Classification of empty lattice 4-simplices

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Empty lattice $d$-simplices

- A $d$-polytope is the convex hull of a finite set of points in some $\mathbb{R}^d$. Its *dimension* is the dimension of its affine span. (E.g., 2-polytopes = Convex polygons, etc.)
- A $d$-polytope is a *$d$-simplex* if its vertices are exactly $d + 1$. Equivalently, if they are affinely independent. (Triangle, tetrahedron, …)
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**Definition**

A *lattice polytope* $P \subset \mathbb{R}^d$ is a polytope with integer vertices. It is:
- **hollow** if it has no integer points in its interior.
- **empty** if it has no integer points other than its vertices.

In particular, an *empty* $d$-*simplex* is the convex hull of $d + 1$ affinely independent integer points and not containing other integer points.

Empty 2 and 3-simplices and hollow 2-polytope.
The normalized volume $\text{Vol}(P)$ of a lattice polytope $P$ equals its Euclidean volume $\text{vol}(P)$ times $d!$. 
Volume, width

- The **normalized volume** $\text{Vol}(P)$ of a lattice polytope $P$ equals its Euclidean volume $\text{vol}(P)$ times $d!$.
  It is always and integer, and for a lattice simplex $\Delta = \text{conv}\{v_1, \ldots, v_{d+1}\} \subset \mathbb{R}^d$ it coincides with its determinant:
  
  $\text{Vol}(\Delta) = \det \begin{vmatrix} v_1 & \cdots & v_{d+1} \\ 1 & \cdots & 1 \end{vmatrix}$
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The width of $P \subset \mathbb{R}^d$ with respect to a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ equals the difference $\max_{x \in P} f(x) - \min_{x \in P} f(x)$.

$$\text{width}(P, f) = 4$$
The \textit{normalized volume} $\text{Vol}(P)$ of a lattice polytope $P$ equals its Euclidean volume $\text{vol}(P)$ times $d!$. It is always an integer, and for a lattice simplex $\Delta = \text{conv}\{v_1, \ldots, v_{d+1}\} \mathbb{R}^d$ it coincides with its determinant:

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The \textit{width} of $P \subset \mathbb{R}^d$ with respect to a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ equals the difference $\max_{x \in P} f(x) - \min_{x \in P} f(x)$. We call \textit{(lattice) width} of $P$ the minimum width of $P$ with respect to integer functionals.

$$\text{width}(P) = 2$$
We call rational (lattice) diameter of $P$ to the maximum length of a rational segment contained in $P$ (with “length” measured with respect to the lattice).

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\[ \text{diam}(P) = 4.5 \]

- It equals the inverse of the *first successive minimum* of $P - P$. In particular, Minkowski’s First Theorem implies:

\[ \text{Vol}(P) \leq d! \text{diam}(P)^d. \]

- Not to be mistaken with the (integer) lattice diameter $= \text{max. lattice length of an integer segment in } P$. 

\[ \delta \]
What do we know about empty lattice $d$-simplices?

We write $P \cong_{\mathbb{Z}} Q$ meaning $Q = \phi(P)$ for some unimodular affine integer transformation, $\phi$. 

The only empty $1$-simplex is the unit segment.

The only empty $2$-simplex is the unimodular triangle (Pick's Theorem).

Empty lattice $3$-simplices are completely classified: Theorem (White 1964) Every empty tetrahedron of determinant $q$ is equivalent to $T(p, q) := \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 0, 1), (p, q, 1)\}$ for some $p \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Moreover, $T(p, q) \cong_{\mathbb{Z}} T(p_0, q)$ if and only if $p_0 = \pm p \pm 1 \pmod{q}$. 

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What do we know about empty lattice 3-simplices

In particular, they all have width 1, i.e., they are between two parallel lattice hyperplanes.

