1		3		1	⋯	
⋯	1		1		1		1		1		1		1	⋯
	⋯	0		0		0		0		0		0	⋯	

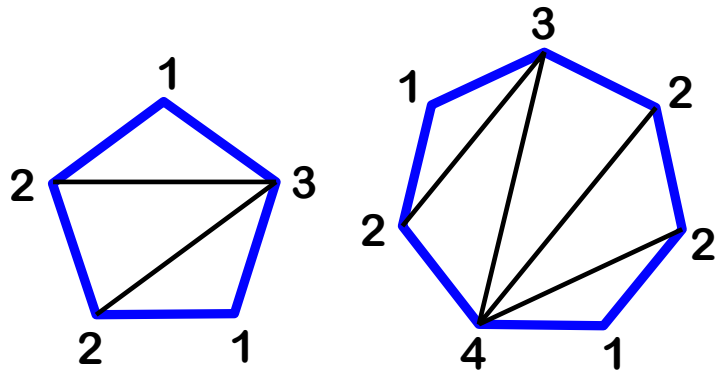
Answer (for periodic 2nd rows):

Theorem

$a_1, a_2, \ldots, a_n$ defines a Coxeter frieze pattern with $n+1$ rows if and only if it is the sequence of the number of triangles at a vertex of a triangulated n-gon.

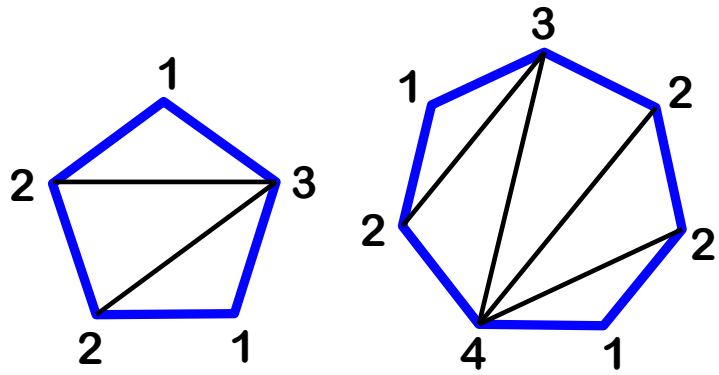


Coxeter Frieze Patterns



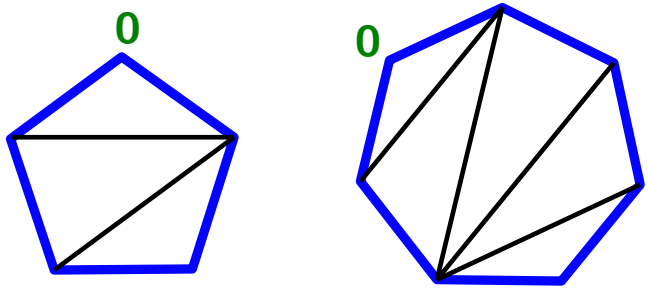
⋯	0	0	0	0	0	0	0	0	0	0	0	0	⋯
⋯	1	1	1	1	1	1	1	1	1	1	1	⋯	
⋯	2	1	3	2	2	1	4	2	1	3	2	2	⋯
⋯	1	2	5	3	1	3	7	1	2	5	3	⋯	
⋯	3	1	3	7	1	2	5	3	1	3	7	1	⋯
⋯	2	1	4	2	1	3	2	2	1	4	1	⋯	
⋯	1	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	0	0	0	0	0	0	0	0	0	0	0	⋯	

Coxeter Frieze Patterns



⋯	0	0	0	0	0	0	0	0	0	0	0	0	⋯
⋯	1	1	1	1	1	1	1	1	1	1	1	⋯	
⋯	2	1	3	2	2	1	4	2	1	3	2	2	⋯
⋯	1	2	5	3	1	3	7	1	2	5	3	⋯	
⋯	3	1	3	7	1	2	5	3	1	3	7	1	⋯
⋯	2	1	4	2	1	3	2	2	1	4	1	⋯	
⋯	1	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	0	0	0	0	0	0	0	0	0	0	0	⋯	

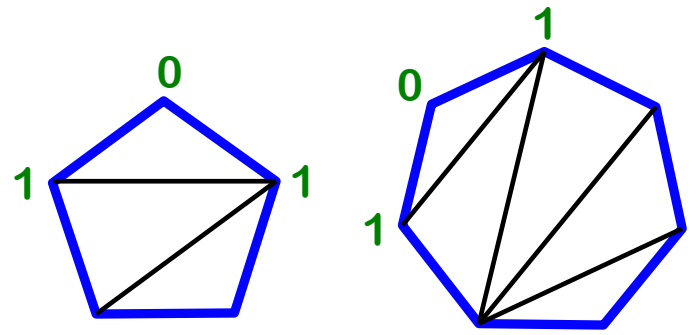
The defining sequence $a_1, \ldots, a_n$ (here: 3221421) is called the **quiddity**.

