THE LONELY VERTEX PROBLEM

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Abstract. In a locally finite tiling of $\mathbb{R}^n$ by convex polytopes, each point $x \in \mathbb{R}^n$ is either a vertex of at least two tiles, or no vertex at all.

1. Introduction

In [F], the following problem was stated in the context of finite local complexity of self-similar substitution tilings, see Section 3 for details. Throughout the text, 'vertex' always means the vertex of a convex polytope in the usual geometric sense, see for instance [Z]. It means neither a combinatorial vertex of a tile, nor the vertex of a tiling in the sense of [GS] (that is, an isolated point of the intersection of finitely many tiles of a tiling).

Question 1: In a locally finite tiling $\mathcal{T}$ of $\mathbb{R}^n$, where all tiles are convex polytopes, is there a point x which is the vertex of exactly one tile?

In other words: Is there a 'lonely vertex' in a locally finite polytopal tiling?

For tilings in dimension $n = 1$ and $n = 2$, it is easy to see that the answer is negative. In the sequel we show that the answer is negative for all dimensions n. In the remainder of this section we will fix the notation and discuss the necessity of the requirement 'locally finite'. In Section 2 we obtain the main results, namely, Theorem 2.1, Theorem 2.4, and the answer to Question 1 in Theorem 2.5. In Section 3 we apply these results to prove a condition for local finite complexity of self-similar substitution tilings with integer factor. Section 4 contains some further remarks.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The n-dimensional unit sphere is denoted by $\mathbb{S}^n$. For two points $x, y \in \mathbb{R}^n$, the line segment with endpoints x and y is denoted by $\overline{xy}$. A (convex) *polyhedron* is the intersection of finitely many closed halfspaces. A (convex) *polytope* is a bounded polyhedron. In the following, only convex polytopes are considered. Thus we drop the word 'convex' in the sequel, the term 'polytope' always means convex polytope. A spherical polytope is the intersection of a sphere with centre x with finitely many halfspaces H_i, where $x \in \bigcap_i H_i$.

Let $\mathbb{X}$ be either a Euclidean or a spherical space. A collection of polytopes $\mathcal{T} = \{T_n\}_{n \geq 0}$ which is a covering of $\mathbb{X}$ — that is, the union of all polytopes T_i equals $\mathbb{X}$ — as well as a packing of $\mathbb{X}$ — that is, the interiors of the polytopes are pairwise disjoint — is called a (polytopal) *tiling*. A tiling $\mathcal{T}$ is called *locally finite* if each bounded set $U \in \mathbb{X}$ intersects only finitely many tiles of $\mathcal{T}$.

If we do not require the tiling to be locally finite, lonely vertices are possible. For instance, consider a tiling in $\mathbb{R}^2$ which contains the following tiles (see Figure 1): A rectangle R with vertices $(1, 0)$, $(-1, 0)$, $(-1, -1)$, $(1, -1)$, a square S with vertices $(0, 0)$, $(0, 1)$, $(-1, 1)$, $(-1, 0)$,

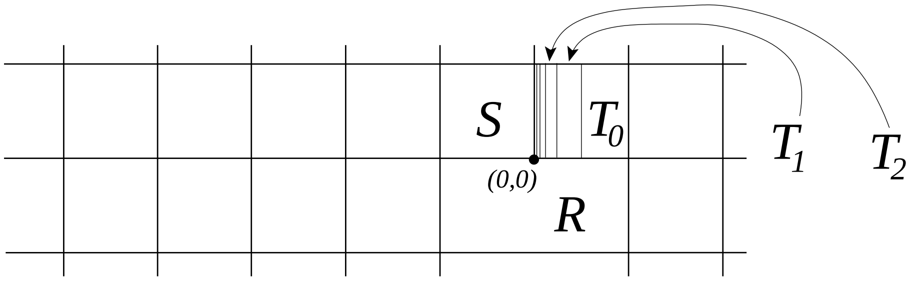

FIGURE 1. A lonely vertex at $(0,0)$ in a tiling which is not locally finite.

