Unusual normal tilings

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Joint work with Christian Richter, Alexey Glazyrin, Zsolt Lángi

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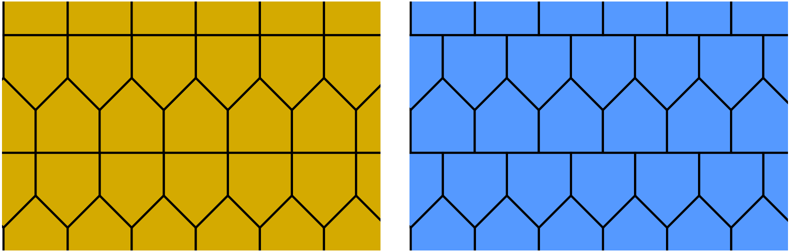
15th November 2021

A *tiling* is a covering of the plane which is a packing of the plane as well.



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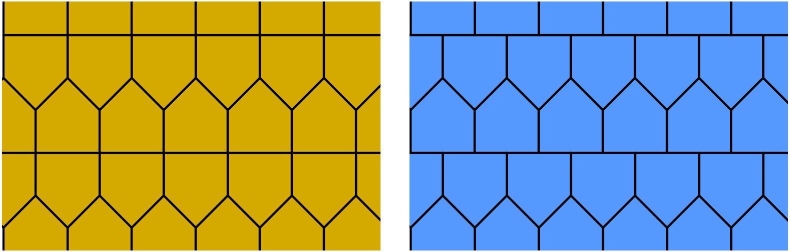

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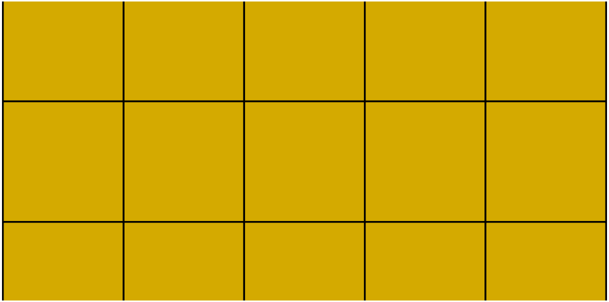
A tiling is called *vertex-to-vertex* if the intersection of any two tiles is a full edge, or a vertex, or empty.

(Left tiling: yes, right tiling: no)

A tiling is called *normal* if there are $r > 0, R > 0$ such that

- Each tile contains in a disk of radius r
- Each tile is contained in a disk of radius R



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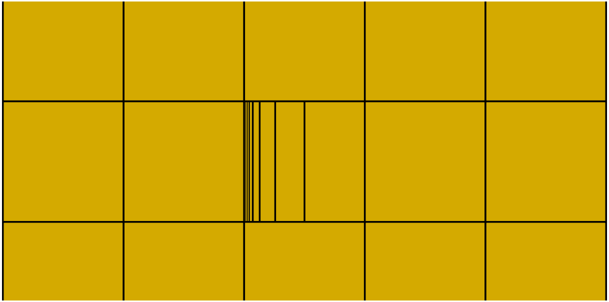

Not normal.

Part 1

(with Christian Richter)

A rich source of interesting problems:

`nandacumar.blogspot.com`

Question: *Is there a tiling of the plane by pairwise noncongruent triangles of equal area and equal perimeter?*