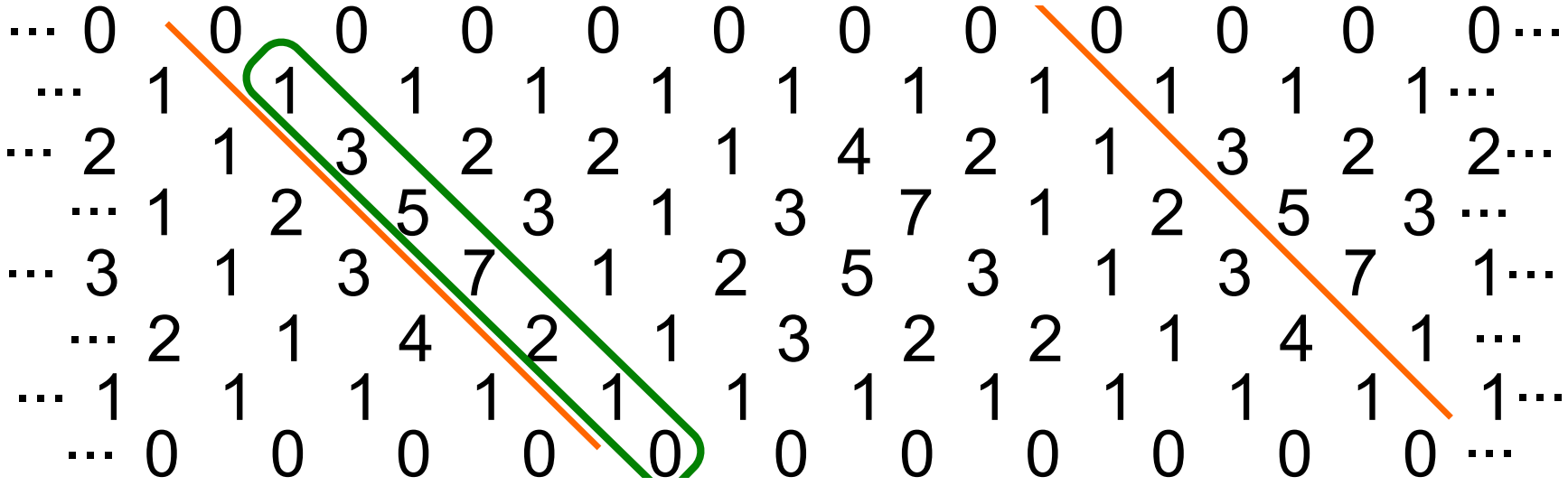


⋯	0		0		0		0		0		0		0		0		0	⋯
	⋯	1		1		1		1		1		1		1		1	⋯	
⋯	2		1		3		2		2		1		4		2		1	2	⋯
	⋯	1		2		5		3		1		3		7		2		3	⋯
⋯	3		1		3		7		1		2		5		3		1	3	⋯
	⋯	2		1		4		2		1		3		2		2		1	⋯
⋯	1		1		1		1		1		1		1		1		1	1	⋯
	⋯	0		0		0		0		0		0		0		0		0	⋯

There is more, for instance:

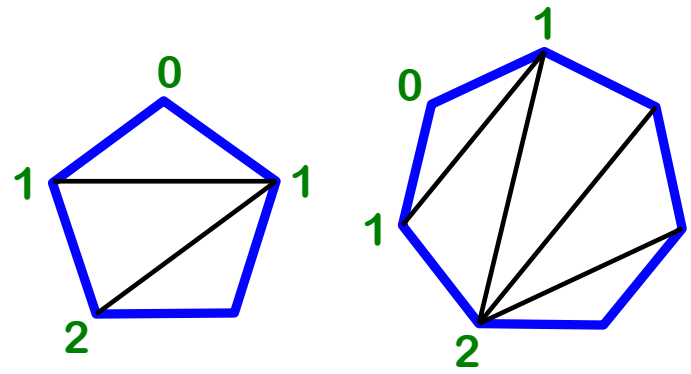
- Label some vertex with 0.

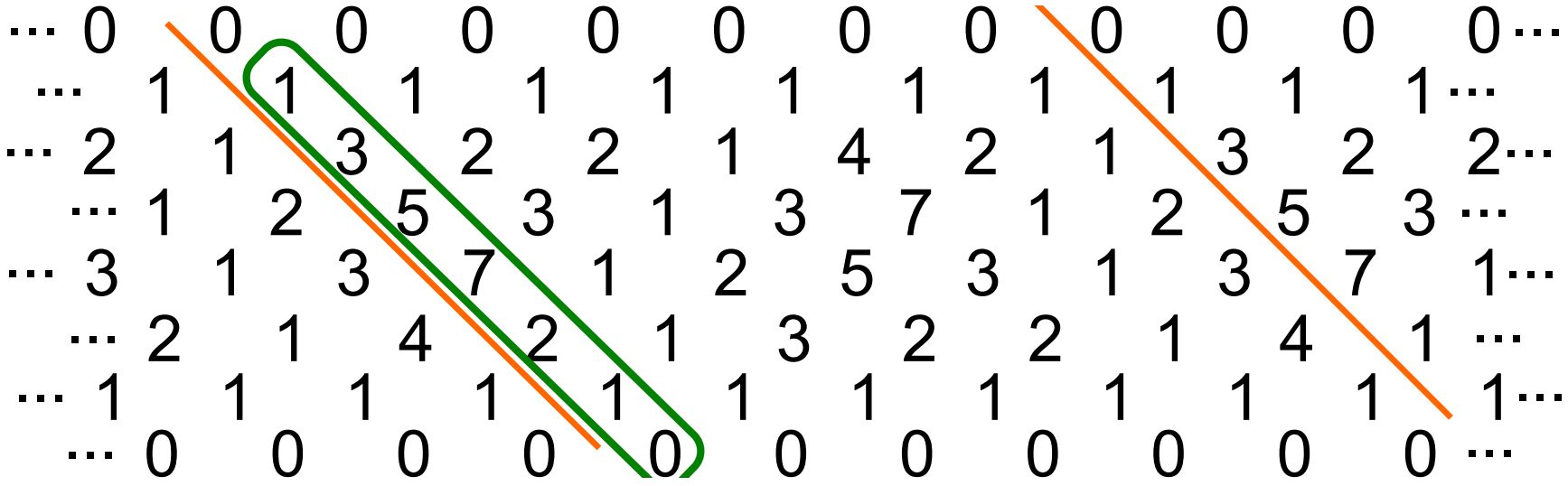




There is more, for instance:

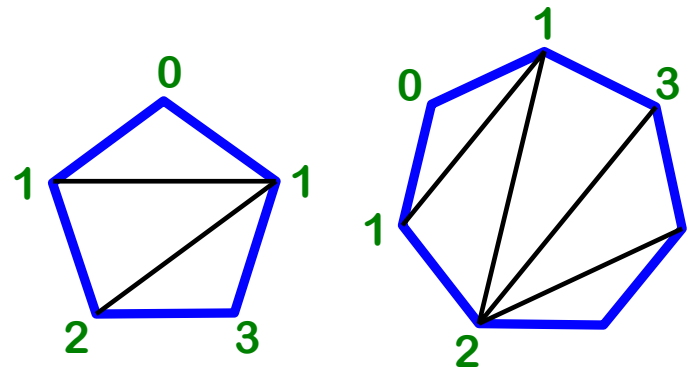
- Label some vertex with 0.
- Label the vertices adjacent to it with 1.





There is more, for instance:

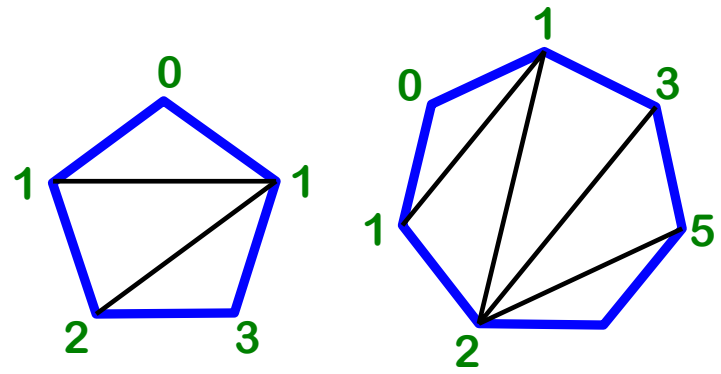
- Label some vertex with 0.
- Label the vertices adjacent to it with 1.
- If a vertex v shares a triangle with two other vertices with labels n, resp., m, label v with $n + m$.

