

INDUCTIVE ROTATION TILINGS

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ABSTRACT. A new method for constructing aperiodic tilings is presented. The method is illustrated by constructing a particular tiling and its hull. The properties of this tiling and the hull are studied. In particular it is shown that these tilings have a substitution rule, that they are nonperiodic, aperiodic, limitperiodic and pure point diffractive.

Dedicated to Nikolai Petrovich Dolbilin on the occasion of his 70th birthday.

1. INTRODUCTION

The discovery of the celebrated Penrose tilings [12], see also [6, 1], and of physical quasicrystals [15] gave rise to the development of a mathematical theory of aperiodic order. Objects of study are nonperiodic structures (i.e. not fixed by any nontrivial translation) that nevertheless possess a high degree of local and global order. In many cases the structures under consideration are either discrete point sets (Delone sets, see for instance [2]) or tilings (tilings are also called tessellations, see [6] for a wealth of results about tilings).

Three frequently used construction methods for nonperiodic tilings are local matching rules, cut-and-project schemes and tile substitutions, see [1] and references therein for all three methods. In this paper we describe an additional way to construct nonperiodic tilings, the “inductive rotation”, found in 2008 by the second author who is not a scientist but an artist. Variants of this construction have been considered before, see for instance [3] → “People” → “Petra Gummelt”, but up to the knowledge of the authors this construction does not appear in the existing literature.

After describing the construction we will prove that the resulting tilings are nonperiodic, aperiodic and limitperiodic, that they can be described as model sets, hence are pure point diffractive, that they possess uniform patch frequencies and that they are substitution tilings. (For an explanation of these terms see below). The latter property will be the key for most of the other results. We end with some remarks and open questions. In order to keep the paper as much self-contained as possible we try to explain all terms here, but only as far as required to state the results. Where needed we provide references for further information.

Let us fix some notation. A *tiling* of \mathbb{R}^2 is a collection of tiles $\{Q_i \mid i \in \mathbb{N}\}$ that is a covering (i.e. $\bigcup_{i \in \mathbb{N}} Q_i = \mathbb{R}^2$) as well as a packing (i.e. the intersection of the interiors of two distinct

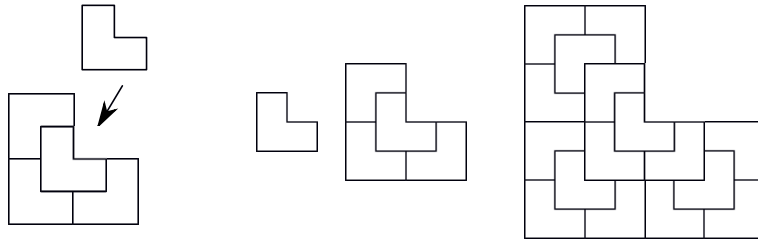


FIGURE 1. The substitution rule for the aperiodic chair tiling (left) and the first two iterates of this rule applied to a single prototile.

tiles Q_i and Q_j is empty). The Q_i are called *tiles* of the tiling. For our purposes it is fine to think of the tiles as nice compact sets like squares or triangles. If all tiles in the tiling belong to finitely many congruence classes $[T_1], \dots, [T_m]$ then we call the representatives T_1, \dots, T_m *prototiles* of the tiling. Any finite subset of a tiling is called a *patch*. Examples of patches of a tiling are obtained by intersecting a tiling \mathcal{T} with a ball $B_r(x)$, i.e. with an open ball of radius r with centre x . The intersection is defined as

$$\mathcal{T} \cap B_r(x) := \{T \in \mathcal{T} \mid T \subset B_r(x)\}.$$

Sometimes we want to equip the tiles with an additional attribute like colour or decoration. Then we consider the prototiles equipped with the different colours or decorations, too. One can formalise this by considering pairs (T, i) , where T is some tile and i encodes the additional attribute. In order to keep notation simple we keep in mind to distinguish a black unit square from a red unit square when necessary without writing down the additional label i .

A vector $t \in \mathbb{R}^2$ such that $\mathcal{T} + t = \mathcal{T}$ is called *period* of \mathcal{T} . (If $\mathcal{T} = \{Q_i \mid i \in \mathbb{N}\}$ then $\mathcal{T} + t$ means $\{Q_i + t \mid i \in \mathbb{N}\}$.) A tiling \mathcal{T} is called *periodic* if it has a nontrivial period, i.e. a period $t \neq 0$. A tiling \mathcal{T} is called *2-periodic* if \mathcal{T} possesses two linear independent periods. A tiling \mathcal{T} is called *nonperiodic* if its only period is the trivial period $t = 0$.

A *substitution rule* σ is a simple method to generate nonperiodic tilings. The basic idea is to substitute each prototile T_i with a patch $\sigma(T_i)$ consisting of congruent copies of some of the prototiles T_1, \dots, T_m . The Penrose tilings can be generated by a substitution rule with two prototiles, see [1], [3] or [17]. A simpler example using just one prototiles is shown in Figure 1: a substitution rule for the *chair tiling*. The chair tiling is nonperiodic, even though it contains large 2-periodic subsets [1].

Given a substitution σ with prototiles T_1, \dots, T_m a patch of the form $\sigma(T_i)$ is called *supertile*. More generally, a patch of the form $\sigma^k(T_i)$ is called *level k supertile*. A substitution rule is called *primitive* if there is $k \in \mathbb{N}$ such that each level k supertile contains congruent copies of all prototiles.