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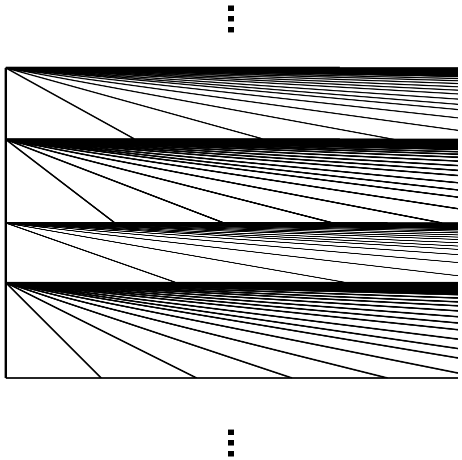
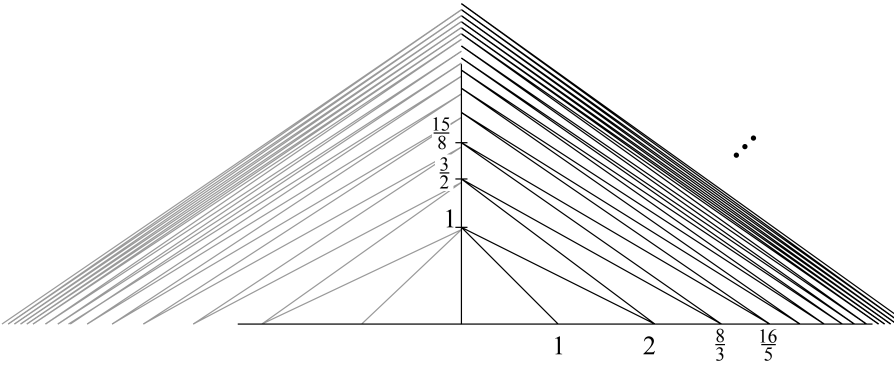


...but this tiling is not normal. (Example due to Jan Bleimling)



...and this tiling is not normal either.

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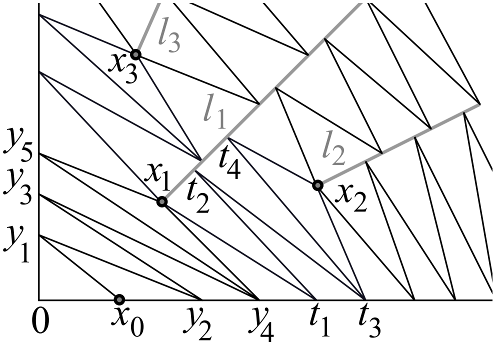

D.F.: Noncongruent equidissections of the plane, *Discrete Geometry and Symmetry*, Springer (2018)

A. Kupavskii, J. Pach, G. Tardos: Tilings of the plane with unit area triangles of bounded diameter, *Acta Math. Hungar.* 155 (2018)

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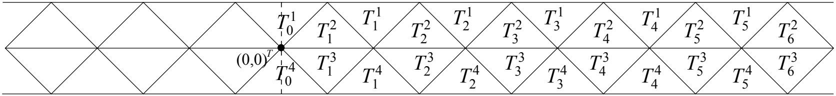

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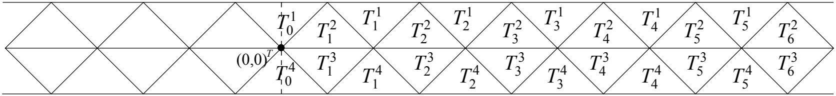
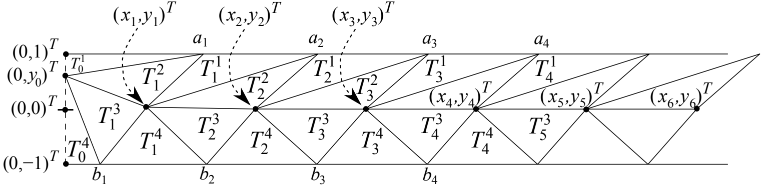
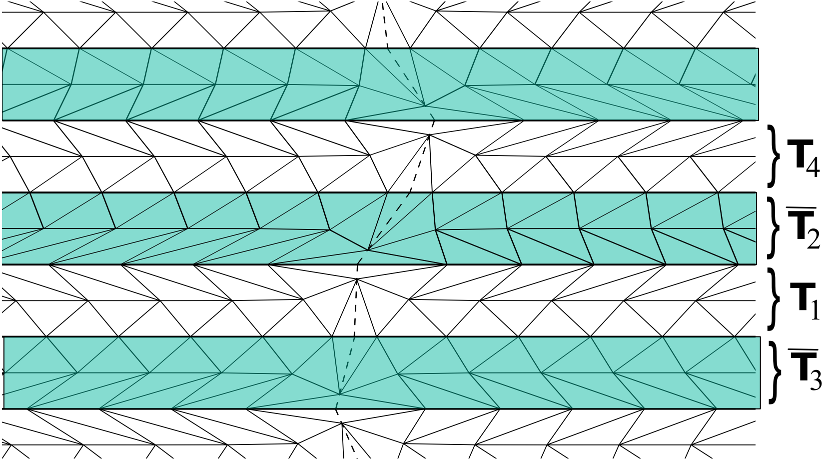