and rectangles T_k with vertices $(\frac{1}{2^k},0)$, $(\frac{1}{2^k},1)$, $(\frac{1}{2^{k+1}},1)$ $(\frac{1}{2^{k+1}},0)$,, where $k \geq 0$. Such a tiling is obviously not locally finite: each sphere with centre $(0,0)$ intersects infinitely many tiles. The tile S has $(0,0)$ as a vertex, and $(0,0)$ is vertex of no other tile. That means, such a tiling contains a lonely vertex at $(0,0)$. The requirement of local finiteness is therefore necessary.

2. The main result

We say that a polytope P and a hyperplane H are *just touching* if $P \cap H \neq \varnothing$, but $\mathrm{int}(P) \cap H = \varnothing$, where $\mathrm{int}(P)$ denotes the interior of P. We define the indicator $\mathbf{I}_P$ for the convex n-dimensional spherical polytope P as the function that equals 1 in all internal points of P and 0 else. In what follows we say that two functions are equal if they are equal in all points except in a set of Lebesgue measure zero. We call a convex n-dimensional spherical polytope a *B-type polytope* if it contains two ends of some diameter of the sphere, and an *A-type polytope* else.

Theorem 2.1. *The indicator of any A-type polytope cannot be equal to the linear combination of indicators of a finite number of B-type polytopes.*

Proof. We will prove this theorem by induction on the dimension n of the embedding space $\mathbb{R}^n \supset \mathbb{S}^{n-1}$. Base of induction: $n = 1$. This case is obvious because the unit sphere in $\mathbb{R}^1$, namely, $\mathbb{S}^0 = \{-1, 1\}$, is the only B-type polytope in $\mathbb{S}^0$.

The step of induction is much more demanding. Let Theorem 2.1 be true for all dimensions less than n. We assume that it's false for n. So there is one A-type polytope P and k B-type polytopes $Q_1, \dots, Q_k$ such that

$$\mathbf{I}_P - \sum_1^k \alpha_i \mathbf{I}_{Q_i} = 0$$

for some $\alpha_i \in \mathbb{R}$. Consider any $(n-1)$-dimensional hyperplane containing the centre x of the sphere, for instance $\{x_1 = 0\}$. For $f : \mathbb{R}^n \to \mathbb{R}$ we define f_0^+ and f_0^-:

$$f_0^+(x_2, \dots, x_n) = \lim_{m \to \infty} f(\frac{1}{2^m}, x_2, \dots, x_n)$$

$$f_0^-(x_2, \dots, x_n) = \lim_{m \to \infty} f(-\frac{1}{2^m}, x_2, \dots, x_n)$$

if these limits exist. Let T be an n-dimensional spherical polytope, and let $T_0 = T \cap \{x_1 = 0\}$. T_0 is a spherical polytope of lesser dimension.

Lemma 2.2. *For $f = \boldsymbol{I}_T$, the function f_0^+ exists in all points of $\mathbb{R}^{n-1}$. Moreover, $f_0^+ = \boldsymbol{I}_{T_0}$ holds if not all the internal points of T are lying in negative semispace, and $f_0^+ = 0$ else.*

Proof. There are three cases: the interior of T intersects $\{x_1 = 0\}$, or T just touches this hyperplane and lies in the positive semispace, or T just touches this hyperplane and lies in the negative semispace. All these cases are rather obvious. $\square$

The same lemma is true for f_0^-.