In the sequel we present a construction method for nonperiodic tilings that is similar but not equal to a substitution rule. Before we give a more precise description let us first illustrate the idea of the construction. We start with a square G_0 of edge length two

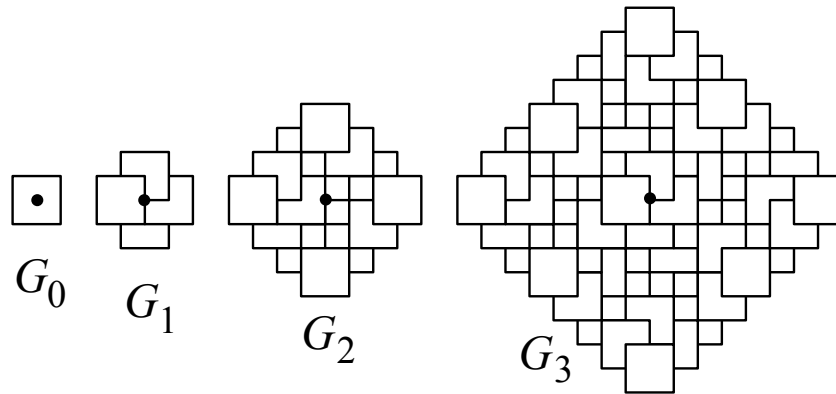


FIGURE 2. The first three iterates of the construction for non-decorated tiles. The black dot indicates the origin.

centred in the origin, see Figure 2 left. (We may imagine this and further squares cut out of cardboard or similar.) In the next step we remove G_0 (but keep its position in mind); we take four translates of G_0 , and we place the first one one unit to the left with respect to G_0 . The next square we rotate by $\frac{-\pi}{2}$ and place it one unit up with respect to G_0 , shuffling it partly *below* the first square. The third square we rotate by $-\pi$, place it one unit to the right with respect to G_0 shuffling it partly below the second square. The last square is rotated by $\frac{-3\pi}{2}$, placed one unit down with respect to G_0 and shuffled below the third square. The result is shown in Figure 2 second from the left. Let us call this constellation of four overlapping squares G_1 .

We proceed in similar way: We take four translates of G_1 and put one translate two units to the left with respect to G_1 , a second translate (rotated by $\frac{-\pi}{2}$) two units up with respect to G_1 and below the first translate, a third translate (rotated by $-\pi$) two units to the right with respect to G_1 and below the second translate, and a fourth translate (rotated by $\frac{-3\pi}{2}$) two units down with respect to G_1 and below the third translate. The result G_2 , as seen from above, is shown in Figure 2 second from the right.

The next generations are constructed analogously, just replace G_1 by G_n and 2 by 2^n in the last paragraph. In this way we can cover arbitrary large parts of the plane. In order to translate this covering into a tiling we consider only the visible (parts of) squares. Thus the tiling has four prototiles: a large square with edge length two, a small square with edge length one, a 1×2 rectangle, and a “chair”, i.e. a non-convex hexagon that is the union of three small squares. The more subtle question how this sequence of finite patterns yields an infinite tiling of the plane is answered below. Briefly, we use the fact that the central patches of the G_i are fixed under the iteration. These patches yield a nested sequence $S_2 \subset S_3 \subset S_4 \cdots$ of patches with a well defined limit which is an infinite tiling.

2. THE TILINGS

Now we give a more precise description of the construction of the sequence of patches. Let φ denote a rotation through $-\pi/2$ about the origin. Start with a square $Q := [-1, 1]^2$. In the following we need to keep track about which squares lie “above” (parts of) other squares. In order to achieve this we may equip each square with an additional label. One may write (Q, i) and consider the label i as the “height” of Q in some orthogonal direction. Anyway, this idea leads to a tedious description of the construction that we omit here. For our purposes it is sufficient just to keep track which squares are higher respectively lower than other squares.

Let $P_0 := \{Q\}$. Let

$$P_1 := \{Q + (-1, 0), \varphi Q + (0, 1), \varphi^2 Q + (1, 0), \varphi^3 Q + (0, -1)\},$$

where the first square is on top of the other squares, the second square is below the first one but above the third and the fourth square, and the fourth square is on bottom. In the n th step let P_n consist of four congruent copies of P_{n-1} :

$$P_n := \{P_{n-1} + (-2^{n-1}, 0), \varphi P_{n-1} + (0, 2^{n-1}), \varphi^2 P_{n-1} + (2^{n-1}, 0), \varphi^3 P_{n-1} + (0, -2^{n-1})\}.$$

All squares in the first set are on top of the squares in the other three sets, the squares in the second set all are below the squares in the first set but on top of all squares in the third set and the fourth set, and the squares in the fourth set are all below the squares in the other three sets. Two squares within one of the sets $P_{n-1} + t$ inherit their above-below relation from the preceding steps in the iteration.

Note that the centres of the squares are contained in the lattice translate

$$(1, 0) + \Lambda := (1, 0) + \langle (1, 1), (1, -1) \rangle_{\mathbb{Z}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \text{ odd}\}$$

Hence, by the construction, the centre of each of the congruent copies of P_1 (i.e. the single point that is the intersection of the four squares in the copy of P_1) is contained in $(2, 0) + 2\Lambda = (2, 0) + \langle (2, 2), (2, -2) \rangle_{\mathbb{Z}}$. Since each $x \in (1, 1) + \Lambda$ has a unique presentation of the form

$$x = y + \begin{cases} (0, \pm 1) \\ (\pm 1, 0) \end{cases}, \quad (y \in (2, 0) + 2\Lambda)$$

the following lemma is immediate.