In this picture, they have width 1 with respect to the functional $f(x, y, z) = z$. 
In contrast, a full classification of empty lattice 4-simplices is not known. If we look at their width, we know that:

1. There are infinitely many of width one.
2. There are infinitely many of width 2 (Haase-Ziegler 2000).
3. The amount of empty 4-simplices of width greater than 2 is finite:

Proposition (Blanco-Haase-Hofmann-Santos, 2016+)

1. For each $d$, there is a $w_1(d)$ such that for every $n \geq 2$ all but finitely many $d$-polytopes with $n$ lattice points have width $\leq w_1(d)$.
2. $w_1(4) = 2$. 
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Proposition (Blanco-Haase-Hofmann-Santos, 2016+)

1. For each $d$, there is a $w^\infty(d)$ such that for every $n \in \mathbb{N}$ all but finitely many $d$-polytopes with $n$ lattice points have width $\leq w^\infty(d)$.
2. $w^\infty(4) = 2$. 
What do we know about empty lattice 4-simplices?

**Theorem (Haase-Ziegler, 2000)**

Among the 4-dimensional empty simplices of determinant $D \leq 1000$,

1. All simplices of width 3 have determinant $D \leq 179$, with a (unique) smallest example, of determinant $D = 41$, and a (unique) example of determinant $D = 179$.
2. There is a unique class of width 4, with determinant $D = 101$,
3. There are no simplices of width $w \geq 5$.

**Conjecture (Haase-Ziegler, 2000)**

The above list is complete. That is, there are no empty 4-simplices of width $w > 2$ and determinant $D > 179$.

**Our main result (I.V.-Santos, 2017+)**

This conjecture is true.
What do we know about empty lattice 4-simplices?

**Theorem (Haase-Ziegler, 2000)**

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**Conjecture (Haase-Ziegler, 2000)**

The above list is complete. That is, there are no empty 4-simplices of width \( > 2 \) and determinant \( > 179 \).

**Our main result (I.V.-Santos, 2017+)**

This conjecture is true.
Our main theorem follows from the combination of a theoretical volume upper bound and an enumeration of all empty lattice 4-simplices of volume less than 7600:
Main theorem

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**Theorem 1 (I.V.-Santos, 2017+)**

There is no hollow 4-simplex of width $> 2$ with determinant greater than 7109.
Our main theorem follows from the combination of a theoretical volume upper bound and an enumeration of all empty lattice 4-simplices of volume less than 7600:

**Theorem 1 (I.V.-Santos, 2017+)**

There is no hollow 4-simplex of width $> 2$ with determinant greater than 7109.

**Theorem 2 (I.V.-Santos, 2017+)**

Up to determinant $\leq 7600$, all empty 4-simplices of width larger than two have determinant in $[41, 179]$ and are as described explicitly by Haase and Ziegler.
Theorem 1: case “\(P\) can be projected to a hollow polytope”

Let \(P\) be a empty lattice 4-simplex of width greater than two. We separate in two cases:

**Case 1** *There is an integer projection \(\pi : P \rightarrow Q\) to a hollow 3-polytope \(Q\).* Then, \(Q\) will also have width greater than two, and there are only the following five hollow 3-polytopes of width greater than two (Averkov, Krümpelmann and Weltge, 2015).

![Figure 1: The \(\mathbb{Z}_3\)-maximal integral lattice-free polytopes with lattice width 2. For further reference, the polytopes are labeled by a pair of indices \((i, j)\), where \(i\) is the number of facets and \(j\) the lattice diameter (defined at the end of the introduction).](image1)

**Figure:** The five hollow lattice 3-polytopes of width greater than two. Their normalized volumes are 27, 25, 27, 27 and 27, respectively.
Theorem 1: case “$P$ can be projected to a hollow polytope”

We can show that in this case:

Proposition

If a hollow 4-simplex $P$ of width at least three can be projected to a hollow lattice 3-polytope $Q$, then

$$\text{Vol}(P) \leq \text{Vol}(Q) \leq 27.$$  

Sketch of proof: The volume of $P$ equals the volume of $Q$ times the length of the maximum fiber in $P$. This fiber is projecting to a lattice point and $P$ is hollow, which implies the fiber to have length at most one.
Thm 1: case “$P$ cannot be projected to a hollow polytope”

Case 2 There is no integer projection of $P$ to a hollow 3-polytope

We use the following lemma:

**Lemma**

Let $\pi : P \rightarrow Q$ be an integer projection of a hollow $d$-simplex $P$ onto a non-hollow lattice $(d - 1)$-polytope $Q$. Let:

- $\delta$ be the maximum length of a fiber ($\pi^{-1}$ of a point) in $P$.
- $0 \leq r < 1$ be the maximum dilation factor such that $Q$ contains a homothetic hollow copy $Q_r$ of itself.