Let us consider now $f = \mathbf{I}_P - \sum_1^k \alpha_i \mathbf{I}_{Q_i}$. Without loss of generality, let one of the $(n-1)$-dimensional faces of P be contained in $\{x_1 = 0\}$, and let P lie in the positive semispace. Then f_0^+ exists, and $f_0^+ = \mathbf{I}_{P_0} - \sum_1^k \alpha_i \mathbf{I}_{Q_i^+}$, where $Q_i^+ = Q_i \cap \{x_1 = 0\}$ if not all the internal points of Q_i are lying in negative semispace, and $Q_i^+ = \varnothing$ else. Likewise, $f_0^- = -\sum_1^k \alpha_i \mathbf{I}_{Q_i^-}$, where Q_i^- are defined analogously. Obviously $f_0^+ = 0$ and $f_0^- = 0$ holds, because they are limits of sequences which are equal to 0. We define $g = f_0^+ - f_0^-$. It follows that $g = 0$ and $g = \mathbf{I}_{P_0} - \sum_1^k \alpha_i (\mathbf{I}_{Q_i^+} - \mathbf{I}_{Q_i^-})$. If the interior of Q_i intersects $\{x_1 = 0\}$, then $Q_i^+ = Q_i^-$, and the corresponding brackets in the sum are equal to 0. (At this point convexity is required.) If Q_i just touches this hyperplane, then one of the members in the corresponding term in brackets is equal to 0. So

$$0 = \mathbf{I}_{P_0} - \sum_1^k \beta_i \mathbf{I}_{S_i},$$

where $S_i = \varnothing$ if the interior of Q_i intersects the hyperplane, $S_i = Q_i^+$ and $\beta_i = \alpha_i$ if Q_i just touches the hyperplane and lies in the positive semispace, $S_i = Q_i^-$ and $\beta_i = -\alpha_i$ if it just touches the hyperplane and lies in the negative semispace.

Lemma 2.3. *If a B-type polytope Q just touches a hyperplane H through the centre x of a sphere, then the polytope $Q \cap H$ is also a B-type polytope.*

Proof. Any B-type polytope contains two ends of some diameter of the n-sphere, say, points k, ℓ. If $\overline{k\ell} \cap H = \{x\}$, then H intersects the interior of the polytope Q. This is impossible, since Q and H are just touching. Therefore $\overline{k\ell} \subset H$. Hence the polytope $Q \cap H$ contains two ends of some diameter of the sphere and is a B-type polytope. $\square$

So all S_i are B-type polytopes, and P_0 is an A-type polytope. We have a contradiction with the proposition of the induction. This completes the proof of Theorem 2.1. $\square$

Theorem 2.4. *Any sphere S in $\mathbb{R}^n$ cannot be partitioned in B-type polytopes and exactly one A-type polytope.*

Proof. We assume there is such a decomposition. P is an A-type polytope and $Q_1, \ldots, Q_k$ are B-type polytopes. Let M_1 and M_2 are two hemispheres such that $M_1 \cup M_2 = S$. Then

$$\mathbf{I}_P + \sum_1^k \mathbf{I}_{Q_i} - \mathbf{I}_{M_1} - \mathbf{I}_{M_2} = 0.$$

This contradicts Theorem 2.1. $\square$

Theorem 2.5. *Let $\mathcal{T}$ be a locally finite polytopal tiling in $\mathbb{R}^n$. There is no point $x \in \mathbb{R}^n$ such that x is a vertex of exactly one polytope of $\mathcal{T}$.*

Proof. We choose a sphere S with centre x such that all faces of the polytopes of $\mathcal{T}$ intersecting S contain x. We can find such a sphere since $\mathcal{T}$ is locally finite. If x is a vertex of a tile T in $\mathcal{T}$, then its intersection with S is an A-type polytope. If $x \in T$ is not a vertex, then the intersection $T \cap S$ is a B-type polytope. Because of Theorem 2.4 there can't be exactly one A-type polytope. So x can't be a vertex of exactly one polytope of the tiling $\mathcal{T}$. □

Remark: The last result generalizes immediately to spherical and hyperbolic tilings: Even though no two of Euclidean space $\mathbb{R}^n$, hyperbolic space $\mathbb{H}^n$ and spherical space $\mathbb{S}^n$ are conformal to each other, they are *locally conformal*: There is a map $f_x : \mathbb{X} \to \mathbb{X}'$ (where $\mathbb{X}, \mathbb{X}' \in \{\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n\}$), such that, for a given point $x \in \mathbb{X}$, lines through x are mapped to lines through $f_x(x)$, and their orientations and the angles between such lines are preserved. This is all we need to generalize the result.