Lemma 2.1. *The set P_n consists of 4^n congruent copies of Q with centres*

$$\{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 \text{ odd}, |x_1| + |x_2| \leq 2^n - 1\}$$

Because of Lemma 2.1 almost any $(x_1, x_2) \in \mathbb{R}^2$ (i.e. any with $x_1, x_2 \notin \mathbb{Z}$) is covered by exactly two squares in P_n (for n large enough). Any (x_1, x_2) with exactly one integer coordinate is covered by three squares in P_n (the interior of one square and some common edge of two further squares; again if n is large enough), and any (x_1, x_2) with two integer coordinates is covered by four squares (midpoints of edges) if $x_1 + x_2$ is even, and by five

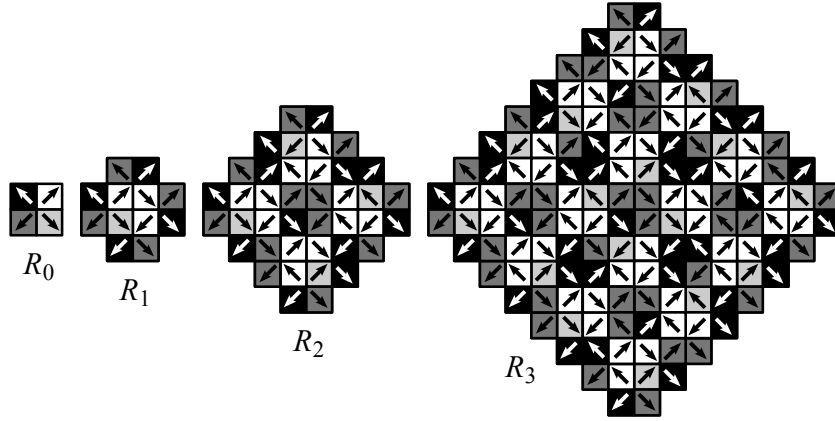


FIGURE 3. The construction of Figure 2 applied to squares with certain decorations. A single square R_0 is shown on the left: the decoration divides R_0 into four unit squares of distinct colours, where each unit square carries an arrow in addition. These decorated unit squares yield the tiles of the tiling \mathcal{A} .

squares (the centre of one square and the common vertex of four further squares) if $x_1 + x_2$ is odd. This yields the following result.

Lemma 2.2. *For all $n \in \mathbb{N}$ the covering degree of P_n is two in $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1 + x_2| \leq 2^n - 1\}$.*

Since we may decide to equip the squares used above with some additional decoration there are several ways in which this construction yields tilings of \mathbb{R}^2 . In the sequel we will mainly consider one particular tiling: the *arrowed tiling* \mathcal{A} that is obtained by decorating the underlying square R_0 with arrows and colours as shown in Figure 3 (left).

The tiling \mathcal{A} is obtained by considering what we see viewing P_n from “above”, where P_n now consists of congruent copies of the decorated square R_0 . Formally, each unit square $K = [k, k + 1] \times [m, m + 1]$ ($k, m \in \mathbb{Z}$) inherits the decoration (colour and arrow) from the top square of the two large squares in P_n that contain K . We want to consider \mathcal{A} as a tiling by unit squares with decoration as prototiles where the unit squares carry an arrow as well as a colour (here: black, dark grey, light grey, white). From now on the term “tile” always denotes such a unit square with decoration, if not mentioned otherwise.

Regarding the point how to obtain a tiling of the entire plane from the finite patches P_n , a very natural approach—and a very useful one—is considering the limit $\lim_{n \rightarrow \infty} P_n$, where the limit is taken with respect to the local topology [1]. In particular, in this topology two tilings are ε -close if they agree in a ball of radius $1/\varepsilon$ centred in the origin. For our purposes we may define the distance $d(\mathcal{T}, \mathcal{T}')$ between two tilings $\mathcal{T}, \mathcal{T}'$ by

$$\tilde{d}(\mathcal{T}, \mathcal{T}') = \inf\{\varepsilon > 0 \mid \exists x, y \in B_\varepsilon : B_{1/\varepsilon} \cap (\mathcal{T} + x) = B_{1/\varepsilon} \cap (\mathcal{T}' + y)\}$$

where B_r denotes the open ball of radius r centred in 0. In order to make \tilde{d} into a metric, i.e. in order to ensure transitivity, we define

$$d(\mathcal{T}, \mathcal{T}') = \min\{\tilde{d}(\mathcal{T}, \mathcal{T}'), \frac{1}{\sqrt{2}}\},$$

compare [9]. We will see in the sequel that the central parts S_n of P_n for $n \geq 2$ form a nested sequence

$$S_2 \subset S_3 \subset \cdots \subset S_n \subset \cdots,$$

where S_n has support $\{(x_1, x_2) \mid |x_1 + x_2| < 2^{n-1} - 1\}$, hence the sequence P_n —or any sequence of tilings $(\mathcal{T}_n)_n$ where \mathcal{T}_n contains P_n as its central patch—converges with respect to the metric d .

For the next steps it will be convenient to consider the arrowed tiling \mathcal{A} . In order to avoid confusion let us call the corresponding sequence R_n , i.e. the sequence P_n where each tile is equipped with the arrow decoration of Figure 3. The first observation is that the arrows on tiles at the boundary of R_n point outwards. In order to make the term “boundary” precise, let $\text{bd}(R_n)$ be the set of the tiles in R_n that have edges on the boundary of $\bigcup_{T \in R_n} T$. These are exactly the tiles that have two edges that are not shared with other tiles of R_n ; or equivalently: these are exactly the tiles with a vertex that is not vertex of another tile of R_n .

Lemma 2.3. *The arrows on all tiles at the boundary $\text{bd}(R_n)$ of R_n point outwards, i.e. in the direction of their vertex that is not a vertex of any other tile of R_n .*

Proof. The claim is true for R_0 and R_1 , hence it follows inductively for all R_n by considering the construction: The boundary of R_n consists of the boundaries of R_{n-1} . \square

Lemma 2.4. *All arrows on the tiles in R_n on the two diagonals (x_1, x_2) with $x_1 = x_2$ or $x_1 = -x_2$ respectively show in the following directions: $(1, -1)$ for $x_1 = x_2$, $(1, 1)$ for $x_1 = -x_2 < 0$, $(-1, -1)$ for $x_1 = -x_2 > 0$.*

This is an immediate consequence of Lemma 2.3 and the construction. Figure 4 (right) illustrates the situation.