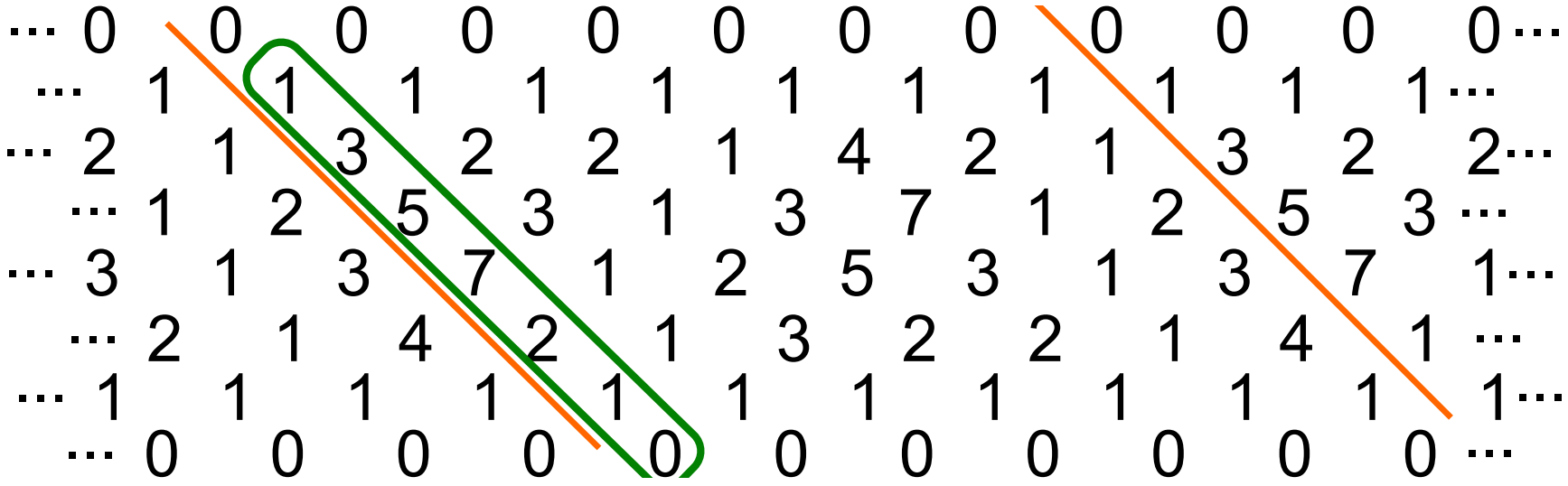


⋯	0		0		0		0		0		0		0		0		0	⋯
	⋯	1		1		1		1		1		1		1		1	⋯	
⋯	2		1		3		2		2		1		4		2		1	2	⋯
	⋯	1		2		5		3		1		3		7		1		2	⋯
⋯	3		1		3		7		1		2		5		3		1	3	⋯
	⋯	2		1		4		2		1		3		2		2		1	⋯
⋯	1		1		1		1		1		1		1		1		1	1	⋯
	⋯	0		0		0		0		0		0		0		0		0	⋯

There is more, for instance:

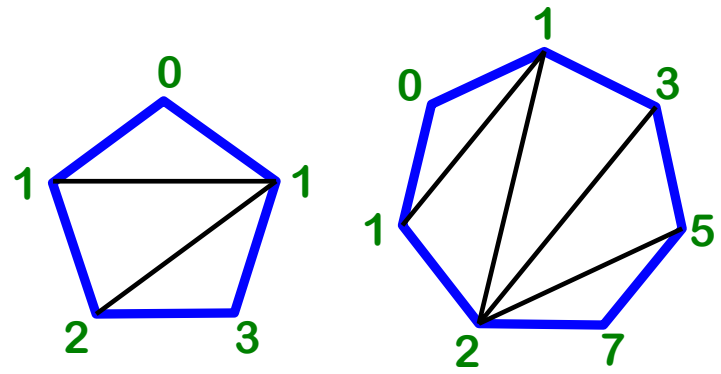
- Label some vertex with 0.
- Label the vertices adjacent to it with 1.
- If a vertex v shares a triangle with two other vertices with labels n, resp., m, label v with $n + m$.

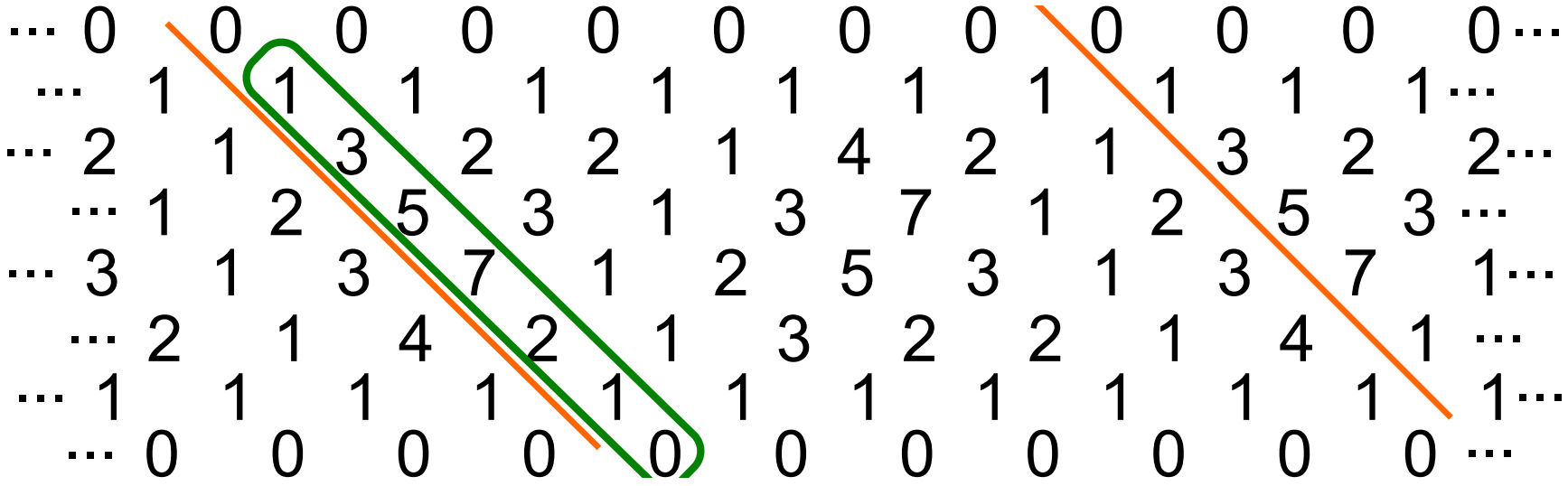




There is more, for instance:

- Label some vertex with 0.
- Label the vertices adjacent to it with 1.
- If a vertex v shares a triangle with two other vertices with labels n, resp., m, label v with $n + m$.





