Perfect colourings of regular graphs

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A *perfect colouring* of the vertices of a graph $G = (V, E)$ with m colours:

Colour V such that for all $v \in V$ with colour i holds: v is adjacent to a_{i1} vertices of colour 1, v is adjacent to a_{i2} vertices of colour 2, ... v is adjacent to a_{im} vertices of colour m.

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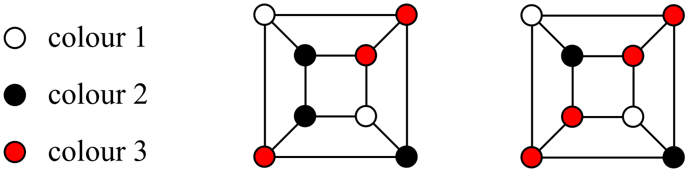
I.e., all white vertices have the same number of white neighbours, of black neighbours, ...



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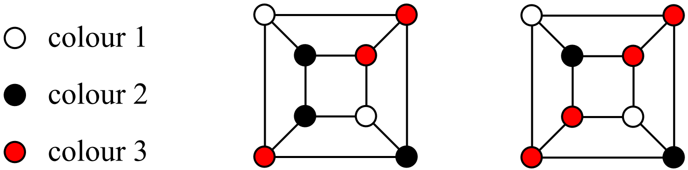

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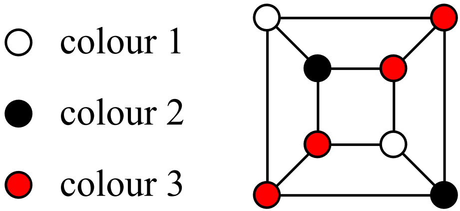
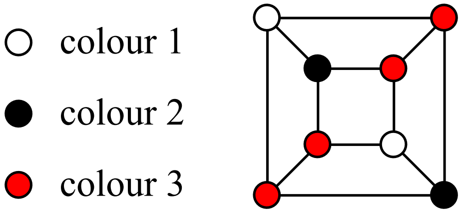
(Note that adjacent vertices are allowed to have the same colour)

All white vertices are adjacent to the same number a_{11} of white vertices, a_{12} of black vertices...



So here: $a_{11} = 0$, $a_{12} = 1$, $a_{13} = 2$, All white vertices are adjacent to the same number a_{11} of white vertices, a_{12} of black vertices...



So here: $a_{11} = 0$, $a_{12} = 1$, $a_{13} = 2$, ... altogether:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Questions:

- Find necessary and sufficient conditions on M to be the matrix of a perfect colouring.
- Find all perfect colourings of a given (class of) graph(s).

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Three simple necessary criteria:

Lemma (weak symmetry)

$a_{ij} = 0$ *iff* $a_{ji} = 0$.

Clear: if each red vertex has a white neighbour then each white vertex has a red neighbour.

OK: $\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$ Not OK: $\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$

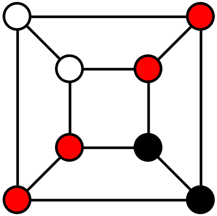
From now on let G be simple, connected, without loops.

Lemma (connected colour graph)

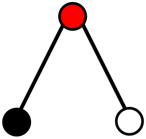
The corresponding colour graph

$$G' = (\{1, \ldots, m\}, \{\{i, j\} \mid a_{ij} \neq 0\})$$

is connected.



G



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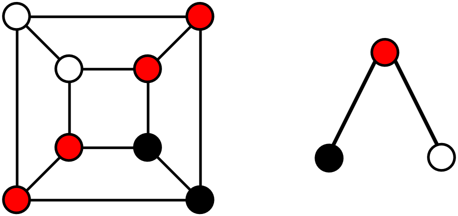

G G'

Clear: Since G is connected, there is a path from each colour to any colour.

Lemma (consistent counting)

For each cycle $v_1, v_2, \ldots, v_k, v_1$ $(k \geq 2)$ *holds:*

$$a_{v_1,v_2} \cdot a_{v_2,v_3} \cdots a_{v_k,v_1} = a_{v_1,v_k} \cdot a_{v_k,v_{k-1}} \cdots a_{v_2,v_1}$$

Simple: n_1 white vertices are adjacent to a_{12} black vertices each, and n_2 black vertices are adjacent to a_{21} white vertices each, hence: $n_1 a_{12} = n_2 a_{21}$.

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Ditto $n_2 a_{23} = n_3 a_{32}$ and $n_1 a_{13} = n_3 a_{31}$. Hence

$$n_1 a_{12} a_{23} a_{31} = n_2 a_{21} a_{23} a_{31} = n_3 a_{21} a_{32} a_{31} = n_1 a_{21} a_{32} a_{13},$$

hence $a_{12} a_{23} a_{31} = a_{13} a_{32} a_{21}$ and so on.