Theorem 2.5. *The sequence R_n is convergent with respect to the metric d . Consequently, any sequence of tilings $(\mathcal{T}_n)_n$ where \mathcal{T}_n contains R_n as its central patch is convergent with respect to the metric d .*

Proof. The idea is to consider three steps of the iteration, i.e. how R_{n+2} is build up from congruent copies of R_{n-1} . This shows that the central patch of R_{n+1} reappears as the central patch of R_{n+2} , compare Figure 5.

For $n \geq 1$ the set $R'_n := R_n \setminus \text{bd}(R_n)$ is nonempty and reappears in R_{n+1} , see Figure 4. Considering how R'_n is contained in R_{n+1} we find that the central patch S of R_{n+1} consists of four congruent copies of R'_{n-1} . (This is indicated in the left part of Figure 5, the area shaded in darker grey.) The central patch of R_{n+2} , indicated in Figure 5 by shading in

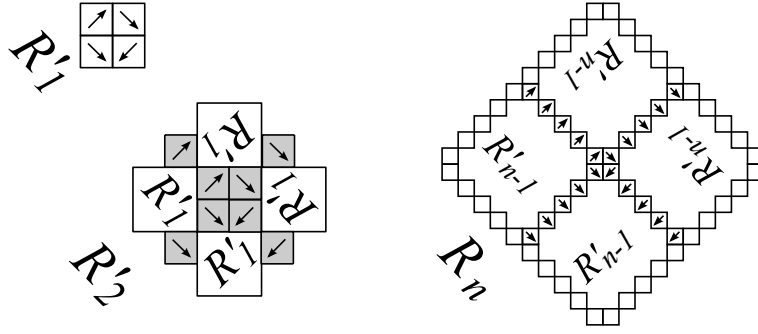


FIGURE 4. The structure of the patches R'_n . Each R'_n is build from four congruent copies of R'_{n-1} and some further tiles on its main diagonals. The arrows in the tiles on the diagonals are pointing in the directions given in Lemma 2.4.

lighter grey, is a translate of the central patch S of R'_{n+1} , since (a) it is build up from congruent copies of R'_{n-1} in the same manner (this yields the bulky part), and (b) the arrows on the diagonal boundaries of the R'_{n-1} coincide by Lemma 2.4 together with the construction (this yields the “skeleton” part). The directions of the arrows on the tiles on the boundaries of the R'_{n-1} are indicated by small black squares in the corresponding corners of the tiles in Figure 5.

Altogether this yields that the central patch of R_{n+1} equals the central patch of R_{n+2} . Let us denote the central patch of R_{n+1} consisting of four congruent copies of R'_{n-1} (plus the tiles on the two diagonals) by S_{n+1} . More precisely, let

$$S_{n+1} := R_{n+1} \cap \{(x_1, x_2) \mid |x_1 + x_2| \leq 2^{n-1} - 1\} = R_{n+2} \cap \{(x_1, x_2) \mid |x_1 + x_2| \leq 2^{n-1} - 1\}.$$

The patches S_{n+1} yield a nested sequence

$$S_2 \subset S_3 \subset S_3 \subset \dots \subset S_n \subset \dots$$

In particular we obtain that for any $n \in \mathbb{N}$ we have that R_{n+1} coincides with all R_k ($k \geq n + 1$) in a ball $B_{1/\varepsilon}$ for $\frac{1}{\varepsilon} \leq \frac{1}{\sqrt{2}}(2^{n-1} - 1)$, i.e. $\varepsilon \geq \frac{\sqrt{2}}{2^{n-1}-1}$. Thus

$$d(R_{n+1}, R_k) \leq \frac{\sqrt{2}}{2^{n-1} - 1} \quad \text{for } k \geq n + 1.$$

Hence the sequence R_n , respectively any sequence of tilings $(\mathcal{T}_n)_n$ where \mathcal{T}_n contains R_n as its central patch, converges for $n \rightarrow \infty$ with respect to the metric d . \square

Note that neither the sets R_n nor the sets R'_n form a nested sequence, there are slight mismatches further away from the centre.

Since the sequence R_n , respectively any sequence of tilings $(\mathcal{T}_n)_n$ where each \mathcal{T}_n contains R_n as its central patch, converges with respect to d , there is a unique limit $\mathcal{A} = \lim_{n \rightarrow \infty} \mathcal{T}_n$. The tiling \mathcal{A} is the desired infinite tiling. Moreover we may now define the *hull* of \mathcal{A} . In

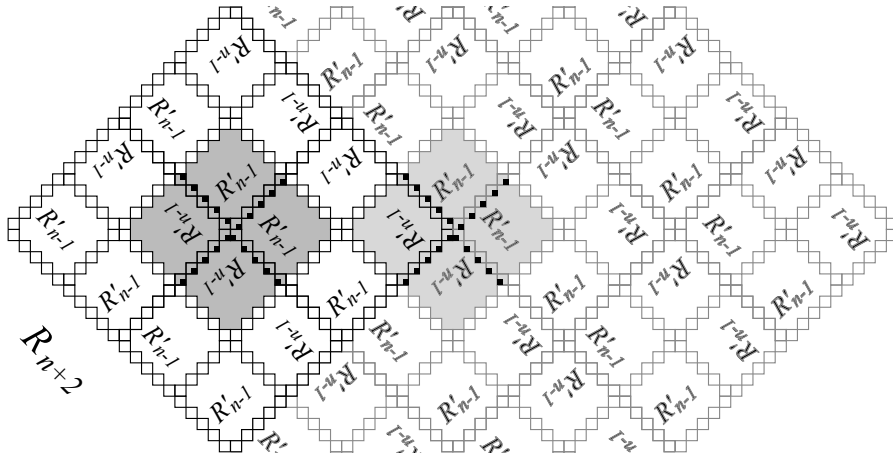


FIGURE 5. How congruent copies of R'_{n-1} are located within R_{n+1} (left part) and R_{n+2} (everything).

the theory of aperiodic tilings the hull turns out to be a central object of study. The hull is the closure of the set of all translates of \mathcal{A} , i.e.

$$\mathbb{X}(\mathcal{A}) = \overline{\{\mathcal{A} + t \mid t \in \mathbb{R}^2\}},$$

where \overline{M} defines the closure of the set M with respect to d , see again [1, Chapter 4] for details.

