Cyclotomic Aperiodic Substitution Tilings

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Visualien der Breitbandkatze

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Motivation

• The Nobel Prize in Chemistry 2011 awarded to Dan Shechtman for the discovery of quasicrystals

[Image of Dan Shechtman]

Dan Shechtman – Facts, Nobelprize.org, 2011
Photo: U. Montan

• Penrose-Tiling
Motivation

- Aperiodic substitution tilings based on Girih Tiles (Islamic Architecture)

Makovicky 1992 + Lu & Steinhardt 2007

- My own version:

Lu, P.J.; Steinhardt, P.J.

Makovicky, E. Hargittai, I. (Ed.)
Motivation

- Do aperiodic substitution tilings with other symmetries exist?

  Yes!
  - 8-fold: Ammann-Beenker-Tiling
  - 12-fold: Shield-Tiling
    Socolar-Tiling
    Stampfli-Gähler / Ship-Tiling
  - 7 and 14-fold: Nischke & Danzer
  - 11-fold: Maloney

D. Frettlöh, F. Gähler, and E. O. Harris. 
Tilings encyclopedia. available at http://tilings.math.uni-bielefeld.de/.


... and references therein ...
Motivation

• Do generalizations for aperiodic substitution tilings with any $n$-fold symmetries exist?

Yes!
- Warrington 1988
- Nischke & Danzer 1996
- Harris 2005 … generalization of a rhomb tiling …
- Hibma 2015 … generalization of Lançon-Billard / Binary-tiling and more …
Motivation

• ... with focus on individual n-fold symmetry?

Yes!

- Gähler, Kwan, Maloney 2015
- Maloney 2015
- Kari & Rissanen 2016

... formal proof that aperiodic substitution tilings with any individual n-fold symmetry exist.
Motivation

Warrington, D.H.

Nischke, K.P.; Danzer, L.

Harriss, E.O.

Hibma, T.


Gähler, F.; Kwan, E.E.; Maloney, G.R.

Maloney, G.R.

Kari, J.; Rissanen, M.
Motivation

• But is there a convinient way to describe all those substitution tilings?

  In detail …
  – Substitution matrix
  – Minimal inflation multiplier

Well …
Definitions

- A “tile” in $\mathbb{R}^d$ is defined as a nonempty compact subset of $\mathbb{R}^d$ which is the closure of its interior.
- A “tiling” in $\mathbb{R}^d$ is a countable set of tiles, which is a covering as well as a packing of $\mathbb{R}^d$. The union of all tiles is $\mathbb{R}^d$. The intersection of the interior of two different tiles is empty.
- A “patch” is a finite subset of a tiling.
- A tiling is called “aperiodic” if no translation maps the tiling to itself.
- “Prototiles” serve as building blocks for a tiling.
- Within this article the term “substitution” means, that a tile is expanded with a linear map—the “inflation multiplier”—and dissected into copies of prototiles in original size—the “substitution rule”.
- A “supertile” is the result of one or more substitutions, applied to a single tile. Within this article we use the term for one substitutions only.
- We use $\zeta_n^k$ to denote the $n$-th roots of unity so that $\zeta_n^k = e^{2\pi i k/n}$ and its complex conjugate $\overline{\zeta_n^k} = e^{-2\pi i k/n}$.
- $\mathbb{Q} (\zeta_n)$ denotes the $n$-th cyclotomic field. Please note that $\mathbb{Q} (\zeta_n) = \mathbb{Q} (\zeta_{2n})$ for odd $n$.
- The maximal real subfield of $\mathbb{Q} (\zeta_n)$ is $\mathbb{Q} (\zeta_n + \overline{\zeta_n})$.
- $\mathbb{Z} [\zeta_n]$ denotes the the ring of algebraic integers in $\mathbb{Q} (\zeta_n)$.
- $\mathbb{Z} [\zeta_n + \overline{\zeta_n}]$ denotes the the ring of algebraic integers (which are real numbers) in $\mathbb{Q} (\zeta_n + \overline{\zeta_n})$.
- We use $\mu_{n,k}$ to denote the $k$-th diagonal of a regular $n$-gon with side length $\mu_{n,1} = \mu_{n,n-1} = 1$.
- $\mathbb{Z} [\mu_n] = \mathbb{Z} [\mu_{n,1}, \mu_{n,2}, \mu_{n,3} \ldots \mu_{n,[n/2]}]$ denotes the ring of the diagonals of a regular $n$-gon.
Diagonals or regular n-gons vs. roots of unity