Then:

1. $\text{Vol}(P) \leq \delta \text{Vol}(Q)$.
2. $\delta^{-1} \geq 1 - r$. 
**Thm 1: case “P cannot be projected to a hollow polytope”**

**Case 2** There is no integer projection of $P$ to a hollow 3-polytope

We use the following lemma:

**Lemma**

Let $\pi : P \to Q$ be an integer projection of a **hollow** $d$-simplex $P$ onto a **non-hollow** lattice $(d-1)$-polytope $Q$. Let:

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- $0 \leq r < 1$ be the maximum dilation factor such that $Q$ contains a **homothetic hollow copy** $Q_r$ of itself.

Then:

1. $\text{Vol}(P) \leq \delta \text{Vol}(Q)$.
2. $\delta^{-1} \geq 1 - r$.

- $r$ measures whether $Q$ is “close to hollow” ($r \approx 1$) or “far from hollow” ($r \approx 0$)
- In what follows we project $P$ along the direction with $\delta=\text{diam}(P)$. Part (2) says “if $Q$ is far from hollow then $\text{diam}(P)$ is small”
Figure: Projection of an empty \((d)\)-simplex into an \((d-1)\)-polytope
So, let $\pi : P \to Q$ be the projection along the direction giving the diameter of $P$, so that the $\delta$ in the theorem equals the lattice diameter of $P$. We have a dichotomy:
The dichotomy

So, let $\pi : P \to Q$ be the projection along the direction giving the diameter of $P$, so that the $\delta$ in the theorem equals the lattice diameter of $P$. We have a dichotomy:

- If $Q$ is “far from hollow” then we use Minkowski’s First Thm.

\[ \text{Vol}(P - P) \leq d!2^d \delta^d. \]
Together with $\text{Vol}(P - P) = \binom{2d}{d} \text{Vol}(P)$ (Rogers-Shephard for a simplex):

\[ \text{Vol}(P) = \frac{\text{Vol}(P - P)}{\binom{8}{4}} \leq \frac{24 \cdot 16}{\binom{8}{4}} \delta^4 = 5.48 \delta^4. \]

E.g., with $r \leq 5/6$, $\delta \leq 6$.

\[ \text{Vol}(P) \leq 5.48 \cdot 6^4 = 7109. \]
The dichotomy

So, let $\pi : P \to Q$ be the projection along the direction giving the diameter of $P$, so that the $\delta$ in the theorem equals the lattice diameter of $P$. We have a dichotomy:

- If $Q$ is “close to hollow” then we use the Lemma:

$$\Vol(P) = \delta \Vol(Q) = \frac{\delta}{r^3} \Vol(Q_r),$$

where :

1. $r$ is bounded away from 0: by the previous case we can assume $r \geq \frac{5}{6}$.
2. $Q_r$ is hollow of width at least $3r \geq 2.5$, which implies
   $$\Vol(Q_r) \leq 32 \frac{5^3}{3^3} = 148.148$$ (see next slide).
3. $\delta \leq 60$ (we skip this).

... so we get an upper bound on $\Vol(P)$. 

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Let $w \geq 2.5$. Then, the following holds for any lattice-free convex body $K$ in dimension three of width at least $w$:

(a)\[ \text{Vol}(K) \leq \frac{48w^3}{(w - 1)^3} \leq 222.22 \ldots. \]

(b) If $K$ is a lattice 3-polytope with at most five points:

\[ \text{Vol}(K) \leq \frac{32w^3}{(w - 1)^3} \leq 148.148 \ldots. \]
Putting the bounds for $r \geq 5/6$, $\text{Vol}(Q_r) \leq 148.148\ldots$ and $\delta \leq 60$ together we get:

$$\text{Vol}(P) \leq \frac{\delta}{r^3} \text{Vol}(Q_r) \leq 60 \frac{6^3}{5^3} 32 \frac{5^3}{6^3} \leq 15360.$$  