Since \mathcal{A} has only finitely many prototiles, and all tiles are vertex-to-vertex, the set of all finite patches in \mathcal{A} up to some given radius $r > 0$ is finite, up to congruence (even up to translations). This property is called *finite local complexity* [1]. It holds for each tiling in $\mathbb{X}(\mathcal{A})$ and for the set $\mathbb{X}(\mathcal{A})$ as a whole. Hence by standard reasoning ([13, 14], see also [1, Chapter 4]), $\mathbb{X}(\mathcal{A})$ is compact with respect to d , hence $(\mathbb{X}(\mathcal{A}), d)$ is a compact metric space.

3. PROPERTIES OF THE TILINGS

With the help of the metric d above we are now able to give a precise definition of aperiodic tilings. A tiling is *aperiodic* if its hull does not contain any periodic tiling. In particular, aperiodicity implies nonperiodicity. (The hull of a periodic tiling \mathcal{T} contains only translates of \mathcal{T} , so in particular it contains *only* periodic tilings.)

Considering Figure 5 one may get the impression that \mathcal{A} is periodic. At least the arrangement of the R'_{n-1} in the image seem to form a periodic pattern. This behaviour is well-known for certain aperiodic structures like the chair tiling, the Robinson square tiling or the one-dimensional period doubling sequence, see [1, 3]. Loosely speaking, a limit-periodic tiling is one that is the union of infinitely many periodic packings with larger and larger periods where the lattices Λ_i of periods are nested sequences $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \dots$, possibly up to a set of density zero. In fact, the exact definition of limitperiodicity uses spectral

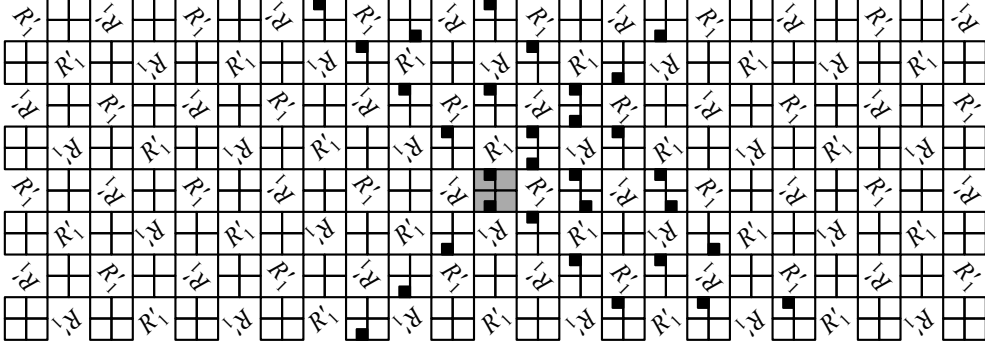


FIGURE 6. The congruent copies of R'_1 in \mathcal{A} form a periodic subset of the tiling with density $\frac{1}{2}$. The centre of the shaded patch is the origin.

properties of the hull. We will not explain this in detail (compare [1]) since there is a simple geometric sufficient condition ensuring limitperiodicity that applies to our examples.

Theorem 3.1. \mathcal{A} is the union of 2-periodic packings M_n and a set \mathcal{Z} of density zero. More precisely, $\mathcal{A} = \mathcal{Z} \cup \bigcup_{n \in \mathbb{N}} M_n$, where

$$M_n := \{R'_n + ((2^n, 0) + 2^{n+1}\Lambda), \varphi R'_n + ((0, -2^n) + 2^{n+1}\Lambda), \varphi^2 R'_n + ((-2^n, 0) + 2^{n+1}\Lambda), \varphi^3 R'_n + ((0, 2^n) + 2^{n+1}\Lambda)\}$$

where $\Lambda = \langle (1, 1), (1, -1) \rangle_{\mathbb{Z}}$ (i.e. the integer span of the two vectors $(1, 1)$ and $(1, -1)$) and \mathcal{Z} is the set of all tiles in \mathcal{A} whose diagonals are contained in $\{(x_1, x_2) \mid x_1 = \pm x_2\}$.

Proof. By the proof of Theorem 2.5 we get that for any $n \in \mathbb{N}$ the central patch of \mathcal{A} is R_n . Considering how patches of type R'_{n-3} are located in R_n we obtain inductively arbitrary large parts of 2-periodic packings consisting of congruent copies of R'_k , $\varphi R'_k$, $\varphi^2 R'_k$ and $\varphi^3 R'_k$ for any $k \in \mathbb{N}$, compare Figure 5. Figure 6 indicates the arrangement of congruent copies of R'_1 . Let $\Lambda = \langle (1, 1), (1, -1) \rangle_{\mathbb{Z}}$. Then the four 2-periodic sets

$$(1) \quad R'_1 + ((2, 0) + 4\Lambda), \varphi R'_1 + ((0, -2) + 4\Lambda), \varphi^2 R'_1 + ((-2, 0) + 4\Lambda), \varphi^3 R'_1 + ((0, 2) + 4\Lambda)$$

yield already half of the tiles of \mathcal{A} . Each of the four sets has period vectors $(4, 4)$ and $(4, -4)$. This situation is indicated in Figure 6. The arrangement of R_n is similar on all levels 2^n , hence further periodic structures in \mathcal{A} are

$$(2) \quad R'_n + ((2^n, 0) + 2^{n+1}\Lambda), \varphi R'_n + ((0, -2^n) + 2^{n+1}\Lambda), \varphi^2 R'_n + ((-2^n, 0) + 2^{n+1}\Lambda), \varphi^3 R'_n + ((0, 2^n) + 2^{n+1}\Lambda)$$