Corollary 2.6. *Each k-face of some tile in a locally finite $\mathcal{T}$ tiling of $\mathbb{R}^n$ by polytopes is covered by finitely many k-faces of some other tiles.*

Proof. We use induction on k. The case $k = 0$ is Theorem 2.5: Any vertex is covered by a vertex of some other tile.

Let the statement be true for $k-1$. Let F be a k-face of some tile $T \in \mathcal{T}$. Let x be a point in the relative interior of F. As above, let S be a sphere with centre x such that

(A) All faces of polytopes in $\mathcal{T}$ intersecting S contain x.

Since F is a k-face, $F' = F \cap S$ is a $(k-1)$-face of $T \cap S$ (in the spherical tiling $\mathcal{T} \cap S$). By the proposition of induction, F' is covered by $(k-1)$-faces F_i. Because of (A), the convex hull $\text{conv}(x, F')$ of x and F' in $\mathbb{R}^n$ is covered by $\text{conv}(x, F_i)$, which are subsets of k-faces in $\mathcal{T}$. This is true for any x in the relative interior of F, thus everywhere. Because of local finiteness, F is covered by finitely many k-faces. □

The following theorem is used in the next section.

Theorem 2.7. *Given a polytopal tiling $\mathcal{T}$, let $G = (V, E)$ be the following undirected graph: V is the set of all vertices of tiles in $\mathcal{T}$. Vertices are identified if they are equal as elements of $\mathbb{R}^n$. E is the set of edges in G, where $(x, y) \in E$ iff the line segment $\overline{xy}$ is an entire edge of some tile in $\mathcal{T}$. Then, all connected components of G are infinite.*

Proof. Obviously, any two vertices of some polytopal tile T are connected by a finite path of edges of T, so they are in the same connected component of G. Therefore, each tile belongs either entirely to a connected component of G or not.

Assume there is a finite connected component C in G. Let $\mathcal{F}$ be the set of all tiles belonging to C. Being finite, the union $\text{supp}(\mathcal{F})$ (which is a polytope, though not necessarily convex) has some outer vertex x.

The vertex x corresponds to an A-type polytope as above. By Theorem 2.4, there is at least one further A-type polytope, belonging to a tile $T \notin \mathcal{F}$. Because T contributes an A-type polytope, x is a vertex of T. This contradicts $T \notin \mathcal{F}$, proving the claim. □

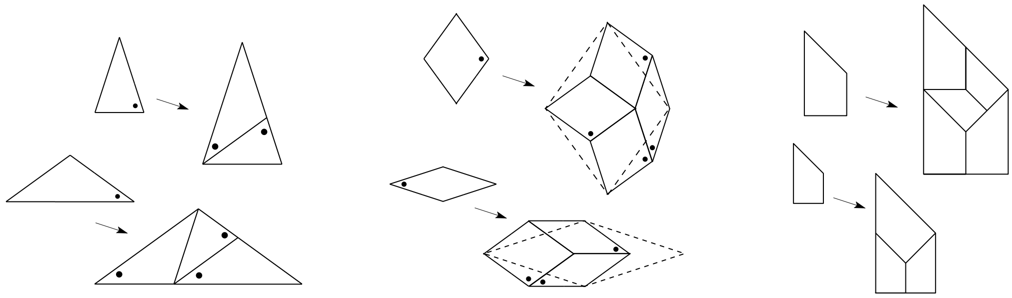

FIGURE 2. Three examples of tile-substitutions: The Penrose substitution rule for triangular tiles (left), the substitution rule for binary tilings (centre), the semi-detached house substitution rule (right).

