PERFECT COLOURINGS OF REGULAR GRAPHS

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Abstract. A vertex colouring of some graph is called *perfect* if each vertex of colour i has exactly a_{ij} neighbours of colour j. Being perfect imposes several restrictions on the colour incidence matrix (a_{ij}). We list several (old and new) necessary conditions for a matrix to be the colour incidence matrix of a perfect colouring. Moreover we show that a certain combination of these conditions is also sufficient. Using this we determine a list of all colour incidence matrices corresponding to perfect colourings of 3-regular, 4-regular and 5-regular graphs with two, three and four colours, respectively. As an application we determine all perfect colourings of the edge graphs of the Platonic solids with two, three and four colours, respectively.

1. Introduction

In this paper an m-colouring of some graph $G = (V, E)$ is a partition of V into disjoint nonempty sets $V_1, \ldots, V_m$. Note that we do not require adjacent vertices to have different colours. A colouring of the vertex set V of some graph $G = (V, E)$ with m colours is called *perfect* if (1) all colours are used, and (2) for all i, j the number of neighbours of colours j of any vertex v of colour i is a constant a_{ij}. See [10, Sec. 9.3], where perfect colourings are called *equitable partitions*. It seems that the term "equitable partition" is used for two different concepts in graph theory: one is what we call perfect colouring above, the second is a colouring where every pair of adjacent vertices has different colours, and where the number of any two colour classes differs by at most one. Hence we will refer to the first concept by the term perfect colouring here. See Figures 1-5 below for some examples of perfect colourings.

The matrix $A = (a_{ij})$ is called the *colour incidence matrix* of the perfect colouring. Any subgroup of the automorphism group of a graph G induces a perfect colouring of G by considering the orbits of the group [10, Sec. 9.3], but not every perfect colouring arises from a graph automorphism.

Perfect colourings have been studied in different contexts, see for instance [4, 5, 8, 9, 12, 13]. This paper is organized as follows: In Section 2 several combinatorial arguments and a little algebraic graph theory yield a list of (old and new) necessary conditions for a matrix being a colour incidence matrix. We show that a combination of these conditions is also sufficient in Section 3. Using these results we computed the list of all colour incidence matrices of perfect colourings with two, three and four colours for k-regular connected graphs for $k \in \{3, 4, 5\}$, respectively (Section 4). To our best knowledge this list has not been published before.

As an application we determine all perfect colourings of the edge graphs of the Platonic solids using two, three and four colours, respectively (Section 5). All perfect 2-colourings of the edge graphs of the Platonic solids have been determined in [3] already. The perfect three-colourings of the edge graphs of the Platonic solids were studied in [1], but some cases were missed. To our knowledge the perfect 4-colourings of the edge graphs of the Platonic solids given in this paper are new.

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2. Necessary conditions for colour incidence matrices

Let G be a k-regular connected graph and $A \in \mathbb{N}^{m\times m}$ be a colour incidence matrix for a perfect colouring of G with m colours. Clearly each row sum of A equals k. Thus for each row there are $\binom{k+m-1}{m-1}$ different possibilities to distribute the entries such that the row sum equals k. Hence there are $\binom{k+m-1}{m-1}^m$ matrices to consider altogether. We use the following lemmas to reduce the number of matrices to consider.

Let $A = (a_{ij})_{m\times m}$ be a colour incidence matrix for a graph $G = (V, E)$. This first observation says A must possess a weak form of symmetry, following from the symmetry of vertex adjacency relations.

Lemma 1. *Suppose $A = (a_{ij})_{m\times m}$ is a colour incidence matrix for a graph $G = (V, E)$. Then, $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $1 \le i, j \le m$.*

Since we are interested in connected regular graphs, we also sought conditions for colour incidence matrices of connected graphs. The following lemma is essentially a rephrasing of the fact that the induced graph $G' = (\{V_1, \dots, V_m\}, E')$ (where the colour classes V_i of G are the vertices of G', and there is an edge between V_i and V_j in G' if there is an edge between V_i and V_j in G) is connected, hence has a spanning tree. The pairs $(n_1, n_2), (n_3, n_4), \dots$ are an enumeration of the edges of a spanning tree of G'. In the lemma, the expressions in curly braces are multisets.

Lemma 2. *Suppose $A = (a_{ij})_{m\times m}$ is a colour incidence matrix for a connected graph $G = (V, E)$. Then there exists an $(m-1)$-tuple $(a_{n_1,n_2}, a_{n_3,n_4}, \dots, a_{n_{2m-3},n_{2m-2}})$ of nonzero components such that $n_1 \ne n_2$, and if $m \ge 3$, $n_{2i-1} \notin \{n_1, \dots, n_{2i-2}\}$ and $n_{2i} \in \{n_1, \dots, n_{2i-2}\}$ for $2 \le i \le m-1$.*

Proof. By connectedness, there exist distinct n_1, n_2 such that $a_{n_1,n_2} \ne 0$. If $m = 2$, this means $a_{12} \ne 0$. For $m \ge 3$, there must also exist n_3 such that $a_{n_3,n_1} \ne 0$ or $a_{n_3,n_2} \ne 0$, otherwise G is the disjoint union of the graph induced by $V_{n_1} \cup V_{n_2}$ and the graph induced by the other vertices. Choose $n_4 \in \{n_1, n_2\}$ such that $a_{n_3,n_4} \ne 0$. The result then follows inductively. □

Remark 1. The condition of Lemma 2 together with the weak symmetry condition of Lemma 1 is equivalent to A not being conjugate via a permutation matrix to a block diagonal matrix.

For $1 \le i, j \le m$, let v_i and v_j denote the number of vertices in V_i and V_j respectively. The central fact that provides the next result is the following:

$$a_{ij}v_i = a_{ji}v_j \text{ holds for all } 1 \le i \ne j \le m. \tag{1}$$

Hence there are several ways to relate v_i and v_j by a_{ij} and a_{ji}.