Theorem

Lemmas 1-3 are necessary and sufficient. I.e., a matrix $M \in \mathbb{N}^{m \times m}$ is the colouring matrix of a connected graph iff it has the properties weak symmetry, connected colour graph and consistent counting.

- Necessary: see above.
- Sufficient: construct graphs for each instance.

Application: List all m-colouring matrices of k-regular graphs.
k-regular: row sum equals k.

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Numbers of colouring matrices among all possible matrices (nonnegative integer entries, all row sums $= k$.)

$m \setminus k$	3	4	5
2	6 of 16	10 of 25	15 of 36
3	18 of 1000	64 of 3375	153 of 9261
4	72 of 16 000	485 of 1 500 625	2042 of 9 834 496

Counting is up to permutation. (This is the computationally most expensive part)

(Computations both in `SageMath` and `scilab`)

Application: List all m-colouring matrices of k-regular graphs.
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$m \setminus k$	3	4	5
2	< 1 sec	< 1 sec	< 1 sec
3	< 1 sec	2 sec	12 sec
4	3 min	55 min	one night

Counting is up to permutation. (This is the computationally most expensive part)

(Times for computations in `SageMath`)

All matrices for perfect 2-colorings...

...of 3-regular graphs

$\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

...of 4-regular graphs

$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

...of 5-regular graphs

$\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 4 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$
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All matrices for perfect 3-colorings...

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$$\begin{pmatrix}0&0&3\\0&0&3\\1&1&1\end{pmatrix} \begin{pmatrix}0&0&3\\0&0&3\\1&2&0\end{pmatrix} \begin{pmatrix}0&0&3\\0&1&2\\1&1&1\end{pmatrix} \begin{pmatrix}0&0&3\\0&1&2\\1&2&0\end{pmatrix} \begin{pmatrix}0&0&3\\0&2&1\\1&1&1\end{pmatrix} \begin{pmatrix}0&0&3\\0&2&1\\2&1&0\end{pmatrix} \begin{pmatrix}0&1&2\\1&0&2\\1&1&1\end{pmatrix} \begin{pmatrix}0&1&2\\1&1&1\\2&1&0\end{pmatrix} \begin{pmatrix}0&1&2\\1&2&0\\1&0&2\end{pmatrix}$$
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...of 4-regular graphs

$$\begin{pmatrix}0&0&4\\0&0&4\\1&1&2\end{pmatrix} \begin{pmatrix}0&0&4\\0&0&4\\1&2&1\end{pmatrix} \begin{pmatrix}0&0&4\\0&0&4\\1&3&0\end{pmatrix} \begin{pmatrix}0&0&4\\0&0&4\\2&2&0\end{pmatrix} \begin{pmatrix}0&0&4\\0&1&3\\1&1&2\end{pmatrix} \begin{pmatrix}0&0&4\\0&1&3\\1&2&1\end{pmatrix} \begin{pmatrix}0&0&4\\0&1&3\\1&3&0\end{pmatrix} \begin{pmatrix}0&0&4\\0&1&3\\2&2&0\end{pmatrix} \begin{pmatrix}0&0&4\\0&2&2\\1&1&2\end{pmatrix}$$
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$$\begin{pmatrix}1&0&4\\0&4&1\\1&1&3\end{pmatrix}\begin{pmatrix}1&0&4\\0&4&1\\2&1&2\end{pmatrix}\begin{pmatrix}1&0&4\\0&4&1\\3&1&1\end{pmatrix}\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}\begin{pmatrix}1&1&3\\1&1&3\\2&2&1\end{pmatrix}\begin{pmatrix}1&1&3\\1&3&1\\3&1&1\end{pmatrix}\begin{pmatrix}1&1&3\\1&4&0\\1&0&4\end{pmatrix}\begin{pmatrix}1&1&3\\1&4&0\\2&0&3\end{pmatrix}\begin{pmatrix}1&1&3\\1&4&0\\3&0&2\end{pmatrix}\begin{pmatrix}1&2&2\\1&0&4\\1&4&0\end{pmatrix}\begin{pmatrix}1&2&2\\1&1&3\\1&3&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&2\\1&2&2\\1&2&2\end{pmatrix}\begin{pmatrix}1&2&2\\1&3&1\\1&1&3\end{pmatrix}\begin{pmatrix}1&2&2\\1&4&0\\1&0&4\end{pmatrix}\begin{pmatrix}1&2&2\\2&1&2\\1&1&3\end{pmatrix}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}\begin{pmatrix}1&2&2\\2&2&1\\2&1&2\end{pmatrix}\begin{pmatrix}1&2&2\\2&3&0\\1&0&4\end{pmatrix}\begin{pmatrix}1&2&2\\2&3&0\\2&0&3\end{pmatrix}\begin{pmatrix}1&4&0\\1&0&4\\0&1&4\end{pmatrix}\begin{pmatrix}1&4&0\\1&0&4\\0&2&3\end{pmatrix}\begin{pmatrix}1&4&0\\1&0&4\\0&3&2\end{pmatrix}$$