3. Application to substitution tilings

The discovery of nonperiodic structures with long range order (for instance, Penrose tilings and quasicrystals) had a large impact to many fields in mathematics, see for instance [Lag]. Tile-substitutions are a simple and powerful tool to generate interesting nonperiodic structures with long range order, namely: substitution tilings. The basic idea is to give a finite set of *prototiles* $T_1, \ldots, T_m$, together with a rule how to enlarge each prototile by a common *inflation factor* λ and then dissect it into — or more general, replace it by — copies of the original prototiles. Figure 2 shows some examples of substitution rules. Note, that a substitution σ maps tiles to finite sets of tiles, finite sets of tiles to (larger) finite sets of tiles, and tilings to tilings. By iterating the substitution rule, increasingly larger portions of space are filled, yielding a tiling of the entire space in the limit. For a more precise definition of substitution tilings, see for instance [F2]. For a collection of substitution tilings, and a glossary of related terminology, see [FH].

A tile-substitution rule with a proper dissection, that is, where

$$\lambda T_i = \bigcup_{T \in \sigma(T_i)} T \qquad (1 \leq i \leq m) \tag{1}$$

(where the union is non-overlapping) is called *self-similar* tile-substitution. If (1) does not hold, as in Figure 2 (centre), one may still speak of a substitution tiling, but not of a self-similar tiling.

The following definition turned out to be useful in the theory of nonperiodic tilings. It rules out certain pathological cases and is consistent with other concepts within this theory, for instance the *tiling space*, or the *hull* of a tiling [So], [KP].

Definition 3.1. *Let σ be a tile-substitution with prototiles $T_1, \ldots, T_m$. The sets $\sigma^k(T_i)$ are called* (k-th order) supertiles*.*
A tiling $\mathcal{T}$ is called substitution tiling *(with tile-substitution σ) if for each finite subset $\mathcal{F} \subset \mathcal{T}$ there are i, k such that $\mathcal{F}$ is congruent to a subset of some supertile $\sigma^k(T_i)$.*
The family of all substitution tilings with tile-substitution σ is denoted by $\mathbb{X}_\sigma$.

Many results in the theory of substitution tilings require the tilings under consideration to be of finite local complexity, compare for instance [So], [So2], [LMS].

Definition 3.2. *A tiling $\mathcal{T}$ has* finite local complexity (FLC) *if for each $r > 0$ there are only finitely many different constellations of diameter less than r in $\mathcal{T}$, up to translation.*

Usually, if a certain substitution tiling has FLC, this is easy to see. For instance, each vertex-to-vertex tiling with finitely many prototiles has FLC. More general, the following condition is frequently used [F].

Lemma 3.3. *A tiling is FLC iff there are only finitely many different constellations of two intersecting tiles, up to translation.*

On the other hand, if a tiling does not have FLC, this can be hard to prove, see [D, FrR]. The following theorem covers a broad class of substitution tilings where the inflation factor λ is an integer number. An example of such a tile-substitution is shown in Figure 2 (right), where the inflation factor is 2. A weaker version of this theorem was proved in [F], and it was realized that a negative answer to Question 1 would yield a stronger result. Thus Question 1 was stated in [F] as an open problem.

Theorem 3.4. *Let $\mathcal{T}$ be a self-similar substitution tiling with polytopal prototiles and integer inflation factor. Without loss of generality, let 0 be a vertex of each prototile. If the $\mathbb{Z}$-span of all vertices of the prototiles is a discrete lattice, then $\mathcal{T}$ is of finite local complexity.*

It is remarkable that a requirement on the shape of the prototiles, without any word about the tile-substitution itself, suffices to guarantee FLC. Note, that we do not require the tiles to be convex at this point. It suffices that they are unions of finitely many convex polytopes.

Proof. We begin by showing that all vertices contained in some supertile $S = \sigma^k(T_i) = \{T, T', T'', \ldots\}$ belong to the same connected component of the graph G, with G as in Theorem 2.7. First we consider vertices on the edge of the support of a supertile. A (super-)edge of the supertile S consists of entire edges of some tiles. Thus, all vertices in a single (super-)edge of S belong to the same component C of G. Consequently, all vertices in the union of the edges of the supertile S belong to C.