Lemma 3. *Suppose $A = (a_{ij})_{m\times m}$ is a colour incidence matrix for a graph $G = (V, E)$. Then for any nontrivial cycle $(n_1, n_2, \dots, n_t)$ in the symmetric group S_m on $\{1, 2, \dots, m\}$,*

$$a_{n_1,n_2}a_{n_2,n_3}\cdots a_{n_{t-1},n_t}a_{n_t,n_1} = a_{n_2,n_1}a_{n_3,n_2}\cdots a_{n_t,n_{t-1}}a_{n_1,n_t}.$$

Proof. When $t = 2$, this is trivial: $a_{n_1,n_2}a_{n_2,n_1} = a_{n_2,n_1}a_{n_1,n_2}$. If one of the a_{ij} is zero then also $a_{ji} = 0$ and the statement is trivial, too.

When $2 < t \le m$, this identity arises from equation (1). Indeed, $a_{n_1,n_2}v_{n_1} = a_{n_2,n_1}v_{n_2}$, and therefore $a_{n_1,n_2}a_{n_2,n_3}v_{n_1} = a_{n_2,n_1}a_{n_2,n_3}v_{n_2} = a_{n_2,n_1}a_{n_3,n_2}v_{n_3}$. It follows by induction that $a_{n_1,n_2}a_{n_2,n_3}\cdots a_{n_{t-1},n_t}v_{n_1} = a_{n_2,n_1}a_{n_3,n_2}\cdots a_{n_t,n_{t-1}}v_{n_t}$. Combining this with $a_{n_1,n_t}v_{n_1} = a_{n_t,n_1}v_{n_t}$ gives the expression above. □

For $m = 2, 3, 4$, the next lemmas count the number of vertices of each colour. Depending on the different ways how the induced graph $G' = (\{V_1, \dots, V_m\}, E')$ (see above) might be connected there are several cases how to express v_i in terms of a_{ij} and v_j. By Cayley's formula the number of possible spanning trees of G' equals m^{m-2}. Hence there is only one case to consider for $m = 2$, there are three cases for $m = 3$ and 16 cases for $m = 4$. The case $m = 2$ (Lemma 4) occurs in [3]. The case $m = 3$ (Lemma 5) occurs in [1]. Moreover, for the case $m = 3$, Proposition 2.1 in [2] is equivalent to Lemmas 2, 3, and 5 combined. Hence we sketch a proof for the case when $m = 3$ only.

Lemma 4. *Let* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *be a colour incidence matrix of some connected graph* $G = (V, E)$*. Then* a_{12} *and* a_{21} *are both nonzero and*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}}}, v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1}.$$

Lemma 5. *Let* $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *be a colour incidence matrix of some connected graph* $G = (V, E)$*. Let* v_i *denote the number of vertices of colour* i*.*

(1) If $a_{12} \neq 0$ *and* $a_{13} \neq 0$*, then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{13}}{a_{31}}}, v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{21}a_{13}}{a_{12}a_{31}}}, v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{31}a_{12}}{a_{13}a_{21}} + 1}.$$

(2) If $a_{12} \neq 0$ *and* $a_{23} \neq 0$*, then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{12}a_{23}}{a_{21}a_{32}}}, v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{23}}{a_{32}}}, v_3 = \frac{|V|}{\frac{a_{32}a_{21}}{a_{23}a_{12}} + 1 + \frac{a_{32}}{a_{23}}}.$$

(3) If $a_{13} \neq 0$ *and* $a_{23} \neq 0$*, then*

$$v_1 = \frac{|V|}{1 + \frac{a_{13}a_{32}}{a_{31}a_{23}} + \frac{a_{13}}{a_{31}}}, v_2 = \frac{|V|}{\frac{a_{23}a_{31}}{a_{32}a_{13}} + 1 + \frac{a_{23}}{a_{32}}}, v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{32}}{a_{23}} + 1}.$$

The proof goes along the lines of counting the total number of vertices as $|V| = v_1 + v_2 + v_3$, considering that in Case 1 we have $a_{1i}v_1 = a_{i1}v_i$, hence $|V| = v_1 + \frac{a_{12}}{a_{21}}v_1 + \frac{a_{13}}{a_{31}}v_1$. Here, a_{23} and a_{32} may equal zero. Hence they cannot necessarily be used to relate v_2 and v_3. But, as in the counting procedure in the proof of Lemma 3, one obtains $a_{21}a_{13}v_2 = a_{12}a_{31}v_3$, and consequently the expressions for v_2 and v_3 above.

We have the following analogue for four colours:

Lemma 6. *Let* $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ *be a colour incidence matrix of some connected graph* $G = (V, E)$*. Let* v_i *denote the number of vertices of colour* i*.*

(1) If $a_{12} \neq 0$*,* $a_{13} \neq 0$*, and* $a_{14} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{13}}{a_{31}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{21}a_{13}}{a_{12}a_{31}} + \frac{a_{21}a_{14}}{a_{12}a_{41}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{31}a_{12}}{a_{13}a_{21}} + 1 + \frac{a_{31}a_{14}}{a_{13}a_{41}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{41}a_{12}}{a_{14}a_{21}} + \frac{a_{41}a_{13}}{a_{14}a_{31}} + 1}.$$

(2) If $a_{12} \neq 0$*,* $a_{13} \neq 0$*, and* $a_{24} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{13}}{a_{31}} + \frac{a_{12}a_{24}}{a_{21}a_{42}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{21}a_{13}}{a_{12}a_{31}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{31}a_{12}}{a_{13}a_{21}} + 1 + \frac{a_{31}a_{12}a_{24}}{a_{13}a_{21}a_{42}}}, \qquad v_4 = \frac{|V|}{\frac{a_{42}a_{21}}{a_{24}a_{12}} + \frac{a_{42}}{a_{24}} + \frac{a_{42}a_{21}a_{13}}{a_{24}a_{12}a_{31}} + 1}.$$