There is more, for instance:

- Label some vertex with 0.
- Label the vertices adjacent to it with 1.
- If a vertex v shares a triangle with two other vertices with labels n, resp., m, label v with $n + m$.

Jan and me learned about this Oberwolfach, during a meeting on representation of algebras. We found it fascinating.

Jan and me learned about this Oberwolfach, during a meeting on representation of algebras. We found it fascinating.

So how to play with this w.r.t. aperiodic order? Two ideas:

1. Aperiodic quiddity, or
2. Play it on some aperiodic tiling.

Jan and me learned about this Oberwolfach, during a meeting on representation of algebras. We found it fascinating.

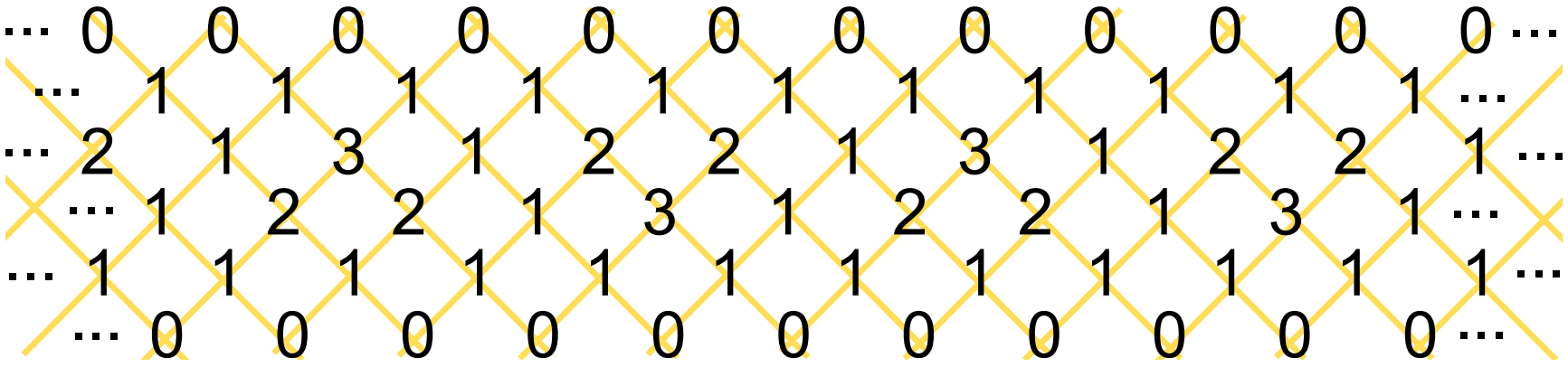
So how to play with this w.r.t. aperiodic order? Two ideas:

1. Aperiodic quiddity, or
2. Play it on some aperiodic tiling.

Number 1 turned out to be less fun. Also, people did look already into it.
xxx Baur, Faber, infinitygon

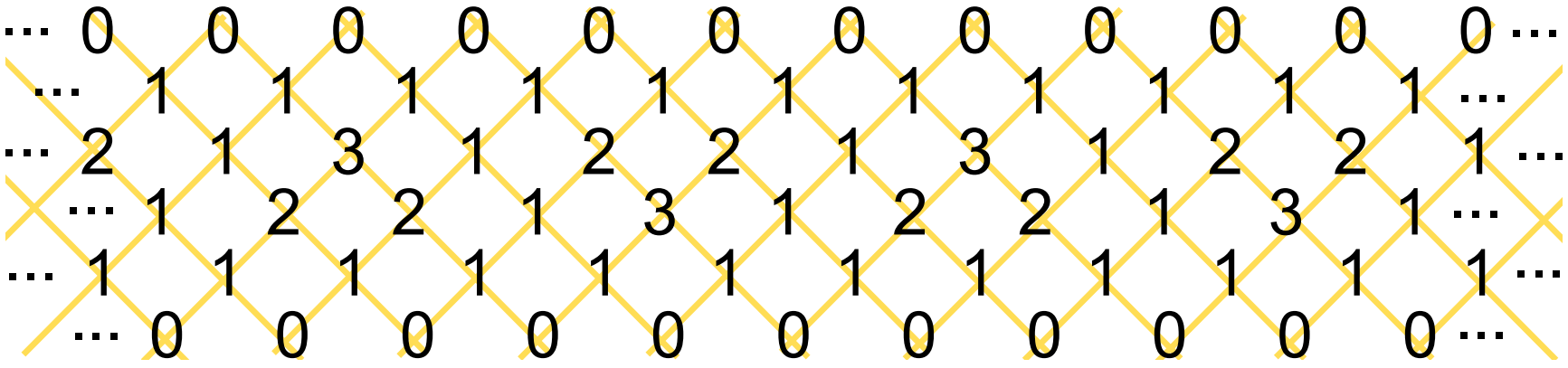
Coxeter friezes on aperiodic tilings

The frieze patterns above live on a square grid:



Coxeter friezes on aperiodic tilings

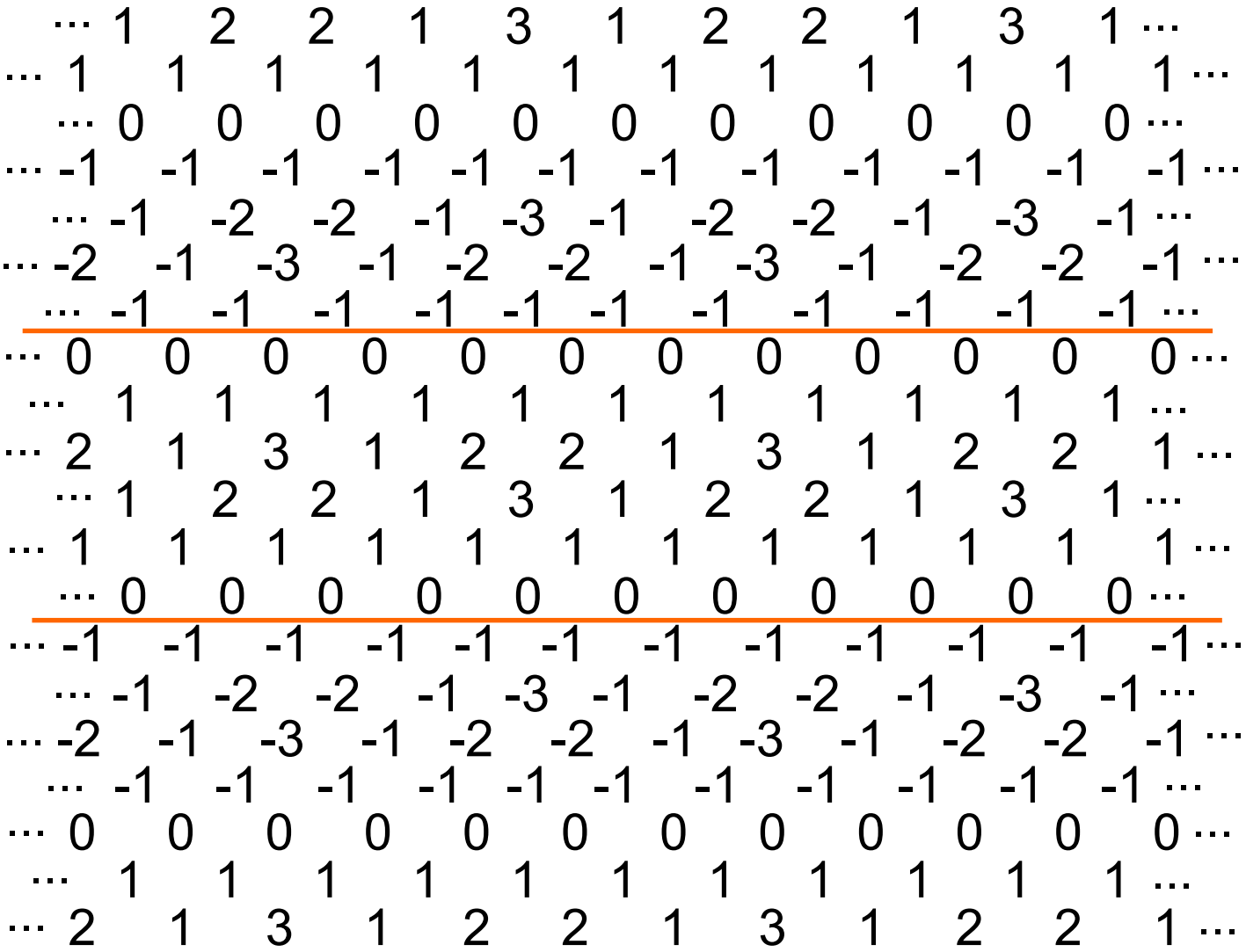
The frieze patterns above live on a square grid:



...OK, only on a finite strip of a square grid. But...

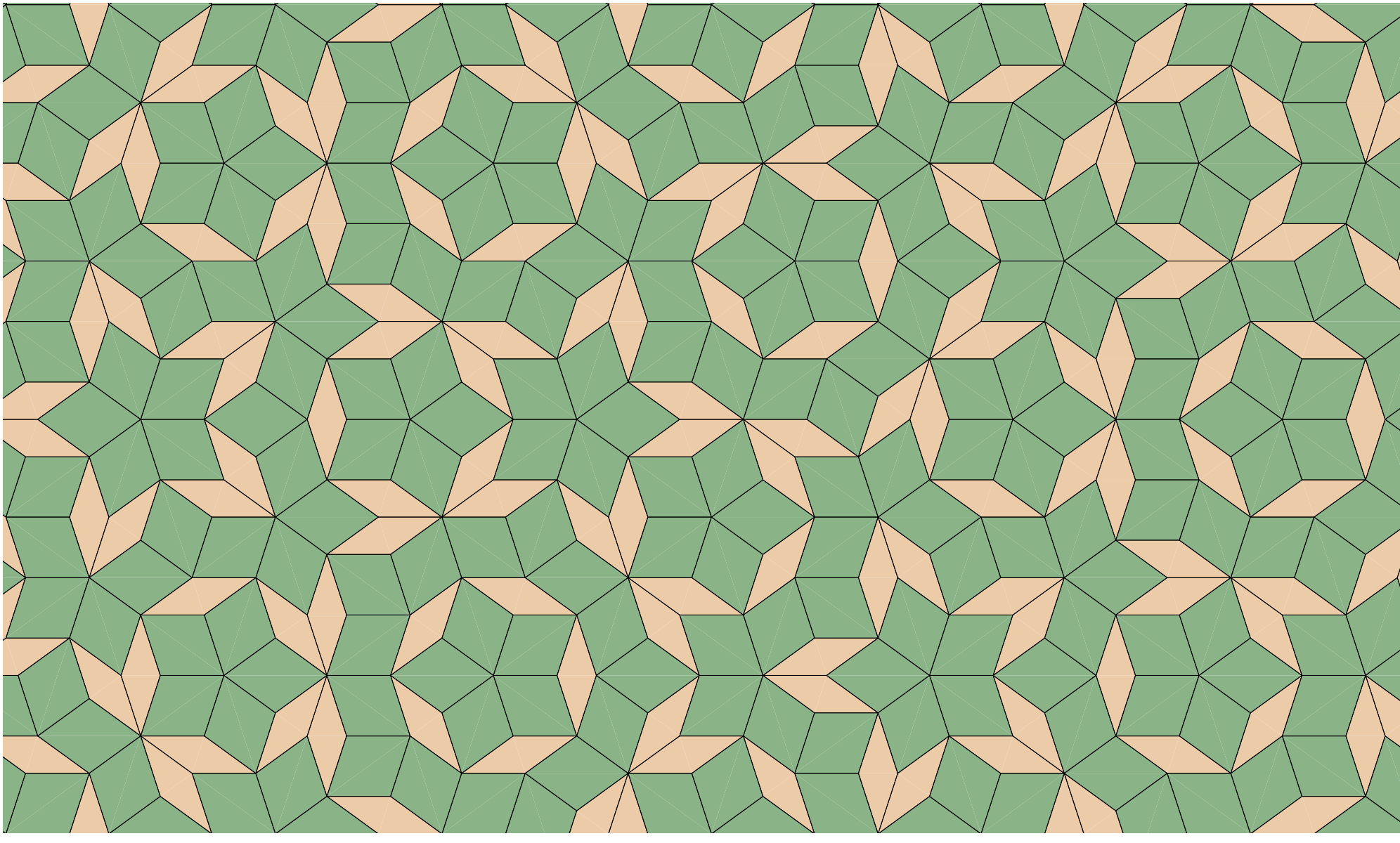
Coxeter friezes on aperiodic tilings

If we allow negative numbers, we can extend it to the entire plane:



...without violating the diamond rule.