But these three bounds are not independent since:

- $1 - r \leq \delta^{-1}$ (e.g., if $r \approx 5/6$ then $\delta \lesssim 6$).
- $\text{Vol}(Q_r) \leq 32 \left( \frac{3r}{3r-1} \right)^3$ (e.g., if $r \approx 1$ then $\text{Vol}(Q_r) \approx 108$).
Putting the bounds for $r \geq 5/6$, $\text{Vol}(Q_r) \leq 148.148 \ldots$ and $\delta \leq 60$ together we get:

$$\text{Vol}(P) \leq \frac{\delta}{r^3} \text{Vol}(Q_r) \leq 60 \frac{6^3}{5^3} \frac{32}{6^3} \frac{5^3}{6^3} \leq 15360.$$ 

But these three bounds are not independent since:

- $1 - r \leq \delta^{-1}$ (e.g., if $r \simeq 5/6$ then $\delta \lesssim 6$).
- $\text{Vol}(Q_r) \leq 32 \left( \frac{3r}{3r-1} \right)^3$ (e.g., if $r \simeq 1$ then $\text{Vol}(Q_r) \simeq 108$).

Optimizing the three parameters together we get

$$\text{Vol}(P) \leq 6992.$$
Summing up:

- If $P$ projects to a hollow 3-polytope then
  
  $$\text{Vol}(P) \leq 27$$

- If $P$ does not project to a hollow 3-polytope we have the following cases:
  
  1. $Q$ is “far from hollow” ($r \leq 5/6$) then
     
     $$\text{Vol}(P) \leq 7109$$
  
  2. $Q$ is “close to hollow” ($r \geq 5/6$) then
     
     $$\text{Vol}(P) \leq 6992$$
Once we have found a bound for the volume, we need to enumerate all empty 4-simplices up to that bound.
We have used two different algorithms to enumerate all empty 4-simplices.
To enumerate all empty 4-simplices of a given determinant $D$ we use one of two algorithms:

Algorithm 1: **If $D$ has less than 5 prime factors.** It is a complete enumeration of all possibilities after fixing one of the facets of the simplex.

Algorithm 2: **If $D$ has at least 2 prime factors.** Create the simplices by decomposing the volume $D = ab$ with $a$ and $b$ relatively prime and combining the simplices with volumes $a$ and $b$.

For some values of $D$ both algorithms can be used, or different factorizations of $D$ can be chosen in Algorithm 2. Experimentally, we observe that Algorithm 2 is much slower than Algorithm 1 if $a \ll b$, and slightly faster than Algorithm 1 if $a \simeq b$. 
Computational data

Computation time (sec.) for the list of all empty lattice 4-simplices of a given determinant
We have identified all empty lattice 4-simplices of with greater than two. How to classify the rest of empty lattice 4-simplices:

- Those of width 1 can be classified as they form a 3-parameter family, similar to the White Theorem in dimension 3.
- Those of width 2?
We have identified all empty lattice 4-simplices of width greater than two. How to classify the rest of empty lattice 4-simplices:

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**Theorem (Not true (Barile et al.))**

All except for finitely many empty 4-simplices belong to the classes (of cyclic quotient singularities) classified by Mori-Morrison-Morrison (1988), and hence have width at most two.

We still have some information of those of width 2.
At the end, we have found some new families that can complete the classification of empty 4-simplices of width 2, and so, the classification of empty 4-simplices.

**Theorem (Blanco-I.V.-Santos, in preparation)**

All except for finitely many empty 4-simplices belong to one of the following cases:

- **The three-parameter family of empty 4-simplices of width one.**
- **Two 2-parameter families of empty 4-simplices projecting to the second dilation of a unimodular triangle (one listed by Mori et al., the other not).**
- **The 29 Mori 1-parameter families (they project to 29 hollow "primitive" 3-polytopes).**
- **23 additional 1-parameter families that project to 23 “non-primitive" hollow 3-polytopes.**
Thanks for your attention

The article is in arXiv:1704.07299

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