For each n the union of the four sets has density $\frac{1}{2}$. Each of the four sets has period vectors $(2^{n+1}, 2^{n+1})$ and $(2^{n+1}, -2^{n+1})$. It is easy to see that the union of these sets for all $n \in \mathbb{N}$ contains all tiles of \mathcal{A} except tiles along the diagonals $x_1 = \pm x_2$, i.e. all tiles in \mathcal{Z} . This set has density zero. \square

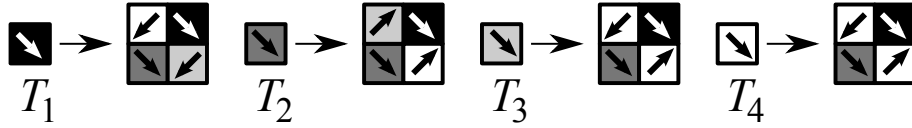


FIGURE 7. The substitution rule σ yielding the tiling \mathcal{A} .

Remark 3.2. Note that Equation (2) yields all tiles in \mathcal{A} except the tiles on the diagonals $x_1 = \pm x_2$, hence (2) together with Lemma 2.4 yields the entire tiling.

The next theorem is the key to all further results. This holds because the theory of substitution tilings is developed pretty well, while there is no theory for the construction above yet.

Theorem 3.3. *\mathcal{A} can be generated by a primitive tile substitution rule.*

Proof. This result is achieved by giving an appropriate substitution rule yielding the tiling \mathcal{A} . More precisely, we will show that this rule yields a substitution tiling that possesses the particular structure (union of periodic packings) described by Remark 3.2.

This substitution rule is shown in Figure 7. Let us denote this substitution rule by σ . Its prototiles are denoted by T_1, T_2, T_3 and T_4 ; they are the prototiles of \mathcal{A} . Figure 8 shows the action of $\sigma, \sigma^2, \sigma^3$ and σ^4 on T_1 . Note that $\sigma(T_3) = \sigma(T_4)$. Note also that there are no reflections involved: all tiles in $\sigma(T_i)$ are direct congruent copies of the T_i , not reflected congruent copies.

To obtain an infinite substitution tiling in \mathbb{X}_σ it is useful to consider a tiling \mathcal{S} that is fixed under σ , i.e. $\sigma(\mathcal{S}) = \mathcal{S}$. The construction of such a tiling is standard: take a legal “seed”, i.e. a patch P that is contained in some $\sigma^n(T)$ for some prototile T , such that $\sigma(P)$ (or $\sigma^2(P)$ or $\sigma^3(P)$...) contains a translate of P in its interior. Here we choose P as the patch of four tiles shown in Figure 9 left, consisting of three dark grey tiles of type T_2 and one black tile of type T_1 . P occurs in $\sigma^4(T_1)$, see Figure 8.

Applying σ to P one gets that P is exactly the central patch of $\sigma(P)$, compare Figure 9. Hence $\sigma(P)$ is the central patch of $\sigma^2(P)$, and inductively we get that $\sigma^n(P)$ is the central patch of $\sigma^{n+1}(P)$. Hence the sequence $\sigma_n(P)$ (or any sequence of tilings \mathcal{S}_n having $\sigma^n(P)$ as their central patch respectively) converge to some tiling in the local topology. Let us denote this tiling by \mathcal{S} .

In order to show that $\mathcal{S} = \mathcal{A}$ we use the particular structure of \mathcal{A} stated in Remark 3.2. Consider $\sigma^3(T_1), \sigma^3(T_2), \sigma^3(T_3)$ and $\sigma^3(T_4)$. The interior of each of the four patches contains four congruent copies of R'_1 . On their boundaries these patches have halves of R'_1 . (See for instance Figure 8, the patch on the right is the union of one copy of $\sigma^3(T_1), \sigma^3(T_2), \sigma^3(T_3)$ and $\sigma^3(T_4)$ each.) The entire constellation of congruent copies of R'_1 and halves of R'_1 agrees on all four supertiles $\sigma^3(T_i)$, and it is invariant under rotation by $\pi/2$ about the centre of each $\sigma^3(T_i)$. All tiles in \mathcal{S} lie edge to edge, hence all “supertiles” $\sigma^3(T_i)$ lie edge-to-edge. Hence \mathcal{S} contains the same periodic arrangement as \mathcal{A} , i.e. the one in Equation (1).

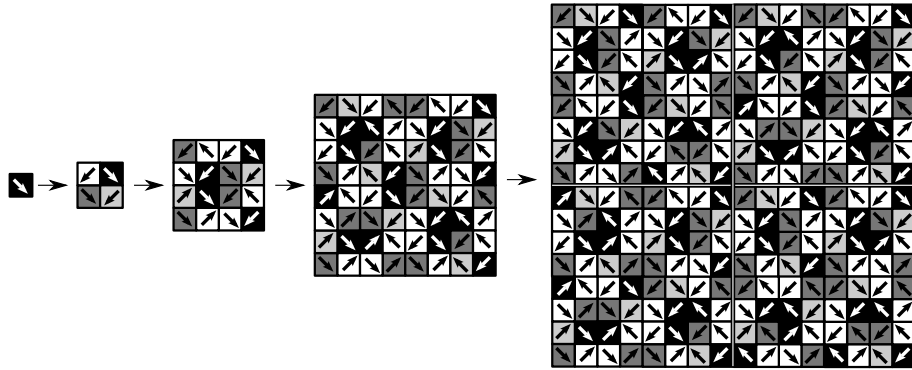


FIGURE 8. The image shows T_1 , $\sigma(T_1)$, $\sigma^2(T_1)$, $\sigma^3(T_1)$, and $\sigma^4(T_1)$ (from left to right). On the right it is indicated how $\sigma^4(T_1)$ consists of congruent copies of $\sigma^3(T_1)$, $\sigma^3(T_2)$, $\sigma^3(T_3)$ and $\sigma^3(T_4)$. A seed for \mathcal{A} is found in $\sigma^4(T_1)$ (e.g. the middle of the fourth and the fifth row, compare Figure 9).