(3) If $a_{12} \neq 0$*,* $a_{13} \neq 0$*, and* $a_{34} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{13}}{a_{31}} + \frac{a_{13}a_{34}}{a_{31}a_{43}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{21}a_{13}}{a_{12}a_{31}} + \frac{a_{21}a_{13}a_{34}}{a_{12}a_{31}a_{43}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{31}a_{12}}{a_{13}a_{21}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{43}a_{31}}{a_{34}a_{13}} + \frac{a_{43}a_{31}a_{12}}{a_{34}a_{13}a_{21}} + \frac{a_{43}}{a_{34}} + 1}.$$

(4) *If* $a_{12} \neq 0$, $a_{14} \neq 0$, *and* $a_{23} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{12}a_{23}}{a_{21}a_{32}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{21}a_{14}}{a_{12}a_{41}}}$$

$$v_3 = \frac{|V|}{\frac{a_{32}a_{21}}{a_{23}a_{12}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{32}a_{21}a_{14}}{a_{23}a_{12}a_{41}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{41}a_{12}}{a_{14}a_{21}} + \frac{a_{41}a_{12}a_{23}}{a_{14}a_{21}a_{32}} + 1}.$$

(5) *If* $a_{12} \neq 0$, $a_{14} \neq 0$, *and* $a_{34} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{14}a_{43}}{a_{41}a_{34}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{21}a_{14}a_{43}}{a_{12}a_{41}a_{34}} + \frac{a_{21}a_{14}}{a_{12}a_{41}}}$$

$$v_3 = \frac{|V|}{\frac{a_{34}a_{41}}{a_{43}a_{14}} + \frac{a_{34}a_{41}a_{12}}{a_{43}a_{14}a_{21}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{41}a_{12}}{a_{14}a_{21}} + \frac{a_{43}}{a_{34}} + 1}.$$

(6) *If* $a_{12} \neq 0$, $a_{23} \neq 0$, *and* $a_{24} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{12}a_{23}}{a_{21}a_{32}} + \frac{a_{12}a_{24}}{a_{21}a_{42}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{32}a_{21}}{a_{23}a_{12}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{32}a_{24}}{a_{23}a_{42}}}, \qquad v_4 = \frac{|V|}{\frac{a_{42}a_{21}}{a_{24}a_{12}} + \frac{a_{42}}{a_{24}} + \frac{a_{42}a_{23}}{a_{24}a_{32}} + 1}.$$

(7) *If* $a_{12} \neq 0$, $a_{23} \neq 0$, *and* $a_{34} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{12}a_{23}}{a_{21}a_{32}} + \frac{a_{12}a_{23}a_{34}}{a_{21}a_{32}a_{43}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{23}a_{34}}{a_{32}a_{43}}}$$

$$v_3 = \frac{|V|}{\frac{a_{32}a_{21}}{a_{23}a_{12}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{43}a_{32}a_{21}}{a_{34}a_{23}a_{12}} + \frac{a_{43}a_{32}}{a_{34}a_{23}} + \frac{a_{43}}{a_{34}} + 1}.$$

(8) *If* $a_{12} \neq 0$, $a_{24} \neq 0$, *and* $a_{34} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}} + \frac{a_{12}a_{24}a_{43}}{a_{21}a_{42}a_{34}} + \frac{a_{12}a_{24}}{a_{21}a_{42}}}, \qquad v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1 + \frac{a_{24}a_{43}}{a_{42}a_{34}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{34}a_{42}a_{21}}{a_{43}a_{24}a_{12}} + \frac{a_{34}a_{42}}{a_{43}a_{24}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{42}a_{21}}{a_{24}a_{12}} + \frac{a_{42}}{a_{24}} + \frac{a_{43}}{a_{34}} + 1}.$$

(9) *If* $a_{13} \neq 0$, $a_{14} \neq 0$, *and* $a_{23} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{13}a_{32}}{a_{31}a_{23}} + \frac{a_{13}}{a_{31}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{23}a_{31}}{a_{13}a_{32}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{23}a_{31}a_{14}}{a_{32}a_{13}a_{41}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{31}a_{14}}{a_{13}a_{41}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{41}a_{13}a_{32}}{a_{14}a_{31}a_{23}} + \frac{a_{41}a_{13}}{a_{14}a_{31}} + 1}.$$

(10) *If* $a_{13} \neq 0$, $a_{14} \neq 0$, *and* $a_{24} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{14}a_{42}}{a_{41}a_{24}} + \frac{a_{13}}{a_{31}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{24}a_{41}}{a_{14}a_{42}} + 1 + \frac{a_{24}a_{41}a_{13}}{a_{42}a_{14}a_{31}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{31}a_{14}a_{42}}{a_{13}a_{41}a_{24}} + 1 + \frac{a_{31}a_{14}}{a_{13}a_{41}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{42}}{a_{24}} + \frac{a_{41}a_{13}}{a_{14}a_{31}} + 1}.$$

(11) *If* $a_{13} \neq 0$, $a_{23} \neq 0$, *and* $a_{24} \neq 0$ *then*

$$v_1 = \frac{|V|}{1 + \frac{a_{13}a_{32}}{a_{31}a_{23}} + \frac{a_{13}}{a_{31}} + \frac{a_{13}a_{32}a_{24}}{a_{31}a_{23}a_{42}}}, \qquad v_2 = \frac{|V|}{\frac{a_{23}a_{31}}{a_{32}a_{13}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{32}a_{24}}{a_{23}a_{42}}}, \qquad v_4 = \frac{|V|}{\frac{a_{42}a_{23}a_{31}}{a_{24}a_{32}a_{13}} + \frac{a_{42}}{a_{24}} + \frac{a_{42}a_{23}}{a_{24}a_{32}} + 1}.$$