A famous aperiodic tiling: the Penrose Tiling



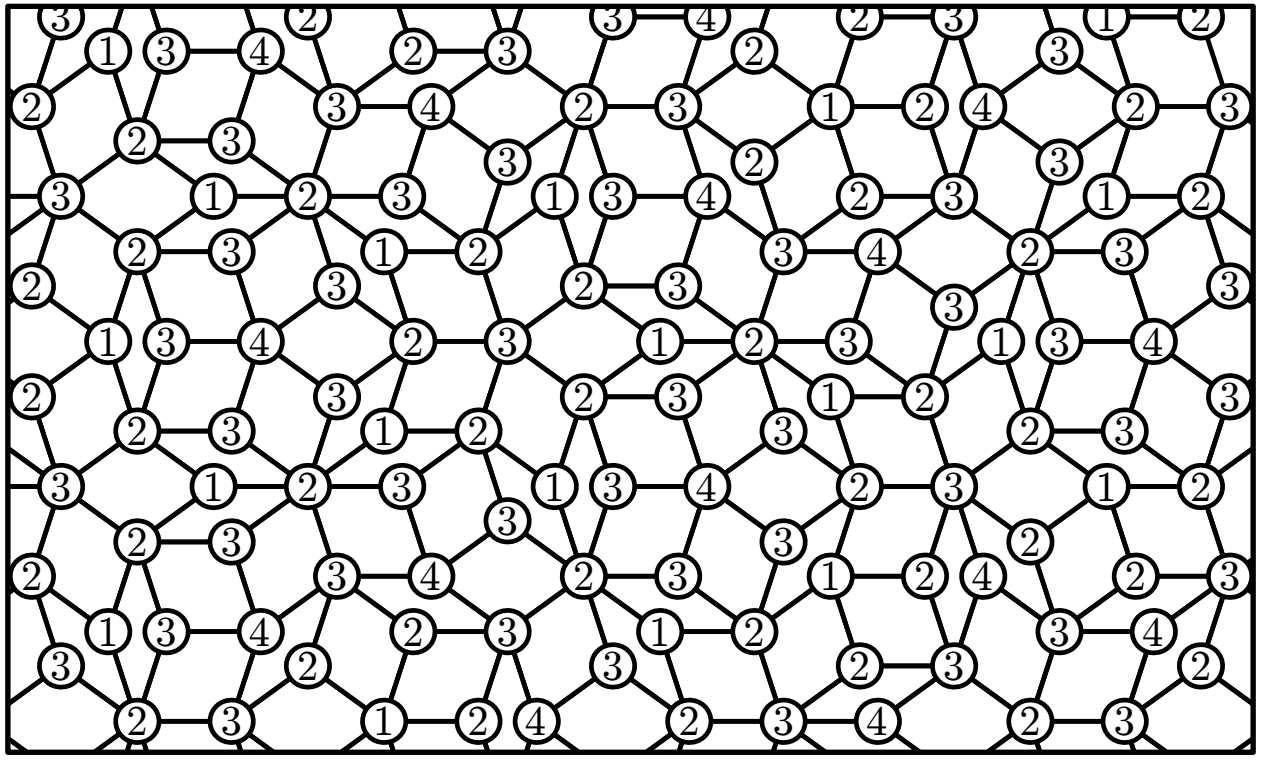


Image: Baake-Grimm, Aperiodic Order, CUP 2013

What do we see?

- diamond rule holds, hence a frieze pattern on a Penrose tiling.

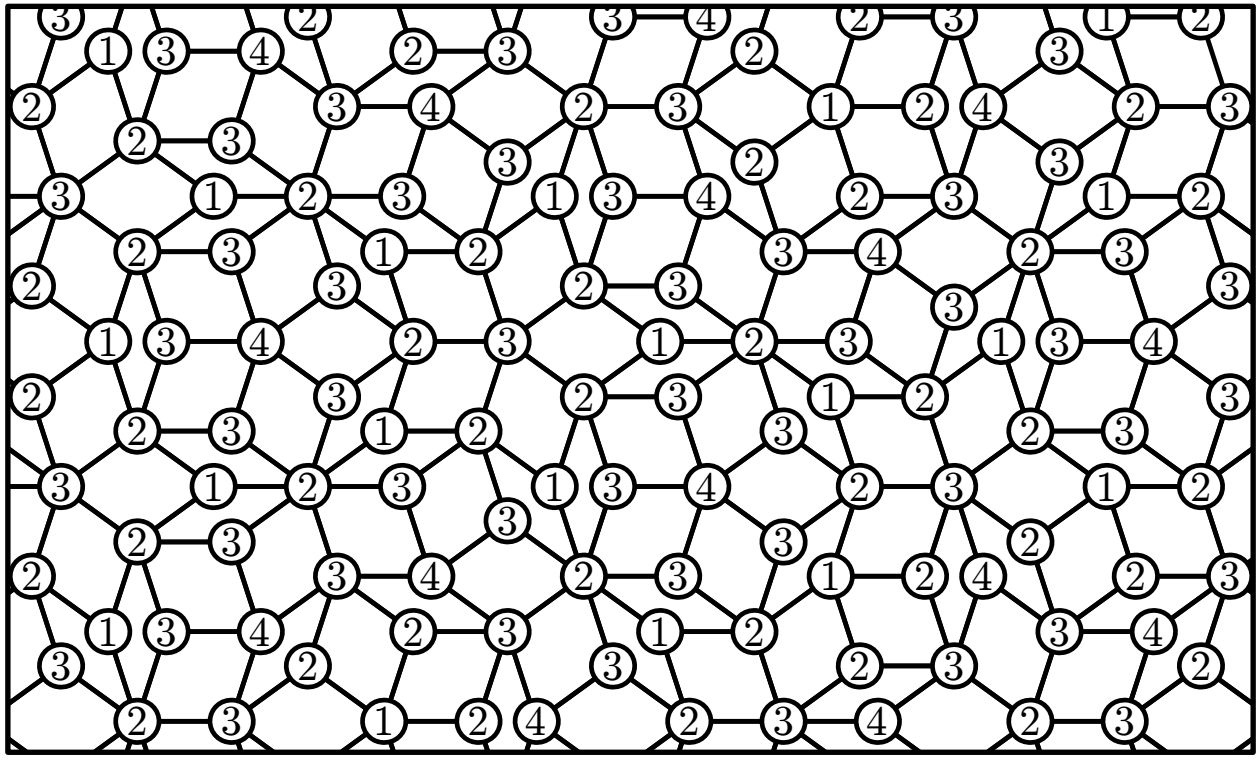


Image: Baake-Grimm, Aperiodic Order, CUP 2013

What do we see?

- diamond rule holds, hence a frieze pattern on a Penrose tiling.
- but this image shows:
  - a "perfect 4-colouring" of the Penrose vertex set,
  - a labelling of the vertices w.r.t the 4-dim lattice A^*.