\[ \mu_{n,1} = \zeta_{2n}^0 = 1 \]
\[ \mu_{n,2} = \zeta_{2n}^1 + \zeta_{2n}^{-1} = \zeta_{2n}^1 + \overline{\zeta_{2n}^1} \]
\[ \mu_{n,3} = \zeta_{2n}^{-2} + \zeta_{2n}^0 + \zeta_{2n}^2 = \zeta_{2n}^0 + \zeta_{2n}^2 + \overline{\zeta_{2n}^2} \quad (n > 4) \]
\[ \mu_{n,4} = \zeta_{2n}^3 + \zeta_{2n}^1 + \zeta_{2n}^{-1} + \zeta_{2n}^{-3} = \zeta_{2n}^1 + \overline{\zeta_{2n}^1} + \zeta_{2n}^3 + \overline{\zeta_{2n}^3} \quad (n > 5) \]

\[ \mu_{n,k} = \sum_{i=0}^{k-1} \zeta_{2n}^{2i-k+1} \quad (n > k \geq 1) \]
Diagonal product formula

- Nischke & Danzer 1996

Steinbach 1997

\[ \mu_{n,1} \mu_{n,k} = \mu_{n,k} \]

\[ \mu_{n,2} \mu_{n,k} = \mu_{n,k-1} + \mu_{n,k+1} \quad (1 < k \leq \lfloor n/2 \rfloor) \]

\[ \mu_{n,3} \mu_{n,k} = \mu_{n,k-2} + \mu_{n,k} + \mu_{n,k+2} \quad (2 < k \leq \lfloor n/2 \rfloor) \]

\[ \mu_{n,4} \mu_{n,k} = \mu_{n,k-3} + \mu_{n,k-1} + \mu_{n,k+1} + \mu_{n,k+3} \quad (3 < k \leq \lfloor n/2 \rfloor) \]

Or, more generally:

\[ \mu_{n,1} \mu_{n,k} = \sum_{i=1}^{h} \mu_{n,k-h+1+2i} \quad (1 \leq h \leq k \leq \lfloor n/2 \rfloor) \]

- Positive sums of \( \mu_{n,k} \) form a commutative semiring ...

\[ \mathbb{N}_0 [\mu_n] \subset \mathbb{Z} [\mu_n] \]

Nischke, K.P.; Danzer, L.

Areas vs. diagonals or regular n-gons

- Areas of proto tiles and super tiles with vertices supported by $\mathbb{Z} [\zeta_n]$ or $\mathbb{Q} (\zeta_n)$ can be described as sums of triangles spanned by roots of unity.

- The areas of triangles spanned by roots of unity and diagonals of regular n-gons have similar proportions:

$$A_{n,k} = \sin \left( \frac{k\pi}{n} \right) \quad \mu_{n,k} = \frac{\sin \left( \frac{k\pi}{n} \right)}{\sin \left( \frac{\pi}{n} \right)}$$
A substitution tiling with $l \geq 2$, $l \in \mathbb{N}$ prototiles and substitution rules is partially characterized by its substitution matrix $M \in \mathbb{N}_0^{l \times l}$ with an eigenvalue $\lambda$ and an left eigenvector $x_A$.

$$\lambda x_A^T = x_A^T M$$

The elements of the left eigenvector $x_A$ contain the areas of the prototiles $A_k = A(P_k)$. Since $M \in \mathbb{N}_0^{l \times l}$, the elements of $x_A$ generate a ring of algebraic integers which are real numbers.

The elements of the right eigenvector $x_f$ represent the frequencies of the prototiles $f_k = f(P_k)$, so that:

$$\lambda x_f = M x_f$$
Eingenwert problem ...

- With $\lambda \in \mathbb{N}_0 [\mu_n]$ we can conclude:

$$x_A = \begin{pmatrix} \mu_{n,n-1} \\ \vdots \\ \mu_{n,2} \\ \mu_{n,1} \end{pmatrix} = \frac{1}{A_{n,1}} \begin{pmatrix} A_{n,n-1} \\ \vdots \\ A_{n,2} \\ A_{n,1} \end{pmatrix}$$

... and derive a general approach for the substitution matrices. Examples for $n=8$ and $n=9$:

$$M_8 = c_4 \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_9 = c_4 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Minimal inflation multiplier ...

- For $n \geq 4$ we can derive ...

\[
|\eta_{\text{min}}| = \left| \zeta_{2n}^{1} + \overline{\zeta_{2n}^{1}} \right| = \mu_{n,2}, \text{ odd } n
\]

\[
|\eta_{\text{min}}| = \left| 1 + \zeta_{2n}^{1} \right| = \sqrt{\mu_{n,2} + 2}, \text{ even } n
\]
Thank you ...
CAST Δ7-1.1.1
Kunstpavillon Munich, Germany 2017-11
Installation + Photo by Stefan Pautze
• B. Grünbaum and G. C. Shephard.  

• D. Frettlöh, F. Gähler, and E. O. Harris.  
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