(12) If $a_{13} \neq 0$, $a_{23} \neq 0$, and $a_{34} \neq 0$ then

$$v_1 = \frac{|V|}{1 + \frac{a_{13}a_{32}}{a_{31}a_{23}} + \frac{a_{13}}{a_{31}} + \frac{a_{13}a_{34}}{a_{31}a_{43}}}, \qquad v_2 = \frac{|V|}{\frac{a_{23}a_{31}}{a_{32}a_{13}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{23}a_{34}}{a_{32}a_{43}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{43}a_{31}}{a_{34}a_{13}} + \frac{a_{43}a_{32}}{a_{34}a_{23}} + \frac{a_{43}}{a_{34}} + 1}.$$

(13) If $a_{13} \neq 0$, $a_{24} \neq 0$, and $a_{34} \neq 0$ then

$$v_1 = \frac{|V|}{1 + \frac{a_{13}a_{34}a_{42}}{a_{31}a_{43}a_{24}} + \frac{a_{13}}{a_{31}} + \frac{a_{13}a_{34}}{a_{31}a_{43}}}, \qquad v_2 = \frac{|V|}{\frac{a_{24}a_{43}a_{31}}{a_{42}a_{34}a_{13}} + 1 + \frac{a_{24}a_{43}}{a_{42}a_{34}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{31}}{a_{13}} + \frac{a_{34}a_{42}}{a_{43}a_{24}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{43}a_{31}}{a_{34}a_{13}} + \frac{a_{42}}{a_{24}} + \frac{a_{43}}{a_{34}} + 1}.$$

(14) If $a_{14} \neq 0$, $a_{23} \neq 0$, and $a_{24} \neq 0$ then

$$v_1 = \frac{|V|}{1 + \frac{a_{14}a_{42}}{a_{41}a_{24}} + \frac{a_{14}a_{42}a_{23}}{a_{41}a_{24}a_{32}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{24}a_{41}}{a_{42}a_{14}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{32}a_{24}a_{41}}{a_{23}a_{42}a_{14}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{32}a_{24}}{a_{23}a_{42}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{42}}{a_{24}} + \frac{a_{42}a_{23}}{a_{24}a_{32}} + 1}.$$

(15) If $a_{14} \neq 0$, $a_{23} \neq 0$, and $a_{34} \neq 0$ then

$$v_1 = \frac{|V|}{1 + \frac{a_{14}a_{43}a_{32}}{a_{41}a_{34}a_{23}} + \frac{a_{14}a_{43}}{a_{41}a_{34}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{23}a_{34}a_{41}}{a_{32}a_{43}a_{14}} + 1 + \frac{a_{23}}{a_{32}} + \frac{a_{23}a_{34}}{a_{32}a_{43}}}$$

$$v_3 = \frac{|V|}{\frac{a_{34}a_{41}}{a_{43}a_{14}} + \frac{a_{32}}{a_{23}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{43}a_{32}}{a_{34}a_{23}} + \frac{a_{43}}{a_{34}} + 1}.$$

(16) If $a_{14} \neq 0$, $a_{24} \neq 0$, and $a_{34} \neq 0$ then

$$v_1 = \frac{|V|}{1 + \frac{a_{14}a_{42}}{a_{41}a_{24}} + \frac{a_{14}a_{43}}{a_{41}a_{34}} + \frac{a_{14}}{a_{41}}}, \qquad v_2 = \frac{|V|}{\frac{a_{24}a_{41}}{a_{42}a_{14}} + 1 + \frac{a_{24}a_{43}}{a_{42}a_{34}} + \frac{a_{24}}{a_{42}}}$$

$$v_3 = \frac{|V|}{\frac{a_{34}a_{41}}{a_{43}a_{14}} + \frac{a_{34}a_{42}}{a_{43}a_{24}} + 1 + \frac{a_{34}}{a_{43}}}, \qquad v_4 = \frac{|V|}{\frac{a_{41}}{a_{14}} + \frac{a_{42}}{a_{24}} + \frac{a_{43}}{a_{34}} + 1}.$$

Again the proof is in complete analogy to the one of Lemma 5: we count $|V|$ by $|V| = v_1+v_2+v_3+v_4$. Then we express for instance v_1 using Equation (1) in terms of $v_2, v_3, \ldots$ using the appropriate (non-zero) a_{ij}, depending on the possible spanning tree for G'.

3. Sufficiency of Lemmas 1, 2, and 3

It turns out that Lemmas 1-3 provide the necessary and sufficient conditions for a matrix to be the colour incidence matrix of a perfect colouring of some connected graph (not necessarily regular).

Theorem 7. *Let $A = (a_{ij}) \in \mathbb{N}^{m\times m}$. Then A is a colour incidence matrix for a perfect colouring of a graph $G = (V, E)$ if and only if*

(1) $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $1 \leq i, j \leq m$, and
(2) for any nontrivial cycle $(n_1, n_2, \ldots, n_t)$ in the symmetric group S_m on $\{1, 2, \ldots, m\}$,

$$a_{n_1,n_2}a_{n_2,n_3}\cdots a_{n_{t-1},n_t}a_{n_t,n_1} = a_{n_2,n_1}a_{n_3,n_2}\cdots a_{n_t,n_{t-1}}a_{n_1,n_t}.$$

Moreover, there exists a connected graph G and a perfect colouring of G having colour incidence matrix A if and only if there exists an $(m-1)$-tuple $(a_{n_1,n_2}, a_{n_3,n_4}, \ldots, a_{n_{2m-3},n_{2m-2}})$ of nonzero components such that $n_1 \neq n_2$, and if $m \geq 3$, $n_{2i-1} \notin \{n_1, \ldots, n_{2i-2}\}$ and $n_{2i} \in \{n_1, \ldots, n_{2i-2}\}$ for $2 \leq i \leq m-1$.

Proof. The necessity of these conditions are Lemmas 1-3. We show the sufficiency of Lemmas 1 and 3 by constructing a graph together with a perfect colouring for any given colour incidence matrix A fulfilling the weak symmetry and cycle conditions. We rely on two (probably well-known) facts:

Lemma 8. *There exists a k-regular graph with n vertices if and only if $n \geq k+1$ and nk is even.*

Proof. This is a simple consequence of the Erdős-Gallai Theorem. But for the if-part it is even simpler to construct a graph $G = (V, E)$ with the required properties. Let $V = \{0, 1, \ldots, n-1\}$.