...by moving y_0 up, and keeping unit area etc.



Stack sheared copies of the strip tiling:



(greenish: strip is upside down)

7 pages of computation show:

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- Determine exact values of y_i
- Deviation of y_i is bounded ($\Rightarrow$ normal)
- All triangles within the strip are pairwise noncongruent
- Exploit uncountably many choices for shear angles ($\Rightarrow$ pairwise noncongruent)

$$|\alpha_i| \overset{(12),(15)}{=} \left| \frac{y_0}{1-y_0} + \sum_{j=1}^{i-1} \frac{2y_j}{1-y_j} \right| \leq 2\sum_{j=0}^{\infty} \frac{|y_j|}{|1-y_j|} \overset{(19)}{\leq} 2\sum_{j=0}^{\infty} \frac{2^{-j}y_0}{1-y_0} = \frac{4y_0}{1-y_0} =: C_\alpha,$$

$$|\beta_i| \overset{(12),(16)}{=} \left| -\frac{y_0}{1+y_0} - \sum_{j=1}^{i-1} \frac{2y_j}{1+y_j} \right| \leq 2\sum_{j=0}^{\infty} \frac{|y_j|}{|1+y_j|} \overset{(19)}{\leq} 2\sum_{j=0}^{\infty} \frac{2^{-j}y_0}{1} = 4y_0 =: C_\beta,$$

$$|\xi_i| \overset{(12),(13)}{=} \left| -\sum_{j=1}^{i} \frac{(\alpha_j-\beta_j)y_{j-1}}{1+\alpha_j+\beta_j} \right| \overset{(17)}{=} \left| -\sum_{j=1}^{i} \frac{(\alpha_j-\beta_j)y_{j-1}}{h_{j-1}} \right| \leq \sum_{j=1}^{i} \frac{(|\alpha_j|+|\beta_j|)|y_{j-1}|}{|h_{j-1}|}$$

$$\overset{(19),(c_j)}{\leq} \sum_{j=1}^{\infty} \frac{(C_\alpha+C_\beta)2^{-(j-1)}y_0}{2} = (C_\alpha+C_\beta)y_0. \qquad \square$$

Variations of the questions for convex n-gons ($3 \leq n \leq 6$)

Is there a noncongruent equal area tiling by.... that is...

Triangles	vtv	not vtv
normal	?	?
equal perimeter	?	?
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Kupavskii-Pach-Tardos 2018a

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F. 2018, Kupavskii-Pach-Tardos 2018b

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F.-Richter 2020

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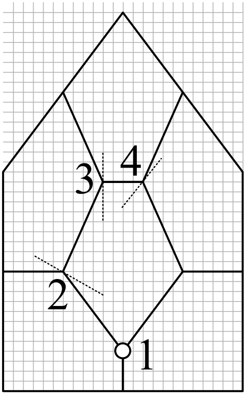
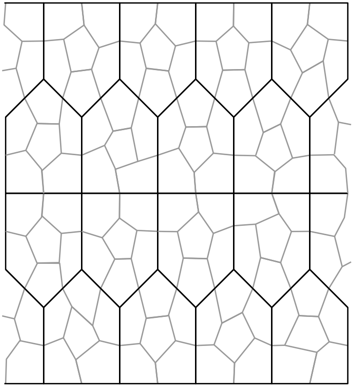
F. 2018