$$\begin{pmatrix}1&4&0\\1&1&3\\0&1&4\end{pmatrix}\begin{pmatrix}1&4&0\\1&1&3\\0&2&3\end{pmatrix}\begin{pmatrix}1&4&0\\1&2&2\\0&1&4\end{pmatrix}\begin{pmatrix}1&4&0\\2&0&3\\0&1&4\end{pmatrix}\begin{pmatrix}1&4&0\\2&0&3\\0&2&3\end{pmatrix}\begin{pmatrix}1&4&0\\2&1&2\\0&1&4\end{pmatrix}\begin{pmatrix}1&4&0\\3&0&2\\0&1&4\end{pmatrix}\begin{pmatrix}2&0&3\\0&0&5\\1&2&2\end{pmatrix}\begin{pmatrix}2&0&3\\0&0&5\\1&3&1\end{pmatrix}\begin{pmatrix}2&0&3\\0&0&5\\1&4&0\end{pmatrix}\begin{pmatrix}2&0&3\\0&1&4\\1&2&2\end{pmatrix}$$

$$\begin{pmatrix}2&0&3\\0&1&4\\1&3&1\end{pmatrix}\begin{pmatrix}2&0&3\\0&1&4\\1&4&0\end{pmatrix}\begin{pmatrix}2&0&3\\0&1&4\\2&3&0\end{pmatrix}\begin{pmatrix}2&0&3\\0&2&3\\1&1&3\end{pmatrix}\begin{pmatrix}2&0&3\\0&2&3\\1&2&2\end{pmatrix}\begin{pmatrix}2&0&3\\0&2&3\\1&3&1\end{pmatrix}\begin{pmatrix}2&0&3\\0&2&3\\2&2&1\end{pmatrix}\begin{pmatrix}2&0&3\\0&2&3\\2&3&0\end{pmatrix}\begin{pmatrix}2&0&3\\0&3&2\\1&1&3\end{pmatrix}\begin{pmatrix}2&0&3\\0&3&2\\1&2&2\end{pmatrix}\begin{pmatrix}2&0&3\\0&3&2\\2&2&1\end{pmatrix}$$

$$\begin{pmatrix}2&0&3\\0&4&1\\1&1&3\end{pmatrix}\begin{pmatrix}2&0&3\\0&4&1\\2&1&2\end{pmatrix}\begin{pmatrix}2&1&2\\1&2&2\\1&1&3\end{pmatrix}\begin{pmatrix}2&1&2\\1&3&1\\2&1&2\end{pmatrix}\begin{pmatrix}2&1&2\\1&4&0\\1&0&4\end{pmatrix}\begin{pmatrix}2&1&2\\1&4&0\\2&0&3\end{pmatrix}\begin{pmatrix}2&3&0\\1&0&4\\0&1&4\end{pmatrix}\begin{pmatrix}2&3&0\\1&0&4\\0&2&3\end{pmatrix}\begin{pmatrix}2&3&0\\1&0&4\\0&3&2\end{pmatrix}\begin{pmatrix}2&3&0\\1&1&3\\0&1&4\end{pmatrix}\begin{pmatrix}2&3&0\\1&1&3\\0&2&3\end{pmatrix}$$

$$\begin{pmatrix}2&3&0\\1&2&2\\0&1&4\end{pmatrix}\begin{pmatrix}2&3&0\\2&0&3\\0&1&4\end{pmatrix}\begin{pmatrix}2&3&0\\2&0&3\\0&2&3\end{pmatrix}\begin{pmatrix}2&3&0\\2&1&2\\0&1&4\end{pmatrix}\begin{pmatrix}3&0&2\\0&0&5\\1&3&1\end{pmatrix}\begin{pmatrix}3&0&2\\0&0&5\\1&4&0\end{pmatrix}\begin{pmatrix}3&0&2\\0&1&4\\1&3&1\end{pmatrix}\begin{pmatrix}3&0&2\\0&1&4\\1&4&0\end{pmatrix}\begin{pmatrix}3&0&2\\0&2&3\\1&2&2\end{pmatrix}\begin{pmatrix}3&0&2\\0&2&3\\1&3&1\end{pmatrix}\begin{pmatrix}3&0&2\\0&3&2\\1&1&3\end{pmatrix}$$

$$\begin{pmatrix}3&0&2\\0&3&2\\1&2&2\end{pmatrix}\begin{pmatrix}3&0&2\\0&4&1\\1&1&3\end{pmatrix}\begin{pmatrix}3&1&1\\1&3&1\\1&1&3\end{pmatrix}\begin{pmatrix}3&1&1\\1&4&0\\1&0&4\end{pmatrix}\begin{pmatrix}3&2&0\\1&0&4\\0&1&4\end{pmatrix}\begin{pmatrix}3&2&0\\1&0&4\\0&2&3\end{pmatrix}\begin{pmatrix}3&2&0\\1&0&4\\0&3&2\end{pmatrix}\begin{pmatrix}3&2&0\\1&1&3\\0&1&4\end{pmatrix}\begin{pmatrix}3&2&0\\1&1&3\\0&2&3\end{pmatrix}\begin{pmatrix}3&2&0\\1&2&2\\0&1&4\end{pmatrix}$$

All matrices for perfect 4-colorings...