For $k = 2m$ let $E = \{\{i, i+j \bmod n\} \mid 0 \leq i \leq n-1, 1 \leq j \leq m\}$. Clearly each vertex is of degree k.

For $k = 2m+1$ (which is possible only if n is even) add edges $\{i, i+\frac{n}{2}\}$ to E $(0 \leq i \leq \frac{n}{2}-1)$. □

A bipartite graph with bipartition (V_1, V_2) is (p, q)-semiregular if each vertex in V_1 (resp. V_2) has degree p (resp. q). Note that $p = 0$ if and only if $q = 0$.

Lemma 9. *Let $p \leq s$, $q \leq r$. If $pr = qs$ then there exists a (p, q)-semiregular bipartite graph with bipartition (U, V) and $|U| = r$, $|V| = s$.*

Proof. We assume p, $q \neq 0$ and construct a graph with the desired properties. Denote the vertices in U by $u_0, u_1, \ldots, u_{r-1}$ and the vertices in V by $v_0, v_1, \ldots, v_{s-1}$. We will use a greedy construction: join vertex u_0 with $v_0, v_1, \ldots, v_{p-1 \bmod s}$, join vertex u_1 with $v_{p \bmod s}, v_{p+1 \bmod s}, \ldots, v_{2p-1 \bmod s}$, and so on. Clearly every vertex in U is adjacent to p vertices in V, and, because of $pr = qs$, every vertex in V is adjacent to exactly q vertices in U. □

Now let A be a matrix satisfying the conditions of Lemma 1 (weak symmetry) and of Lemma 3 (cycles condition). We construct a graph G with colour incidence matrix $A \in \mathbb{N}^{m \times m}$, where the colour classes are denoted by $V_1, \ldots, V_m$.

- Using the non-zero non-diagonal entries we obtain the ratio of v_i to v_j. (If the matrix is a sum of blocks, then these ratios are only between each pair of vertices in the same corresponding block). Because of the cycles condition, we know these ratios are consistent, and there exists an ordered m-tuple of positive integers $(v'_1, \ldots, v'_m)$ satisfying all required relations.
- There is a large enough multiple $(v_1, \ldots, v_m)$ of the m-tuple above such that for each i, $v_i \geq \max\{a_{ii}+1, a_{ji}(j \neq i)\}$ and $a_{ii}v_i$ is even.
- Let V_i have v_i elements. With the vertices in V_i, form an a_{ii}-regular graph (according to Lemma 8). Between distinct cells V_i and V_j form the edge set of an (a_{ij}, a_{ji})-semiregular bipartite graph (according to Lemma 9).

The constructed graph fulfils the conditions of Lemmas 1 and 3. Suppose A also satisfies the connectedness condition of Lemma 2. The constructed graph G is not necessarily connected yet: G may consist of more than one connected component. But by construction, each connected component already fulfils the conditions of Lemma 1 and Lemma 3. □

Remark 2. The regularity is not required above. But the construction ensures that the vertices in V_i have common degree equal to the i-th row sum of A. Hence if the row sum is constant, the graph constructed above is regular.

4. Implementation

Altogether the necessary conditions above yield the following procedure to enumerate all colour incidences matrices for connected regular graphs. We need $(m-1)^2$ nested loops to go through all matrices $A = (a_{ij})_{m \times m} \in \mathbb{N}^{m \times m}$ with constant row sum k. In these loops each matrix A needs to pass the following five tests. The order of the tests is in part arbitrary: we implemented them before we realized the sufficiency of the conditions in Theorem 7. But the most time-consuming test is the last one, so it is desirable to exclude as many matrices as possible before that one.

(1.) Check if for $i \neq j$, $a_{ij} = 0$ if and only if $a_{ji} = 0$.
(2.) Ensure connectedness through Lemma 2. For $m = 2$, this means a_{12} is nonzero; for $m = 3$ and $m = 4$, as mentioned above, at least one of the three pairs in Lemma 5 must be all nonzero; and at least one of the sixteen triples in Lemma 6 must be all nonzero.
(3.) Use Lemma 3 to ensure that the various ways of counting v_i in terms of v_j for distinct i and j are consistent with one another. For example, if $m = 4$ and all non-diagonal entries of A are nonzero, then equality of all sixteen expressions in Lemma 6 is guaranteed by Lemma 3. Thus later it will suffice to consider just one case. Furthermore, several of the products in Lemma 3 may be zero, but by connectedness there is a way to relate any v_r and v_s by products of nonzero a_{ij}'s.
(4.) Without loss of generality, we assume that $v_i \leq v_{i+1}$ for $i < m$. Moreover, in each case in Lemma 4, 5, and 6, we have $|V| = v_1 + \cdots + v_m$. Therefore, for $m = 2, 3, 4$, we identify a suitable case that A satisfies and replace the v_i in the above equation with the corresponding expressions. Dividing the resulting equation by $|V|$ leaves right-hand side expressions involving only the entries a_{ij}. We check whether these expressions are in non-decreasing order and do sum up to 1.
The analogue for two colours needs not to be checked at this step, since the sum is always 1. Hence it suffices to check if $a_{12} \leq a_{21}$. But in the next section we need that these expressions for v_1 and v_2 are integers.
(5.) Finally we want to consider colourings that differ only by a permutation of colours as identical. Hence we identify matrices if they differ only by a permutation, i.e. we omit a matrix A' if $PA'P^{-1} = A$ for some permutation matrix P.

The tests (1.)-(5.) can be implemented in a straight-forward manner in a computer algebra system. We implemented them both in `scilab` and `sagemath`. The `sagemath` worksheets are available for download [15]. There are three worksheets, one for each number of colours. The worksheets are organized in sections, one for each degree k of regularity ($k \in \{3, 4, 5\}$). The comments in the code indicate the different cases and tests. After executing all cells in all sections in the worksheet the list `l` contains all colour incidence matrices passing the tests (1.)-(5.) for the respective value of k. Each section contains further code to determine all perfect colourings for Platonic graphs, see Section 5.