Quadrangles are easier than triangles. Pentagons and hexagons are still easier. E.g.:





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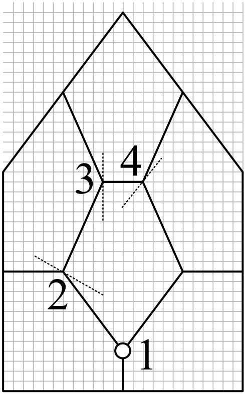

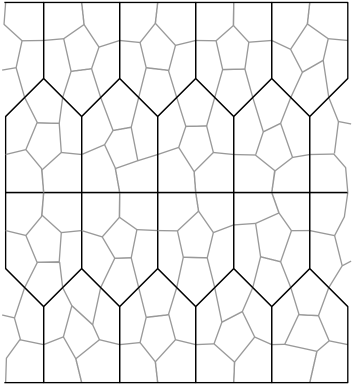

Triangle tilings turn out to be the most restrictive case (equal area *and* equal perimeter impossible).

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F.-Richter 2021+ (preprint)

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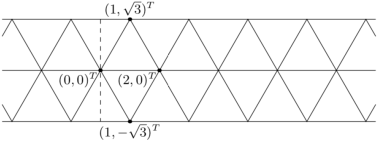
F.-Richter *Journal of Combinatorial Theory A* 2021

Lengthy and technical proof. Similar approach: distort



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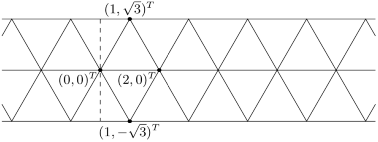

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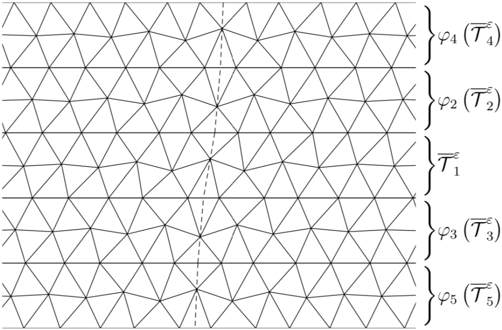
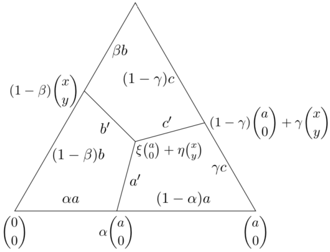
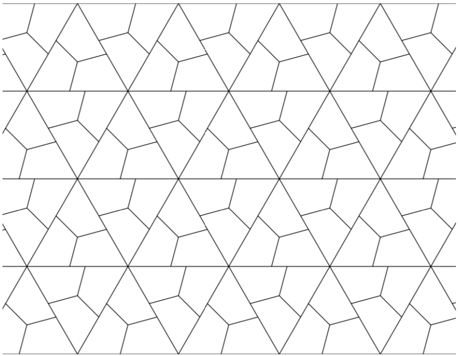


... as close to the regular triangle tiling as desired.

Subdivide triangles into incongruent quadrangles (equal perimeter)





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F.-Richter *Journal of Combinatorial Theory A* 2021

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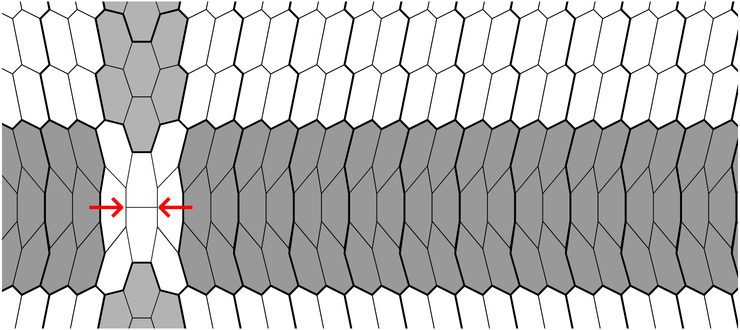
F.-Richter 2020

Recall: quadrangles, pentagons and hexagons are easier.