...of 3-regular graphs

$$\begin{pmatrix}0&0&0&3\\0&0&0&3\\0&0&0&3\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&0&3\\0&0&1&2\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&0&3\\0&0&2&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&1&2\\0&1&0&2\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&1&2\\0&1&2&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&1&2\\0&1&2&0\\1&2&0&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&2&1\\0&2&0&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&2&1\\0&2&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&2&1\\0&2&1&0\\2&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&3\\0&0&3&0\\0&1&0&2\\1&0&2&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&3&0\\0&1&1&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&3&0\\0&2&0&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&0&3&0\\0&2&0&1\\2&0&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&1&0&2\\0&0&1&2\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&1&0&2\\0&0&2&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&1&1&1\\0&1&2&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&1&1&1\\0&1&2&0\\2&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&3\\0&1&2&0\\0&1&0&2\\1&0&2&0\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&1&2&0\\0&1&1&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&3\\0&2&0&1\\0&0&2&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&0&1&2\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&0&2&1\\1&2&0&0\\2&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&1&0&2\\1&0&0&2\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&1&0&2\\1&0&2&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&1&1&1\\1&1&1&0\\2&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&1&2&0\\1&2&0&0\\1&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&2\\0&1&2&0\\1&2&0&0\\2&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&2&0&1\\1&0&2&0\\2&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&2&1&0\\1&1&0&1\\2&0&1&0\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&2&1&0\\1&1&1&0\\1&0&0&2\end{pmatrix}\begin{pmatrix}0&0&1&2\\0&2&1&0\\1&1&1&0\\2&0&0&1\end{pmatrix}\begin{pmatrix}0&0&3&0\\0&0&0&3\\1&0&0&2\\0&1&1&1\end{pmatrix}\begin{pmatrix}0&0&3&0\\0&0&1&2\\1&1&1&0\\0&1&0&2\end{pmatrix}\begin{pmatrix}0&0&3&0\\0&0&1&2\\2&1&0&0\\0&1&0&2\end{pmatrix}\begin{pmatrix}0&0&3&0\\0&1&0&2\\1&0&0&2\\0&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&2\\1&0&0&2\\0&0&2&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&1&0&2\\1&0&2&0\\0&1&0&2\\1&0&2&0\end{pmatrix}\begin{pmatrix}0&1&0&2\\1&0&2&0\\0&1&1&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}0&1&0&2\\1&0&2&0\\0&1&2&0\\1&0&0&2\end{pmatrix}\begin{pmatrix}0&1&0&2\\1&2&0&0\\0&0&0&3\\1&0&2&0\end{pmatrix}\begin{pmatrix}0&1&0&2\\1&2&0&0\\0&0&1&2\\1&0&2&0\end{pmatrix}\begin{pmatrix}0&1&0&2\\1&2&0&0\\0&0&2&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&1\\1&1&0&1\\1&0&2&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{pmatrix}\begin{pmatrix}0&1&2&0\\1&2&0&0\\1&0&0&2\\0&0&1&2\end{pmatrix}\begin{pmatrix}0&3&0&0\\1&0&0&2\\0&0&0&3\\0&1&2&0\end{pmatrix}\begin{pmatrix}0&3&0&0\\1&0&0&2\\0&0&1&2\\0&1&2&0\end{pmatrix}\begin{pmatrix}0&3&0&0\\1&0&0&2\\0&0&2&1\\0&1&1&1\end{pmatrix}\begin{pmatrix}0&3&0&0\\1&0&2&0\\0&1&0&2\\0&0&1&2\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&0&1&2\\0&1&2&0\\1&2&0&0\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&0&2&1\\0&2&0&1\\1&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&2\\0&0&2&1\\0&2&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&0&3&0\\0&2&0&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&0&2\\0&0&1&2\\1&1&1&0\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&0&2\\0&0&2&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&1&1\\0&1&2&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&2&0\\0&1&0&2\\1&0&2&0\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&2&0\\0&1&1&1\\1&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&2&0&1\\0&0&2&1\\1&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1&0&1&1\\0&2&0&1\\1&0&2&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1&0&2&0\\0&0&0&3\\1&0&0&2\\0&1&1&1\end{pmatrix}\begin{pmatrix}1&0&2&0\\0&0&1&2\\1&1&1&0\\0&1&0&2\end{pmatrix}\begin{pmatrix}1&0&2&0\\0&1&0&2\\1&0&0&2\\0&1&1&1\end{pmatrix}\begin{pmatrix}1&2&0&0\\1&0&0&2\\0&0&0&3\\0&1&2&0\end{pmatrix}\begin{pmatrix}1&2&0&0\\1&0&0&2\\0&0&1&2\\0&1&2&0\end{pmatrix}\begin{pmatrix}1&2&0&0\\1&0&0&2\\0&0&2&1\\0&1&1&1\end{pmatrix}\begin{pmatrix}1&2&0&0\\1&0&2&0\\0&1&0&2\\0&0&1&2\end{pmatrix}$$