Theorem

There are infinitely many frieze patterns on Penrose tilings with four distinct values.

Proof.

Theorem

There are infinitely many frieze patterns on Penrose tilings with four distinct values.

Proof.

"one frieze pattern": see above.

Theorem

There are infinitely many frieze patterns on Penrose tilings with four distinct values.

Proof.

"one frieze pattern": see above.

Several: $n^2 - (n+1)(n-1) = 1$.

Theorem

There are infinitely many frieze patterns on Penrose tilings with four distinct values.

Proof.

"one frieze pattern": see above.

Several: $n^2 - (n+1)(n-1) = 1$.

So we may add any $k \in \mathbb{N}$ to the labels.

Theorem

There are infinitely many frieze patterns on Penrose tilings with four distinct values.

Proof.

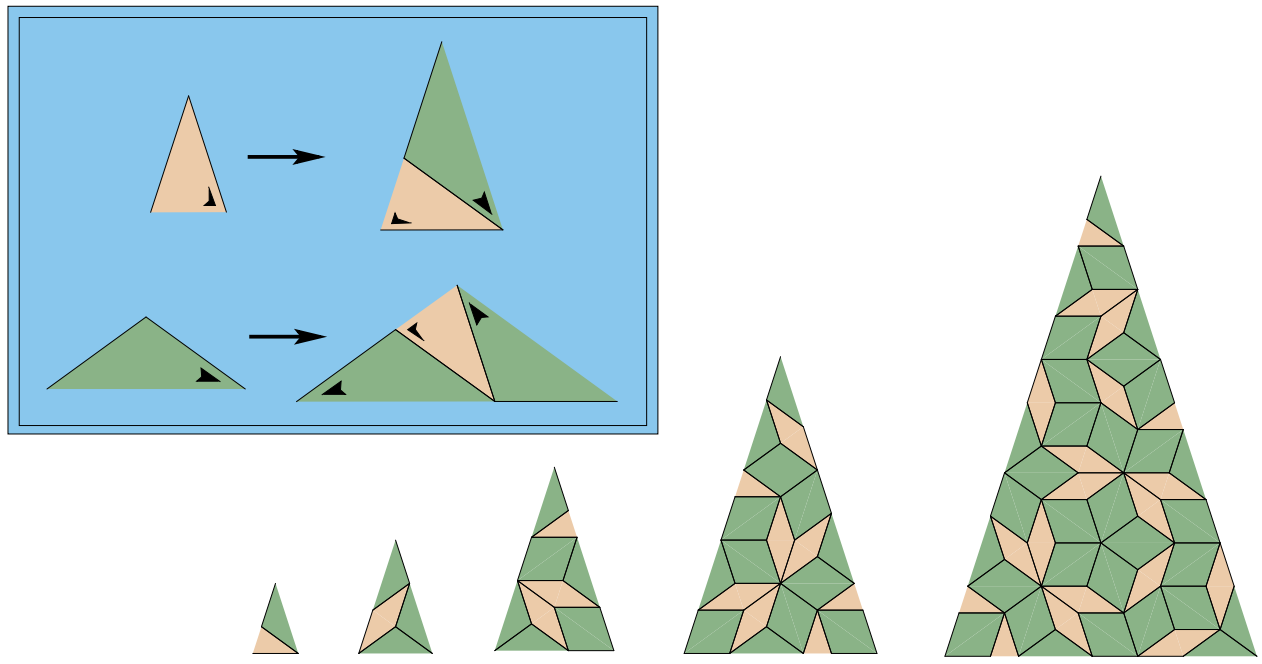
"one frieze pattern": see above.

Several: $n^2 - (n+1)(n-1) = 1$.

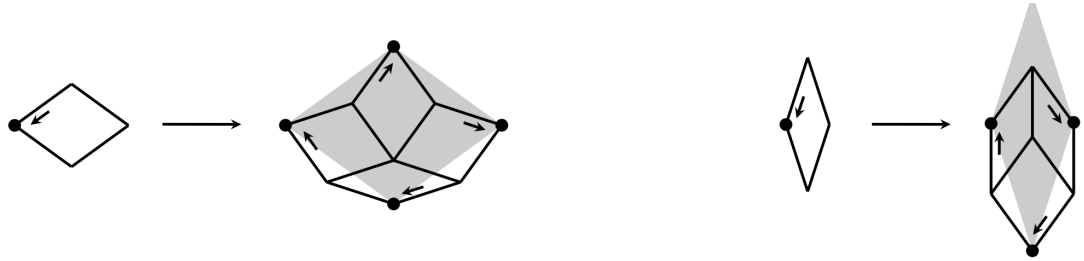
So we may add any $k \in \mathbb{N}$ to the labels. Any $k \in \mathbb{R}$, in fact. $\square$

What about other examples?

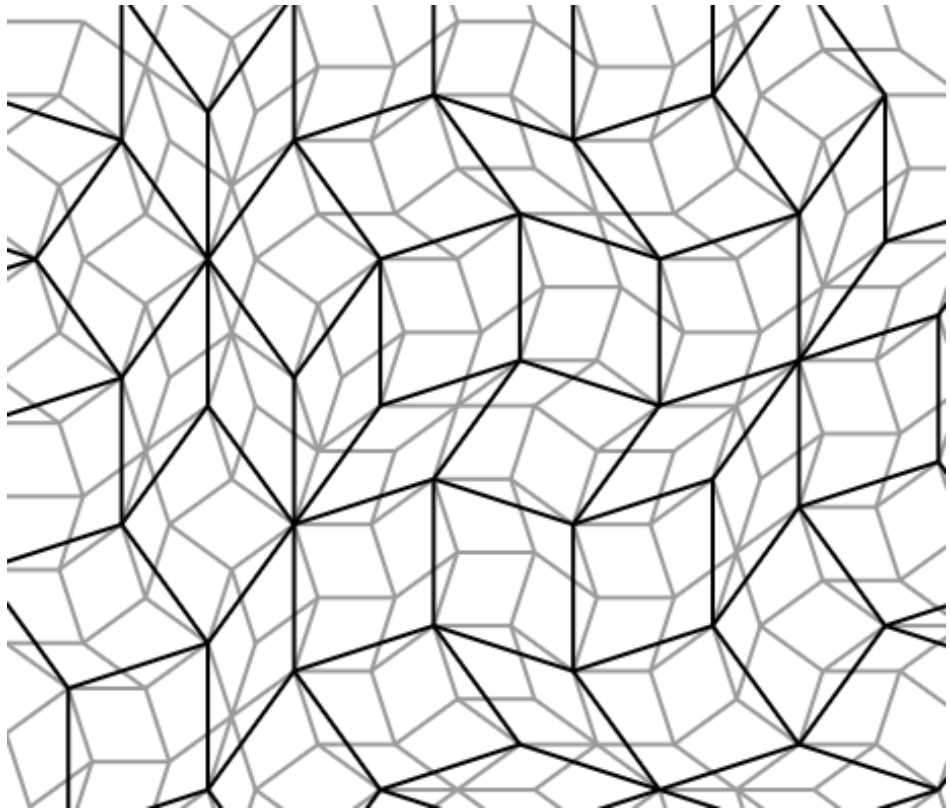
The Penrose tiling can be defined by a "substitution rule".