And usually "not vertex-to-vertex" is less restrictive than "vertex-to-vertex".

But for hexagons it is the other way around: it is harder to find non-vertex-to-vertex tilings by hexagons than vertex-to-vertex ones.



Here: only two non-vertex-to-vertex situations. This leads to...

Part 2

(with Alexey Glazyrin and Zsolt Lángi)

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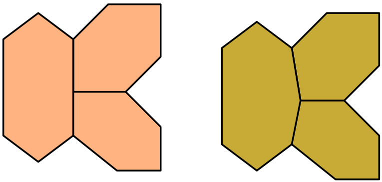
Very similar question: How many heptagons can a tiling by convex n-gons have, if $n \geq 6$?



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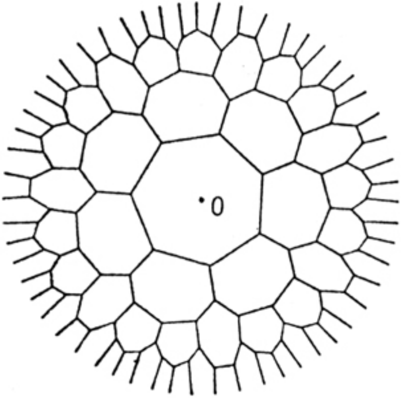
Answer: a lot.



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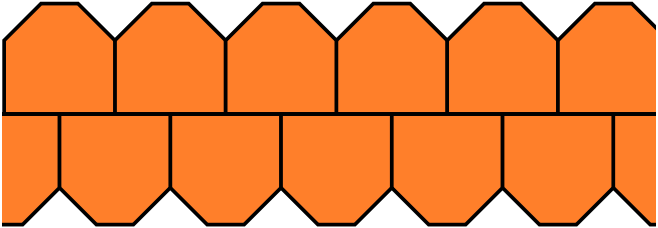

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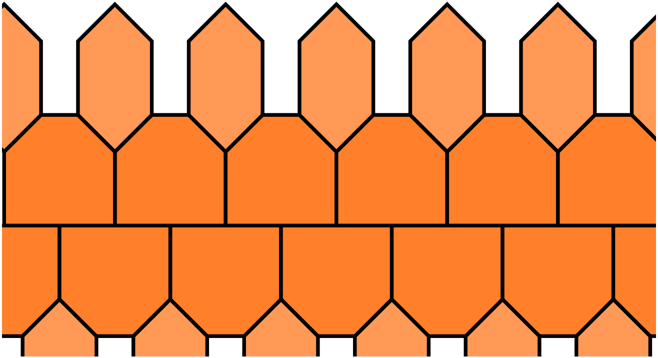

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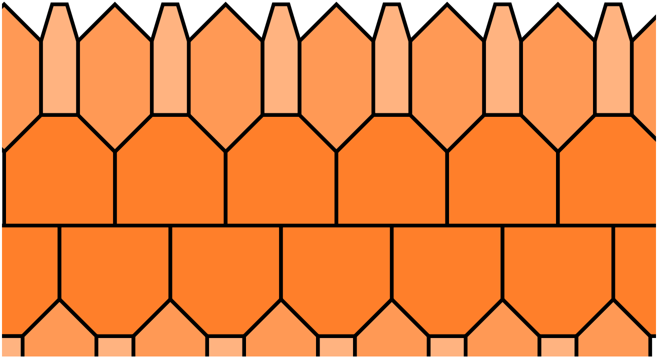