...of 4-regular graphs

$$\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&0&4\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&0&4\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&1&3\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&1&3\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&2&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&2&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&2&2\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&3&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&3&1\\1&2&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&0&3\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\1&2&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\1&3&0&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&1&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&2&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&2&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&3&0\\1&1&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&3&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&0&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&0&2\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&1&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\1&2&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&0&1\\1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&0&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&1&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&1&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&1&0\\3&1&0&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&1&0&3\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&1&1&2\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&1&2&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&0&2\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&0&2\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&0&2\\2&0&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&1&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&1&1\\2&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&3&0&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&3&0&1\\2&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&3&0&1\\3&0&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&0&4\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&1&3\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&1&3\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&2&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&2&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&3&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&3&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&1&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&1&2\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&2&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\1&2&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&1&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&1&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&2&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&2&0\\3&1&0&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&1&0&3\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&1&1&2\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&1&2&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&0&2\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&0&2\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&0&2\\2&0&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&1&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&1&1\\2&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&1&3\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&2&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&2&2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&2&2\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&3&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&3&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&3&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&2&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&2&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&3&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&3&0\\3&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&0&3\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&0&3\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&1&2\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&1&2\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&1&2\\2&0&2&0\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&2&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&2&1\\2&0&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&3&0&1\\0&0&3&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&0&4\\0&3&0&1\\0&0&3&1\\2&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&3\\0&0&1&3\\1&1&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&0&1&3\\1&1&2&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&0&3&1\\1&3&0&0\\3&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&1&0&3\\1&0&0&3\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&1&0&3\\1&0&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&1&0&3\\1&0&3&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&1&2&1\\1&2&1&0\\3&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&1&3&0\\1&3&0&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&1&3&0\\1&3&0&0\\2&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&3\\0&1&3&0\\1&3&0&0\\3&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&2&1&1\\1&1&2&0\\3&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&0&1\\3&0&1&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&1&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&1&0\\2&0&0&2\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&1&0\\3&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&3&0&1\\1&0&3&0\\3&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&1&1\\3&0&1&0\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&2&0\\1&0&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&2&0\\2&0&0&2\end{pmatrix}\begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&2&0\\3&0&0&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&0&4\\1&0&3&0\\1&3&0&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&1&3\\1&1&2&0\\1&3&0&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&2&2\\1&1&0&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&2&2\\1&1&1&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&2&2\\1&1&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&2&2\\1&2&0&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&2&2\\1&2&1&0\\1&2&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&0&2&2\\2&2&0&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&2&2\\2&2&0&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&0&3\\1&0&2&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&0&3\\1&0&3&0\\1&2&0&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&0&3\\1&0&3&0\\1&3&0&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&1&2\\1&1&1&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&1&2\\1&1&2&0\\1&2&0&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&1&2\\2&1&1&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&1&1&2\\2&1&1&0\\2&2&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&1&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&2&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&2&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&3&0\\1&2&0&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\2&0&0&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\2&0&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&0&2\\2&0&2&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&1&1\\1&1&0&2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&2&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&1&1\\1&1&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&1&1\\2&1&0&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&1&1\\2&1&1&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&2&2&0\\2&2&0&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&3&0&1\\1&0&1&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&3&0&1\\1&0&2&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&3&0&1\\1&0&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&3&0&1\\2&0&1&1\\2&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&3&0&1\\2&0&2&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&3&1&0\\2&1&1&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&0&3\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&0&3\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&0&3\\0&2&2&0\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&1&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&1&2\\0&2&1&1\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&0&4\\2&0&0&2\\0&2&1&1\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&1&3\\1&1&2&0\\0&1&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&4&0\\0&0&1&3\\1&1&2&0\\0&2&0&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&1&3\\2&1&1&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&1&3\\2&1&1&0\\0&2&0&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&1&3\\3&1&0&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&1&3\\3&1&0&0\\0&2&0&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&2&2\\1&2&1&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&2&2\\2&2&0&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&0&4&0\\1&1&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&0&3\\0&1&1&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&0&3\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&0&3\\0&2&2&0\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&1&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&0&3\\2&0&0&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&1&2\\1&1&0&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&1&2\\1&1&2&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&1&2\\2&1&1&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&1&2\\3&1&0&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&1&3&0\\1&1&0&2\\0&0&1&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&4&0\\0&2&0&2\\1&0&0&3\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&2&0&2\\1&0&1&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&2&0&2\\2&0&0&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&2&2&0\\1&1&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&0&4&0\\0&3&1&0\\1&1&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&0&4\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&1&3\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&2&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&2&2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&3&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&0&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&1&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&2&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&3&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&0&2\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&0&2\\2&0&2&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&1&1\\2&0&1&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&2&0\\1&0&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&2&0\\2&0&0&2\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&1&2&0\\0&1&0&3\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&1&2&0\\0&1&3&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&0&4\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&0&4\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&1&3\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&1&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&2&2\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&2&2\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&2&2\\2&0&2&0\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&3&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&3&1\\2&0&1&1\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&0&1&2\\1&1&0&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&0&1&2\\1&1&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&0&2&1\\1&2&0&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&0&2&1\\1&2&1&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&0&3&0\\1&3&0&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&1&0&2\\1&0&1&2\\1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&2\\1&1&0&2\\1&0&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&1&1&1\\1&1&1&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&1&1&1\\1&1&2&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&1&2&0\\1&2&1&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&1&2&0\\1&2&1&0\\2&0&0&2\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&2&0&1\\1&0&2&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&2&0&1\\1&0&3&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&2&1&0\\1&1&2&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&2&1&0\\1&1&2&0\\2&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&2\\1&3&0&0\\1&0&3&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&1&1&2\\1&3&0&0\\1&0&3&0\\2&0&0&2\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&0&3&0\\1&1&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&1&0&2\\2&0&2&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&1&0&2\\3&0&1&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&3&0&0\\1&0&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&3&0&0\\1&0&0&3\\0&0&2&2\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&3&0&0\\1&0&1&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&1&3&0\\1&3&0&0\\2&0&0&2\\0&0&1&3\end{pmatrix}$$