The worksheets for two and three colours will need at most a few minutes computing time on a common laptop or desktop computer. The worksheets for four colours need several hours of computation on a modern laptop. The most time is needed for test (5.). Therefore we provide the list containing all colour incidence matrices for download, too [15]. One can download the files, store them in some folder (e.g. `/home/user/sage`) and load the content into any sage worksheet using `open('/home/user/sage/4col-list.sage')`, for instance. After executing the above command, the list `l43` contains all colour incidence matrices for perfect 4-colourings of 3-regular graphs, `l44` contains all colour incidence matrices for perfect 4-colourings of 4-regular graphs, and `l45` the corresponding list for perfect 4-colourings of 5-regular graphs. These lists can then be proceeded further, for instance like in the examples in Section 5.

Using these criteria all colour incidence matrices A for perfect 2-colourings of k-regular graphs with $k \in \{3, 4, 5\}$ are only the ones listed in the Table 1. All colour incidence matrices A for

k	A
3	$\begin{pmatrix}0&3\\1&2\end{pmatrix}, \begin{pmatrix}0&3\\2&1\end{pmatrix}, \begin{pmatrix}0&3\\3&0\end{pmatrix}, \begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}, \begin{pmatrix}2&1\\1&2\end{pmatrix}.$
4	$\begin{pmatrix}0&4\\1&3\end{pmatrix}, \begin{pmatrix}0&4\\2&2\end{pmatrix}, \begin{pmatrix}0&4\\3&1\end{pmatrix}, \begin{pmatrix}0&4\\4&0\end{pmatrix}, \begin{pmatrix}1&3\\1&3\end{pmatrix}, \begin{pmatrix}1&3\\2&2\end{pmatrix}, \begin{pmatrix}1&3\\3&1\end{pmatrix}, \begin{pmatrix}2&2\\1&3\end{pmatrix}, \begin{pmatrix}2&2\\2&2\end{pmatrix}, \begin{pmatrix}3&1\\1&3\end{pmatrix}.$
5	$\begin{pmatrix}0&5\\1&4\end{pmatrix}, \begin{pmatrix}0&5\\2&3\end{pmatrix}, \begin{pmatrix}0&5\\3&2\end{pmatrix}, \begin{pmatrix}0&5\\4&1\end{pmatrix}, \begin{pmatrix}0&5\\5&0\end{pmatrix}, \begin{pmatrix}1&4\\1&4\end{pmatrix}, \begin{pmatrix}1&4\\2&3\end{pmatrix}, \begin{pmatrix}1&4\\3&2\end{pmatrix}, \begin{pmatrix}1&4\\4&1\end{pmatrix}, \begin{pmatrix}2&3\\1&4\end{pmatrix},$
	$\begin{pmatrix}2&3\\2&3\end{pmatrix}, \begin{pmatrix}2&3\\3&2\end{pmatrix}, \begin{pmatrix}3&2\\1&4\end{pmatrix}, \begin{pmatrix}3&2\\2&3\end{pmatrix}, \begin{pmatrix}4&1\\1&4\end{pmatrix}.$

TABLE 1. All colour incidence matrices for k-regular graphs with two colours.

perfect 3-colourings of k-regular graphs with $k \in \{3,4,5\}$ are given in Appendix A. There are 18 possible matrices for 3-regular graphs, 64 for 4-regular graphs, and 153 for 5-regular graphs.

The lists of all colour incidence matrices A for perfect 4-colourings of k-regular graphs with $k \in \{3,4,5\}$ are quite long: there are 72 matrices for 3-regular graphs, 485 for 4-regular graphs, and 2042 for 5-regular graphs. They are available online at [15] in two forms: as a list in `pdf`, and as a loadable `sage` data file. Table 2 compares the number of all matrices in $\mathbb{N}^{m\times m}$ with all row sums equal to k with the number of all colour incidence matrices for perfect colourings for 4-colourings of k-regular graphs with $k \in \{3,4,5\}$.

$m \setminus k$	3	4	5
2	6 of 16	10 of 25	15 of 36
3	18 of 1000	64 of 3375	153 of 9261
4	72 of 16 000	485 of 1 500 625	2042 of 9 834 496

TABLE 2. This table compares the number of all colour incidence matrices (passing the tests (1.)-(5.)) with the entire number of matrices in $\mathbb{N}^{m\times m}$ with all row sums equal to k.

5. PERFECT COLOURINGS OF PLATONIC GRAPHS

A very useful fact in the context of perfect colourings is the following result [10, Theorem 9.3.3].

Theorem 10. *Let M be the adjacency matrix of some graph G and let A be the colour incidence matrix of some perfect colouring of G. Then the characteristic polynomial of A divides the characteristic polynomial of M. In particular, each eigenvalue of A is an eigenvalue of M.*

This can be used to as a further necessary criterion for possible colour incidence matrices for some particular graph G. We illustrate this with the edge graphs of the five Platonic solids. The eigenvalues of those are given in Table 3. An entry a^n means that a is an eigenvalue of algebraic multiplicity n.

G	tetrahedron	cube	octahedron	dodecahedron	icosahedron
	$-1^3, 3$	$-3, -1^3, 1^3, 3$	$-2^2, 0^3, 4$	$-\sqrt{5}^3, -2^4, 0^4, 1^5, \sqrt{5}^3, 3$	$-\sqrt{5}^3, -1^5, \sqrt{5}^3, 5$

TABLE 3. The eigenvalues of the Platonic graphs. A superscript denotes the multiplicity of the respective eigenvalue.