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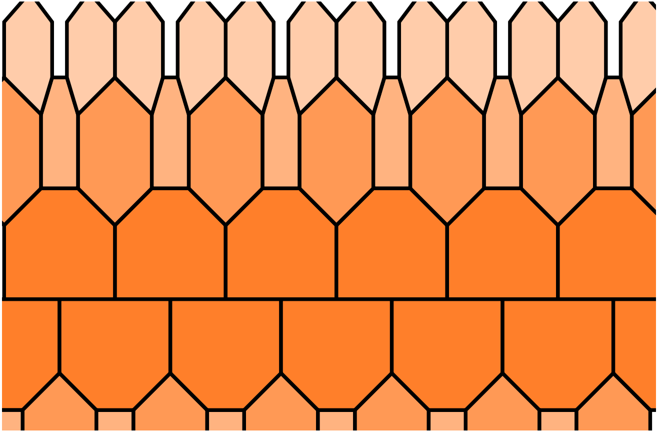

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Theorem (Stehling, Akopyan) In a normal tiling of $\mathbb{R}^2$ by convex polygons, all of them having at least six edges, there are on finitely many n-gons with $n \geq 7$.

Thomas Stehling: Über kombinatorische und graphentheoretische Eigenschaften normaler Pflasterungen, PhD thesis, Dortmund (1989)

Arseniy Akopyan: On the number of non-hexagons in a planar tiling, Comptes Rendus Math. Acad. Sci. Paris 356 (2018).

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Akopyan also provides an upper bound:

$$\#\text{ heptagons } \leq 2\pi\frac{D}{A} - 6$$

D: maximal diameter, A: minimal area.

(so D/A is a measure for how "normal" the tiling is)

Answer: Arbitrarily many. (Even of unit area)

D.F., Alexey Glazyrin, Zsolt Lángi: Hexagon tilings of the plane that are not edge-to-edge, *Acta Math. Hung.* 2021

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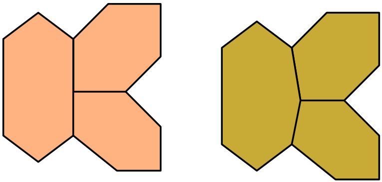
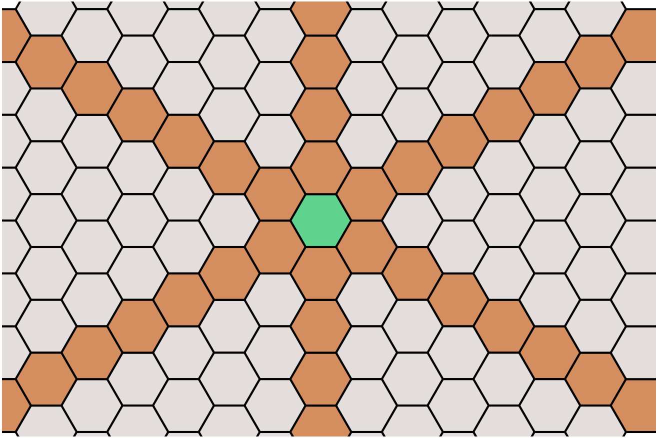
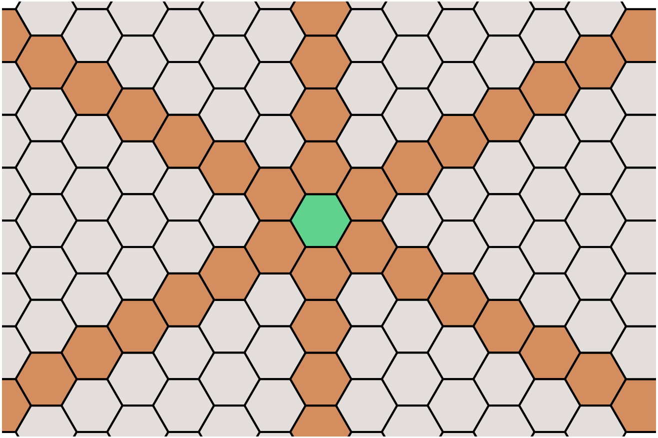
Corollary: A hexagon tiling can have arbitrarily many non-vertex-to-vertex situations (but not infinitely many)