$$\begin{pmatrix}0&2&0&2\\2&0&0&2\\0&0&1&3\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&0&0&2\\0&0&2&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&0&0&2\\0&0&3&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&0&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&1&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&2&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&3&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&0&4\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&1&3\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&1&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&2&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&3&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&0&3\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&0&3\\0&2&2&0\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&3&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&3&0\\0&2&0&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&1&2\\1&1&0&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&1&2\\1&1&2&0\\0&1&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&2&2&0\\1&1&0&2\\1&0&1&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&1&0&2\\1&0&3&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&0&2&0\\1&1&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&0&3\\0&0&2&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&1&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&0&4\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&0&4\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&1&3\\0&1&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&1&3\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&2&2&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&3&1\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&3&1\\0&2&1&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&0&3\\0&0&2&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&1&2\\0&0&1&3\end{pmatrix}$$

$$\begin{pmatrix}0&4&0&0\\1&0&3&0\\0&2&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&0&4\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&1&3\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&1&3\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&2&2\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&3&1\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&2&0\\0&1&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&2&0\\0&1&0&3\\0&0&2&2\end{pmatrix}$$

...and of 5-regular graphs:

$$\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&1&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&2&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&3&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&0&3\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&0&3\\0&2&2&0\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&3&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&3&0\\0&2&0&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&1&2\\1&1&0&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&0&1&2\\1&1&2&0\\0&1&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&2&2&0\\1&1&0&2\\1&0&1&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\1&1&0&2\\1&0&3&0\\0&1&0&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&0&2&0\\1&1&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&0&3\\0&0&2&2\end{pmatrix} \begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&1&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&0&4\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&0&4\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&1&3\\0&1&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&1&3\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&2&2&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&3&1\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&3&1\\0&2&1&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&0&3\\0&0&2&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&1&2\\0&0&1&3\end{pmatrix}$$

$$\begin{pmatrix}0&4&0&0\\1&0&3&0\\0&2&0&2\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&0&4\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&1&3\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&1&3\\0&1&3&0\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&2&2\\0&1&2&1\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&3&1\\0&1&1&2\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&2&0\\0&1&0&3\\0&0&1&3\end{pmatrix} \begin{pmatrix}0&4&0&0\\1&1&2&0\\0&1&0&3\\0&0&2&2\end{pmatrix}$$

...and of 5-regular graphs:

[2042 matrices]

Perfect colourings seem to be not well-studied. But there is an early theorem of H. Sachs from 1966. Can be found in

C. Godsil, G. Royle: *Algebraic graph theory*, Springer (2001)

Theorem

Let M be the adjacency matrix of some graph G and let A be the matrix of the colour graph of some perfect colouring of G. Then each eigenvalue of A is an eigenvalue of M (w.r.t multiple counting).

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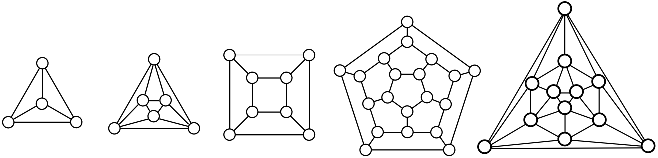
Application of the application: find all 2-, 3-, 4-colourings of the Platonic graphs



Perfect colurings of the Platonic graphs

The eigenvalues of the adjacency matrices of the Platonic graphs:

G	tetrahedron	cube	octahedron
	$-1^3, 3$	$-3, -1^3, 1^3, 3$	$-2^2, 0^3, 4$

G	dodecahedron	icosahedron
	$-\sqrt{5}^3, -2^4, 0^4, 1^5, \sqrt{5}^3, 3$	$-\sqrt{5}^3, -1^5, \sqrt{5}^3, 5$

In order to find candidates for perfect colourings: browse the lists, collect all matrices with the correct eigenvalues.