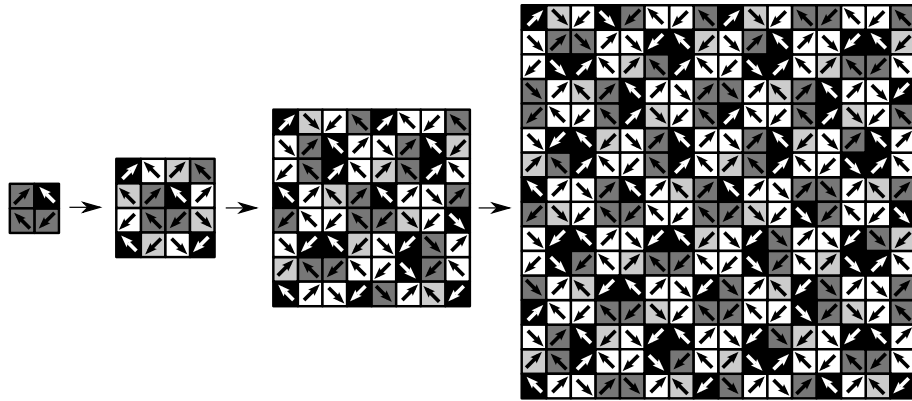


FIGURE 9. The iterates under σ of the small patch on the left converge to \mathcal{A} .

Observe that $\sigma(R_1)$ contains R'_2 , and—more generally— $\sigma(R_n)$ contains R'_{n+1} . (To see this one may use Figure 3 together with Figure 7.) Since $\sigma(\mathcal{S}) = \mathcal{S}$ holds, the tiling \mathcal{S} contains also the periodic arrangements from Equation (2). This shows that \mathcal{S} and \mathcal{A} coincide everywhere except on the diagonals $\{(x, y) \mid x = \pm y\}$. Considering the action of σ it is easy to see that \mathcal{S} has on each of the four branches of this set tiles with arrows that show in the same directions, and that these directions coincide with the ones in \mathcal{A} . Altogether we obtain $\mathcal{S} = \mathcal{A}$. □

Note that there is also the concept of the hull \mathbb{X}_σ of a substitution. Since \mathcal{A} can be generated by a primitive substitution we have that $\mathbb{X}(\mathcal{A}) = \mathbb{X}_\sigma$ (see [1] for details). More important, we now obtain the following result easily.

Theorem 3.4. *All $\mathcal{T} \in \mathbb{X}(\mathcal{A})$ are aperiodic. In particular \mathcal{A} is aperiodic (hence nonperiodic).*

Proof. Since σ is a primitive substitution and $\mathbb{X}(\mathcal{A})$ is of finite local complexity, one can apply a classical result on substitution tilings [6, Thm 10.1.1]: a primitive substitution tiling is aperiodic if in this tiling the level 1 supertiles can be identified in a unique way. (See also Solomyak [16, Thm 1.1] for the proof of the “if and only if” version of this result.)

For the tilings in $\mathbb{X}(\mathcal{A})$ this is particularly simple: any level 1 supertile consists of four tiles sharing a common vertex. These four tiles have at least three distinct colours, and the arrows on the tiles do neither point to this common vertex nor do they point away from it. This leaves only one possibility: the vertices of the congruent copies of R'_1 are the centres of the supertiles. \square

There are two geometric properties of a tiling \mathcal{T} that have strong consequences on the dynamical properties of the hull $\mathbb{X}(\mathcal{T})$ of \mathcal{T} . A tiling \mathcal{T} is *repetitive* if for each $r > 0$ there is $R > 0$ such that each patch of radius less than r is contained in each patch of radius R . If $R \in O(r)$ then \mathcal{T} is called *linearly repetitive*.

Moreover, a tiling \mathcal{T} has *uniform patch frequencies* if the frequencies of all patches are well-defined. More precisely: if P is a patch in \mathcal{T} then let $N_P(B_r(x))$ denote the number of congruent copies of P in $\mathcal{T} \cap B_r(x)$, where $B_r(x)$ denotes the open ball of radius r centred in x . If for all patches P in \mathcal{T} the limit

$$\lim_{r \rightarrow \infty} \frac{1}{\pi r^2} N_P(B_r(x))$$

exists uniformly in x then \mathcal{T} has *uniform patch frequencies*. For a more thorough discussion of repetitivity or uniform patch frequencies see [1] or [4]. We omit it here since we need the terms only to state the following result.

Corollary 3.5. *All $\mathcal{T} \in \mathbb{X}(\mathcal{A})$ are linearly repetitive and have uniform patch frequencies.*

Proof. By a result of Solomyak [16, Lemma 2.3] each primitive substitution tiling in \mathbb{R}^2 with finite local complexity is linearly repetitive.