Now, consider a k-face F of S, where $k \geq 2$. Let all vertices on the boundary of F (of dimension $k-1$) be in the same component C of G. If there is a vertex x in F with $x \notin C$, it belongs to a finite component of G in F which is disjoint with the boundary of F. Thus, F can be extended to a k-dimensional polytopal tiling with the finite component C in the corresponding graph G. But this contradicts Theorem 2.7. Consequently, all vertices in F belong to C. Inductively — by finite induction on k — all vertices contained in the supertile S belong to C.

Now, let Γ be the lattice spanned by the vertices of the prototiles. Since the inflation factor is an integer, the vertices of each supertile S are elements of Γ. All tile-vertices contained in S belong to the same connected component of G, thus — by definition of G — they are connected by a finite path of entire tile edges $\overline{xy}$ with some vertex of S. By the condition in the theorem, $x - y \in \Gamma$ for all such edges $\overline{xy}$. Therefore, all vertices in the supertile are contained in Γ. Consequently, all vertices of $\mathcal{T}$ are elements of Γ.

In particular, if two tiles in $\mathcal{T}$ have nonempty intersection, there is only a finite number of possible position of the vertices of these tiles, by the discreteness of Γ. By Lemma 3.3, $\mathcal{T}$ has FLC. □

4. Remarks

We have established the impossibility of a lonely vertex in a locally finite polytopal tiling in Euclidean, spherical and hyperbolic space of any dimension. Some consequences are discussed in this paper. Naturally, further questions arise. For instance, what can be said about lonely vertices in locally finite tilings with non-convex tiles?

Another natural question is: What can be said about exactly two vertices? Since a lonely vertex is impossible, there may be restrictions for constellations around a point which is a vertex of exactly two tiles T, T'. Indeed, one obtains the following result. Roughly spoken, it means that edges of T and T' either are coincident or opposite. In particular, the number of edges of T containing x equals the number of edges of T' containing x. For clarity, we state the result in terms of A-type and B-type polytopes.

Theorem 4.1. *Let a locally finite tiling of the unit sphere $\mathbb{S}^n$ by polytopes contain exactly two A-type polytopes P, P'. Let x be a vertex of P. Then either x or $-x$ is a vertex of P'.*

Proof. The cases $n = 0$ and $n = 1$ are obvious. So, let $n > 1$.

We proceed by considering possible shapes of B-type polytopes. Any B-type polytope is cut out of the unit sphere $\mathbb{S}^n$, embedded in $\mathbb{R}^{n+1}$, by halfspaces $H_1^+, \ldots, H_m^+$, where $x \in \bigcap_i H_i^+$. Each such halfspace H_i^+ can be represented by a vector c_i which is normal to the bounding hyperplane $H_i = \partial H_i^+$: $H_i^+ = \{x : c_i x \geq 0\}$. We can assume the set of hyperplanes to be minimal. That is, the normal vectors of these hyperplanes are linearly independent (otherwise there would be a superfluous defining inequality $c_i x \geq 0$; that means, a superfluous halfspace). Therefore, the intersection $M := \bigcap_i H_i$ is an $(n+1-m)$-dimensional linear subspace. Since the considered polytope is B-type, it contains two endpoints of some diameter of the sphere. Thus M has to be at least of dimension one. It follows $m \leq n$, and the intersection $\mathbb{S}^n \cap M$ (which is the boundary of the considered B-type polytope), is an $(n-m)$-dimensional unit sphere. In particular, a B-type polytope has a vertex x if and only if it is defined by exactly n halfspaces. Then, $-x$ is also a vertex of this B-type polytope.

By Theorem 2.5, the vertex x of P is a vertex of some further polytope. Either A-type (then P'), or B-type, say, P''. In the latter case, by the reasoning above, $-x$ is a vertex of P'', too.

If $-x$ would be surrounded entirely by B-type polytopes, x also would, which is impossible. Thus, $-x$ is a vertex of an A-type polytope. The only possibility is that $-x$ is a vertex of P'. □

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