Here is another substitution rule:

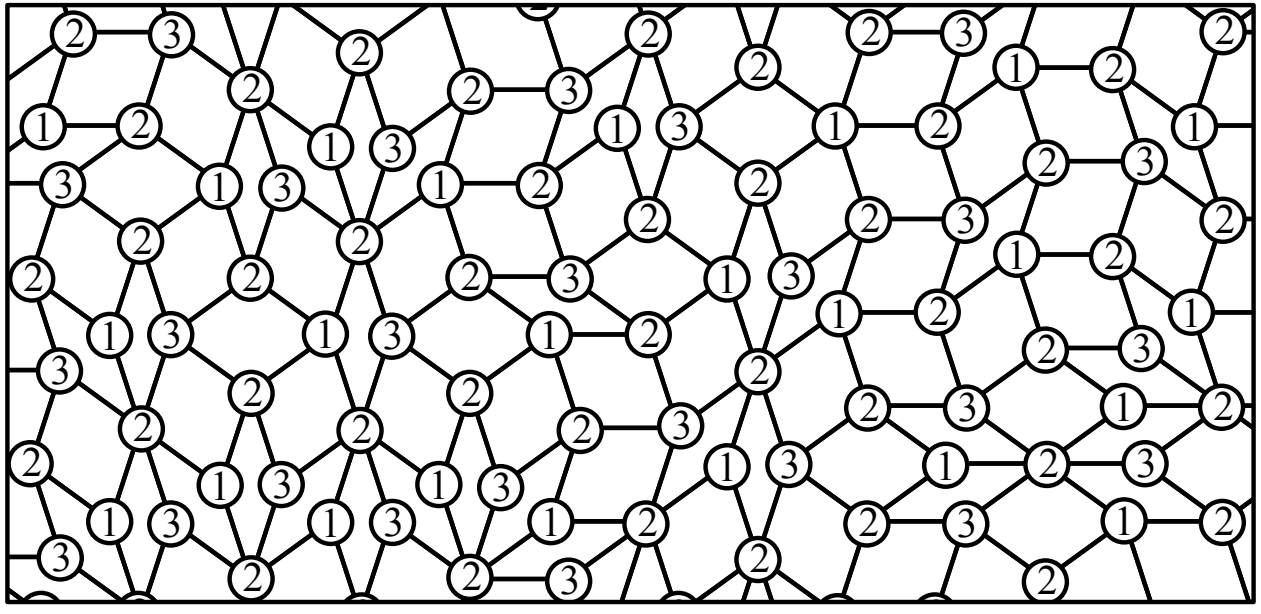


This yields the Godreche-Lanćon-Billard (GLB) tiling:



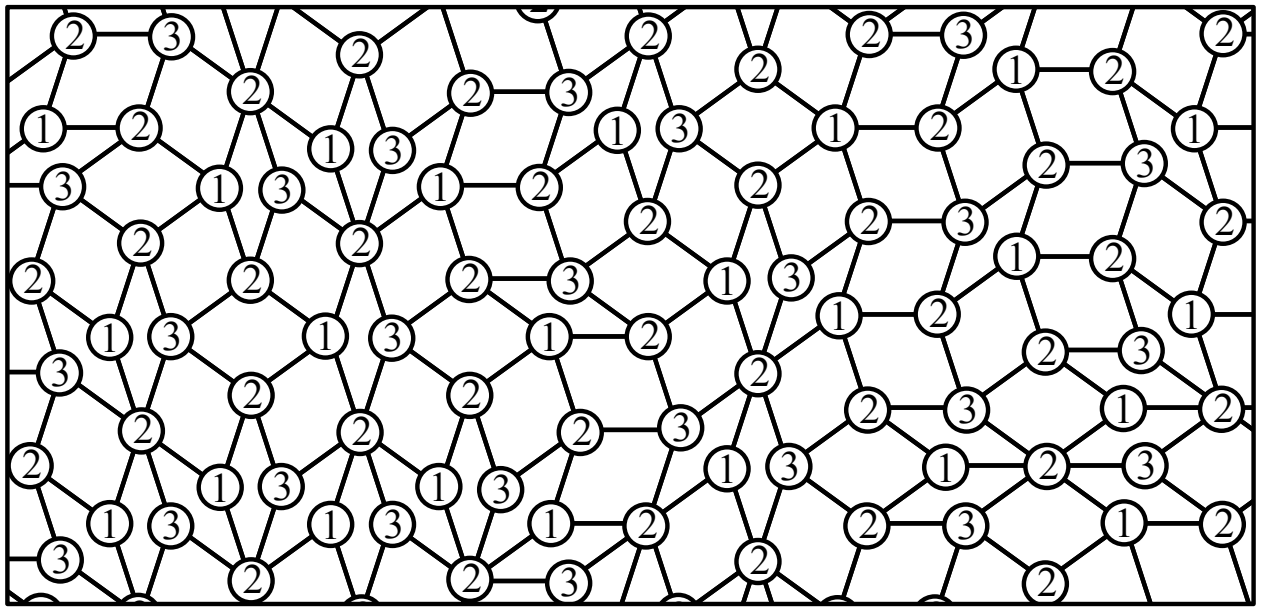
Theorem

There are infinitely many frieze patterns on GLB with three distinct values.



Theorem

There are infinitely many frieze patterns on GLB with three distinct values.



Here the substitution rule allows to prove "there is one":

- put a 1 on each supertile vertex,
- put a 2 on each vertex corresponding to a supertile edge,
- put a 3 on the others.

Now check the finitely many cases.

This implies several further questions. For instance

- Other examples? (Ammann-Beenker?)
- Other number of values (than 4 for Penrose, 3 for GLB)?
- General classification of rhomb tilings (periodic or aperiodic) for allowing frieze patterns.

This implies several further questions. For instance

- Other examples? (Ammann-Beenker?)
- Other number of values (than 4 for Penrose, 3 for GLB)?
- General classification of rhomb tilings (periodic or aperiodic) for allowing frieze patterns.

* *

*

Thank you!