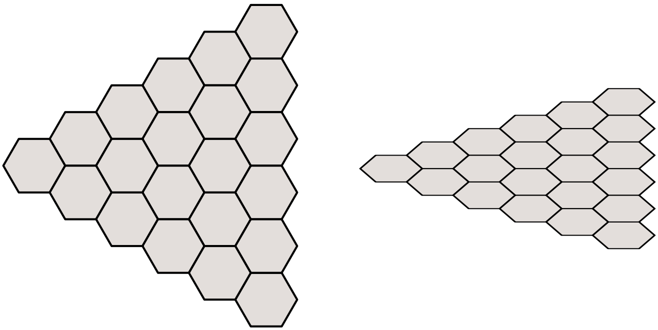


How to obtain "arbitrary many"?

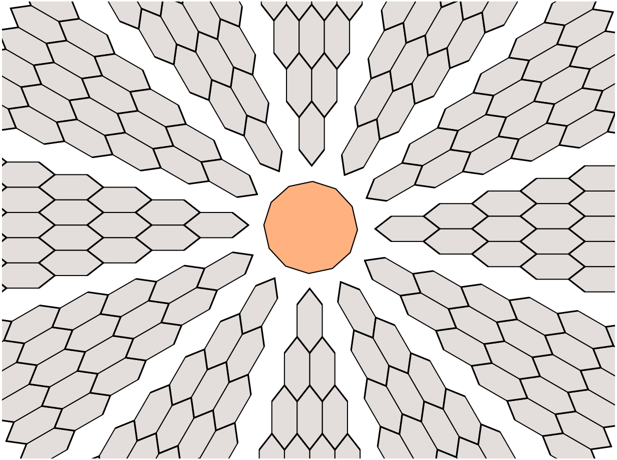


How to obtain "arbitrary many"?

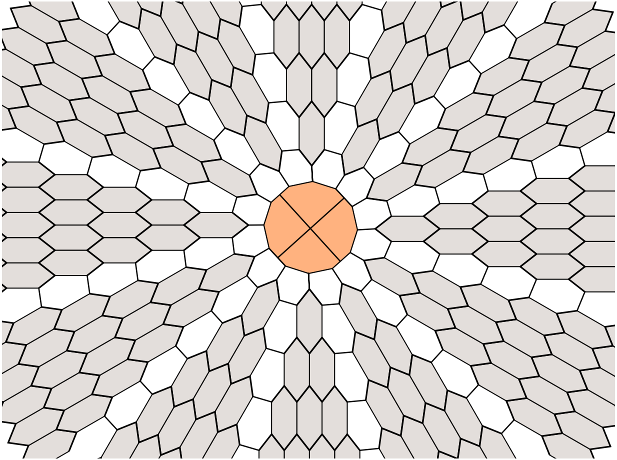
Divide a regular hexagon tiling into six infinite wedges...



...squeeze the wedges...



…arrange them around a central $3n$-gon… (here $n = 4$)



...cut the 3n-gon into n heptagons, and fill the gaps.

We can do the maths in order to compare with Akopyan's bound:

$$\#\text{ heptagons } \leq 2\pi\frac{D}{A} - 6$$

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Our construction yields tilings $\mathcal{T}_n$, with parameters D_n, A_n such that

$$\lim_{n\to\infty} \frac{\#\text{ heptagons in } \mathcal{T}_n}{2\pi\frac{D_n}{A_n} - 6} = \frac{3}{4}$$

Hence we achieve 3/4 of Akopyan's bound.

We can do the maths in order to compare with Akopyan's bound:

$$\# \text{ heptagons } \leq 2\pi \frac{D}{A} - 6$$

Our construction yields tilings $\mathcal{T}_n$, with parameters D_n, A_n such that

$$\lim_{n \to \infty} \frac{\# \text{ heptagons in } \mathcal{T}_n}{2\pi \frac{D_n}{A_n} - 6} = \frac{3}{4}$$

Hence we achieve 3/4 of Akopyan's bound.

It follows that his bound is asymptotically tight

By which I mean, correct up to leading constant, i.e. linear in D/A.

Thank you.