By a result of Lagarias and Pleasants [8, Theorem 6.1] linear repetitivity implies uniform patch frequencies. \square

The fact that our tilings possess the particular structure (union of periodic packings) described by Remark 3.2 is a hint that they are possibly *limitperiodic*. In this particular case this means that they can be generated by a certain cut-and-project method using a lattice in $\mathbb{R}^2 \times (\mathbb{Q}_2)^2$, where \mathbb{Q}_2 denotes the field of 2-adic numbers. Structures of this kind are called *model sets (with 2-adic internal space)*. Model sets are relevant since by a result of Hof [7, 14, 1] all model sets show pure point diffraction. It is beyond the scope of this paper to give details on this, but we may formulate the result and prove it using a simple to check sufficient condition.

Properly speaking, model sets are discrete point sets, not tilings. Hence our tilings are not model sets, but they are strongly related: they are mutually locally derivable (mld) with model sets, meaning that there is a local rule transforming one into the other (see [1]).

Theorem 3.6. *All $\mathcal{T} \in \mathbb{X}(\mathcal{A})$ are limitperiodic. Consequently, all $\mathcal{T} \in \mathbb{X}(\mathcal{A})$ are mutually locally derivable with model sets, hence pure point diffractive.*

Proof. Since \mathcal{A} is a primitive substitution tiling with integer scaling factor (in this case 2) and σ is a *block substitution* (since each unit square is mapped to four unit squares in a 2×2 grid) we can apply a result in [9] and [10]: we need to show that σ has a modular coincidence. By a result in [5] this is the case if there is a *coincidence* in the supertiles. For this it suffices to note that the upper right tile in $\sigma(T_1), \sigma(T_2), \sigma(T_3)$ and $\sigma(T_4)$ is a black tile T_1 with its arrow pointing down right, compare Figure 7. (The lower left tile in all four cases is T_2 , yielding a second coincidence.) For further details see [9, 10, 5, 1]. \square

The fact that \mathcal{A} is pure point diffractive has consequences if one imagines \mathcal{A} as a physical solid: assume that in such a solid the four different types of atoms (or molecules) are arranged in the same pattern as the four different tiles in \mathcal{A} . A diffraction experiment would then show a diffraction image with bright spots (“Bragg peaks”) and (ideally) no diffuse parts.

Now that we have obtained several results on the arrowed tiling \mathcal{A} we turn our attention to the naked tiling using no decoration at all, compare Figure 2. Let us call the tiling obtained by the same construction but with undecorated squares \mathcal{N} .

A tiling \mathcal{T}_1 is called *locally derivable* from a tiling \mathcal{T}_2 if there is a local rule transforming \mathcal{T}_2 into \mathcal{T}_1 . Two tilings $\mathcal{T}_1, \mathcal{T}_2$ are called *mutually locally derivable* if \mathcal{T}_1 is locally derivable from \mathcal{T}_2 and vice versa [1].

Lemma 3.7. *The naked tiling \mathcal{N} is locally derivable from the arrowed tiling \mathcal{A} .*

Proof. It suffices to give a local rule to transform \mathcal{A} into \mathcal{N} . Considering that in the arrow decoration of \mathcal{A} the arrows point always away from the centre of the large square, the arrows determine locally the edges of the large (overlapping) squares in a unique way. The edges of the overlapping squares determine \mathcal{N} . \square

Two tilings that are mutually locally derivable share a lot of properties (aperiodicity, repetitivity, pure point diffraction, uniform patch frequency...) Thus it would suffice to give a local rule that transforms \mathcal{N} into \mathcal{A} . We probably found such a rule, but unfortunately we were yet unable to prove that it really works.

Problem: Are \mathcal{A} and \mathcal{N} mutually locally derivable?

As an intermediate step, one may consider whether \mathcal{A} is mutually locally derivable with the tiling obtained from \mathcal{A} by deleting the colours. It is a simple exercise to see that this is indeed true.

4. REMARKS AND OUTLOOK

The original construction of inductive rotation tilings found by the second author used to place the origin close to the leftmost part of the patches P_n . Hence the “limit” of

this sequence fills only a quarter plane. Four copies of this limit could be used to fill the entire plane. The resulting tilings are the same as the ones described in this paper. The construction in this paper is more adapted to the notion of convergence of sequences of tilings used here.

Here we showed that the tiling \mathcal{A} is nonperiodic by showing that \mathcal{A} is a substitution tiling and then applying the result of Solomyak that a substitution tiling is nonperiodic if σ^{-1} is unique. We found an alternative proof of nonperiodicity by direct means, using only the limitperiodic structure of \mathcal{A} . For the sake of brevity we omit the alternative proof here.

The construction presented here uses squares and rotations by $\frac{\pi}{2}$. The second author found similar constructions for triangles and rotations by $\frac{2\pi}{3}$ as well as for hexagons and rotations by $\frac{\pi}{3}$. For an artistic application of these constructions see <http://hofstetterkurt.net/ip>, see also [11].

The substitution rule σ used to generate the tiling \mathcal{A} uses four prototiles, but $\sigma(T_3) = \sigma(T_4)$. Probably one of these tiles is redundant and everything works for a substitution for three prototiles as well. We prefer to use the four-colour version since it carries more information, so it might make some arguments more clear.

We studied the tilings \mathcal{A} and \mathcal{N} . We have good evidence (but no proof so far) that \mathcal{A} and \mathcal{N} are mutually locally derivable. What about other decorations? There are decorations of the large square Q leading to 2-periodic tilings. So the construction yields at least two distinct mld classes (one aperiodic, one 2-periodic). Are there more mld classes?

The tilings \mathcal{N} and \mathcal{A} are obtained by looking on R_n from “above”. Are the tilings obtained by looking from below congruent to \mathcal{N} respectively \mathcal{A} ? Are they at least mutually locally derivable with \mathcal{N} respectively \mathcal{A} ?

We may realise the sets R_n also with tiles with “thickness” in \mathbb{R}^3 . Now above and below have a precise meaning. The construction for the sets R_n now has to be described in three dimensions. Is there such a construction such that the height of all R_n is bounded by a common constant? Are there arbitrary high stairs?

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