In order to determine all perfect colourings of the Platonic graphs with two colours one can check which matrices in Table 1 have all eigenvalues in $\{-1,3\}$ (for the tetrahedral graph), respectively in $\{-3,-1,1,3\}$ (for the cube), and so on. This is the actual test we implemented in `sage`. One could refine it in order to include counting the multiplicities, but we found that the latter condition does not exclude further matrices.

5.1. The 2-colourings of Platonic graphs.

By the methods in the previous section we obtained a list of all colour incidence matrices for perfect 2-colourings of k-regular graphs for $k \in \{3,4,5\}$. For each matrix in each of these lists we now check whether the corresponding expressions in Lemma 4 are integers, and whether the matrix has the correct eigenvalues, according to the Platonic graph under consideration. For example, since the icosahedron is 5-regular, we check for all fifteen matrices in the last row of Table 1 whether the expressions in Lemma 4 are all integers, and whether all eigenvalues of the matrix are contained in $\{-\sqrt{5}, -1, \sqrt{5}, 5\}$. In this manner we obtained the following candidates for colour incidence matrices for 2-colourings of the Platonic graphs, respectively.

(1) Tetrahedron: $\left(\begin{smallmatrix} 0 & 3 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$

(2) Cube: $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
(3) Octahedron: $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}, {\color{gray}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.
(4) Dodecahedron: $\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
(5) Icosahedron: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.

For the tetrahedron, the cube, the dodecahedron, and the icosahedron, all possible colour incidence matrices in the list above actually do belong to perfect 2-colourings. These colourings are shown in Figures 1, 2, 4 and 5 on page 11. For the octahedron there are only two perfect 2-colourings, shown in Figure 3. In this case, one of the matrices above does not belong to a perfect 2-colouring: the matrix in grey can be checked to be impossible in a straightforward manner by attempting to colour a graph according to these colour incidences. This list confirms the results in [3].

5.2. **The 3-colourings of Platonic graphs.** Applying the analogous procedure, and with Lemma 5 rather than Lemma 4, we obtained a list of all colour incidence matrices for 3-colourings of the Platonic graphs, respectively. In this case, all candidates are valid colour incidence matrices for perfect colourings of Platonic graphs.

(1) Tetrahedron: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$
(2) Cube: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.
(3) Octahedron: $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.
(4) Dodecahedron: $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.
(5) Icosahedron: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.

The perfect colourings corresponding to the colour incidence matrices above are shown in Figures 1-5. This list corrects [1] by providing the three cases missing there, namely $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ for the octahedron and $\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ for the dodecahedron.

5.3. **The 4-colourings of Platonic graphs.** We obtained in a similar manner the following candidates for colour incidence matrices for 4-colourings of the Platonic graphs, respectively.

(1) Tetrahedron: $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.
(2) Cube: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.
(3) Octahedron: $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$
(4) Dodecahedron: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
(5) Icosahedron: $\begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$

All of these candidates have corresponding perfect colourings, and these are shown in Figures 1-5, respectively.

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Appendix A: All colour incidence matrices for 3-colourings

All colour incidence matrices A for perfect 3-colourings of k-regular graphs with $k \in \{3, 4, 5\}$:

3-regular graphs: $\begin{pmatrix}0&0&3\\0&0&3\\1&1&1\end{pmatrix}$ $\begin{pmatrix}0&0&3\\0&0&3\\1&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&3\\0&1&2\\1&1&1\end{pmatrix}$ $\begin{pmatrix}0&0&3\\0&1&2\\1&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&3\\0&2&1\\1&1&1\end{pmatrix}$ $\begin{pmatrix}0&0&3\\0&2&1\\2&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&2\\1&0&2\\1&1&1\end{pmatrix}$ $\begin{pmatrix}0&1&2\\1&1&1\\2&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&2\\1&2&0\\1&0&2\end{pmatrix}$ $\begin{pmatrix}0&1&2\\1&2&0\\2&0&1\end{pmatrix}$
$\begin{pmatrix}0&3&0\\1&0&2\\0&1&2\end{pmatrix}$ $\begin{pmatrix}1&0&2\\0&0&3\\1&2&0\end{pmatrix}$ $\begin{pmatrix}1&0&2\\0&1&2\\1&1&1\end{pmatrix}$ $\begin{pmatrix}1&0&2\\0&1&2\\1&2&0\end{pmatrix}$ $\begin{pmatrix}1&0&2\\0&2&1\\1&1&1\end{pmatrix}$ $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ $\begin{pmatrix}1&1&1\\1&2&0\\1&0&2\end{pmatrix}$ $\begin{pmatrix}1&2&0\\1&0&2\\0&1&2\end{pmatrix}$