Candidates for perfect 2-, 3- and 4-colourings of the tetrahedron:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

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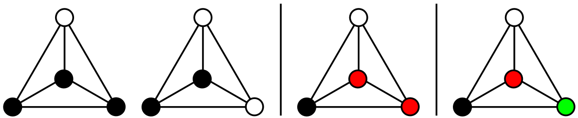
All are matrices for perfect colourings of the tetrahedron:



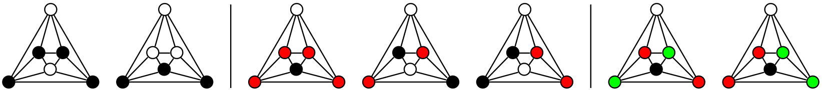
Candidates for perfect 2-, 3- and 4-colourings of the octahedron:

1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.
3. 4 colours: $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

Candidates for perfect 2-, 3- and 4-colourings of the octahedron:

1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.
3. 4 colours: $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

All but one are matrices for perfect colourings of the octahedron::



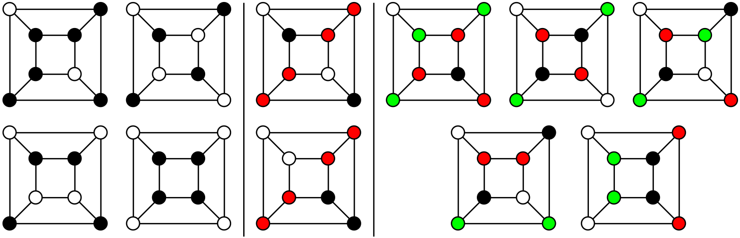
Candidates for perfect 2-, 3- and 4-colourings of the cube:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.
3. 4 col's: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

Candidates for perfect 2-, 3- and 4-colourings of the cube:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.
3. 4 col's: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

All are matrices for perfect colourings of the cube:



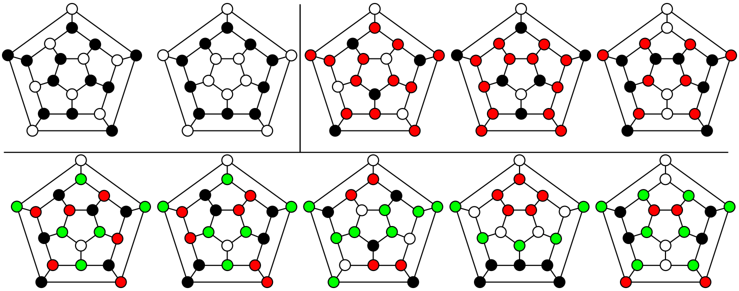
Candidates for perfect 2-, 3- and 4-colourings of the dodecahedron:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.
3. 4 col's: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Candidates for perfect 2-, 3- and 4-colourings of the dodecahedron:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.
3. 4 col's: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

All are matrices for perfect colourings of the dodecahedron:



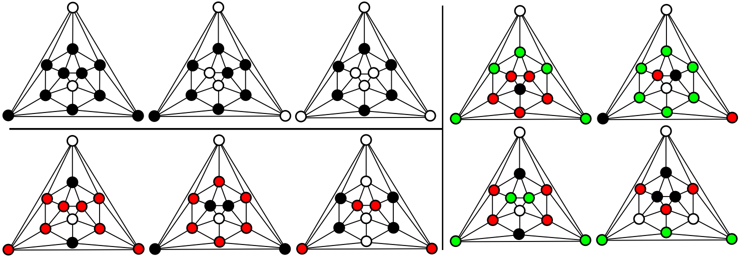
Candidates for perfect 2-, 3- and 4-colourings of the icosahedron:

1. 2 colours: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.
3. 4 colours: $\begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$

Candidates for perfect 2-, 3- and 4-colourings of the icosahedron:

1. 2 colours: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.
3. 4 colours: $\begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$

All are matrices for perfect colourings of the icosahedron:



That was all done in the paper with Joseph (published in 2021).

Recent results:

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Recent results:

The eigenvalues of the graphs of the simplex and the cube for in dimension 4 and 5:

G	eigenvalues
4-simplex	$-1^4, 4$
5-simplex	$-1^5, 5$
4-cube	$-4, -2^4, 0^6, 2^4, 4$
5-cube	$-5, -3^5, -1^{10}, 1^{10}, 3^5, 5$

Perfect colurings of the 4-simplex and the 5-simplex

Candidates for perfect 2-, 3- and 4-colourings of the 4-simplex:

1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

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Candidates for perfect 2-, 3- and 4-colourings of the 4-simplex:

1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

...and for the 5-simplex:

1. 2 colours: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

Perfect colurings of the 4-simplex and the 5-simplex

Candidates for perfect 2-, 3- and 4-colourings of the 4-simplex:

1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

...and for the 5-simplex:

1. 2 colours: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

All of them correspond to perfect colourings — trivially.

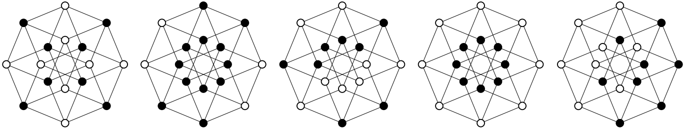
Perfect colurings of the 4-cube

Candidates for the perfect 2-, 3-, (and 4-colourings) of the 4-cube:

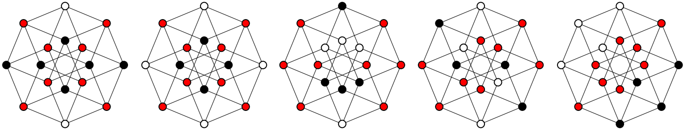
1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$
2. 3 colours:
$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix},$
$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 2 \end{pmatrix}$
3. 4 colours: [23 matrices]

Here it is more interesting:

5 (out of 6) perfect 2-colourings exist:



5 (out of 10) perfect 3-colourings exist:



Showing that a perfect colouring with matrix A does *not* exist requires more ideas.

- Exclude $\left(\begin{smallmatrix} 0 & 4 \\ 2 & 2 \end{smallmatrix}\right)$ by brute force.

Showing that a perfect colouring with matrix A does *not* exist requires more ideas.

- Exclude $\left(\begin{smallmatrix} 0 & 4 \\ 2 & 2 \end{smallmatrix}\right)$ by brute force.
- Identify two colours in

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 2 \end{pmatrix}$$

  yields 2-colourings with matrix $\left(\begin{smallmatrix} 0 & 4 \\ 2 & 2 \end{smallmatrix}\right)$, which we just excluded.

Perfect colurings of the 5-cube

Candidates for the perfect 2-, 3-, (and 4-colourings) of the 5-cube:

1. 2 colours
   $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$
2. 3 colours
   $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & 3 \end{pmatrix},$

   $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix},$

   $\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix}$
3. 4 colours: [57 matrices]

Again it is more interesting:

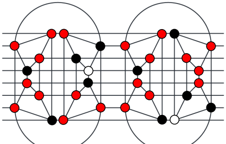
Perfect colurings of the 5-cube

Candidates for the perfect 2-, 3-, (and 4-colourings) of the 5-cube:

1. 2 colours
   $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$
2. 3 colours
   $$\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 1 & 3 \end{pmatrix},$$
   $$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix},$$
   $$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix}$$
3. 4 colours: [57 matrices]

Again it is more interesting:

First challenge: find a nice image of the edge graph of the 5-cube:



Result:

- 6 (out of 9) perfect 2-colourings exist,
- 8 (out of 16) perfect 3-colourings exist.

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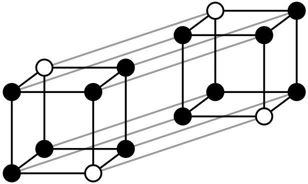
A perfect 3-colouring of a 4-cube with colour adjacency matrix A yields a perfect 3-colouring of a 5-cube with colour adjacency matrix $A + I$:



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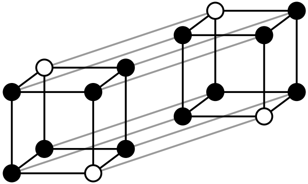

The sixth one $\left(\begin{smallmatrix} 0 & 5 \\ 5 & 0 \end{smallmatrix}\right)$ follows from the fact that the cube graph is bipartite.

In order to exclude the three impossible 2-colourings:

- Two violate a corollary of the consistent counting condition
- Third one: brute force

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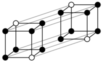
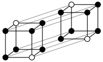
- Existence of five of them: 
- Number 6 and 7: variation of 

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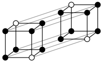
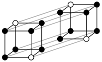
- Existence of five of them: 
- Number 6 and 7: variation of 
- Eighth one: three slides back

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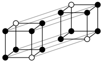
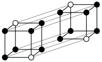
- Existence of five of them: 
- Number 6 and 7: variation of 
- Eighth one: three slides back
- Nonexistence of the remaining ones: as before, reduce to impossible 2-colourings.

More in:

Joseph R.C. Damasco, D.F.:
Perfect colourings of regular graphs, *AMS Contemporary Mathematics* 764 (2021)

D.F.: Perfect colourings of simplices and hypercubes in dimension four and five with few colours, *Bulletin Mathematique de la Société des Sciences Mathématiques de Roumanie* 67 (2024) 191-201.

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Thank you!