4-regular graphs: $\begin{pmatrix}0&0&4\\0&0&4\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&0&4\\1&2&1\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&0&4\\1&3&0\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&0&4\\2&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&1&3\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&1&3\\1&2&1\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&1&3\\1&3&0\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&1&3\\2&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&2&2\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&2&2\\1&2&1\end{pmatrix}$
$\begin{pmatrix}0&0&4\\0&2&2\\2&1&1\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&2&2\\2&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&3&1\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&3&1\\2&1&1\end{pmatrix}$ $\begin{pmatrix}0&0&4\\0&3&1\\3&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&3\\1&0&3\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&1&3\\1&0&3\\2&2&0\end{pmatrix}$ $\begin{pmatrix}0&1&3\\1&2&1\\3&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&3\\1&3&0\\1&0&3\end{pmatrix}$ $\begin{pmatrix}0&1&3\\1&3&0\\2&0&2\end{pmatrix}$ $\begin{pmatrix}0&1&3\\1&3&0\\3&0&1\end{pmatrix}$ $\begin{pmatrix}0&2&2\\1&0&3\\1&3&0\end{pmatrix}$
$\begin{pmatrix}0&2&2\\1&1&2\\1&2&1\end{pmatrix}$ $\begin{pmatrix}0&2&2\\1&2&1\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&2&2\\1&3&0\\1&0&3\end{pmatrix}$ $\begin{pmatrix}0&2&2\\2&0&2\\1&1&2\end{pmatrix}$ $\begin{pmatrix}0&2&2\\2&0&2\\2&2&0\end{pmatrix}$ $\begin{pmatrix}0&2&2\\2&1&1\\2&1&1\end{pmatrix}$ $\begin{pmatrix}0&2&2\\2&2&0\\1&0&3\end{pmatrix}$ $\begin{pmatrix}0&2&2\\2&2&0\\2&0&2\end{pmatrix}$ $\begin{pmatrix}0&4&0\\1&0&3\\0&1&3\end{pmatrix}$ $\begin{pmatrix}0&4&0\\1&0&3\\0&2&2\end{pmatrix}$ $\begin{pmatrix}0&4&0\\1&1&2\\0&1&3\end{pmatrix}$ $\begin{pmatrix}0&4&0\\2&0&2\\0&1&3\end{pmatrix}$
$\begin{pmatrix}1&0&3\\0&0&4\\1&2&1\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&0&4\\1&3&0\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&1&3\\1&1&2\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&1&3\\1&2&1\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&1&3\\1&3&0\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&1&3\\2&2&0\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&2&2\\1&1&2\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&2&2\\1&2&1\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&2&2\\2&2&0\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&3&1\\1&1&2\end{pmatrix}$ $\begin{pmatrix}1&0&3\\0&3&1\\2&1&1\end{pmatrix}$ $\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$
$\begin{pmatrix}1&1&2\\1&2&1\\2&1&1\end{pmatrix}$ $\begin{pmatrix}1&1&2\\1&3&0\\1&0&3\end{pmatrix}$ $\begin{pmatrix}1&1&2\\1&3&0\\2&0&2\end{pmatrix}$ $\begin{pmatrix}1&3&0\\1&0&3\\0&1&3\end{pmatrix}$ $\begin{pmatrix}1&3&0\\1&0&3\\0&2&2\end{pmatrix}$ $\begin{pmatrix}1&3&0\\1&1&2\\0&1&3\end{pmatrix}$ $\begin{pmatrix}1&3&0\\2&0&2\\0&1&3\end{pmatrix}$ $\begin{pmatrix}2&0&2\\0&0&4\\1&3&0\end{pmatrix}$ $\begin{pmatrix}2&0&2\\0&1&3\\1&2&1\end{pmatrix}$ $\begin{pmatrix}2&0&2\\0&1&3\\1&3&0\end{pmatrix}$ $\begin{pmatrix}2&0&2\\0&2&2\\1&1&2\end{pmatrix}$ $\begin{pmatrix}2&0&2\\0&2&2\\1&2&1\end{pmatrix}$
$\begin{pmatrix}2&0&2\\0&3&1\\1&1&2\end{pmatrix}$ $\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$ $\begin{pmatrix}2&1&1\\1&3&0\\1&0&3\end{pmatrix}$ $\begin{pmatrix}2&2&0\\1&0&3\\0&1&3\end{pmatrix}$ $\begin{pmatrix}2&2&0\\1&0&3\\0&2&2\end{pmatrix}$ $\begin{pmatrix}2&2&0\\1&1&2\\0&1&3\end{pmatrix}$

5-regular graphs

$\begin{pmatrix}0&0&5\\0&0&5\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&0&5\\1&2&2\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&0&5\\1&3&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&0&5\\1&4&0\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&0&5\\2&2&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&0&5\\2&3&0\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&1&4\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&1&4\\1&2&2\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&1&4\\1&3&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&1&4\\1&4&0\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&1&4\\2&2&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&1&4\\2&3&0\end{pmatrix}$
$\begin{pmatrix}0&0&5\\0&2&3\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&2&3\\1&2&2\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&2&3\\1&3&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&2&3\\2&2&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&2&3\\2&3&0\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&2&3\\3&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&3&2\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&3&2\\1&2&2\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&3&2\\2&1&2\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&3&2\\2&2&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&3&2\\3&2&0\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&4&1\\1&1&3\end{pmatrix}$
$\begin{pmatrix}0&0&5\\0&4&1\\2&1&2\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&4&1\\3&1&1\end{pmatrix}$ $\begin{pmatrix}0&0&5\\0&4&1\\4&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&0&4\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&0&4\\2&2&1\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&2&2\\2&1&2\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&3&1\\4&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&4&0\\1&0&4\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&4&0\\2&0&3\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&4&0\\3&0&2\end{pmatrix}$ $\begin{pmatrix}0&1&4\\1&4&0\\4&0&1\end{pmatrix}$ $\begin{pmatrix}0&2&3\\1&1&3\\1&2&2\end{pmatrix}$
$\begin{pmatrix}0&2&3\\1&4&0\\1&0&4\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&0&3\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&0&3\\2&2&1\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&1&2\\3&2&0\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&2&1\\3&1&1\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&3&0\\1&0&4\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&3&0\\2&0&3\end{pmatrix}$ $\begin{pmatrix}0&2&3\\2&3&0\\3&0&2\end{pmatrix}$ $\begin{pmatrix}0&3&2\\2&3&0\\1&0&4\end{pmatrix}$ $\begin{pmatrix}0&3&2\\3&0&2\\1&1&3\end{pmatrix}$ $\begin{pmatrix}0&3&2\\3&2&0\\1&0&4\end{pmatrix}$ $\begin{pmatrix}0&5&0\\1&0&4\\0&1&4\end{pmatrix}$
$\begin{pmatrix}0&5&0\\1&0&4\\0&2&3\end{pmatrix}$ $\begin{pmatrix}0&5&0\\1&0&4\\0&3&2\end{pmatrix}$ $\begin{pmatrix}0&5&0\\1&1&3\\0&1&4\end{pmatrix}$ $\begin{pmatrix}0&5&0\\1&1&3\\0&2&3\end{pmatrix}$ $\begin{pmatrix}0&5&0\\1&2&2\\0&1&4\end{pmatrix}$ $\begin{pmatrix}0&5&0\\2&0&3\\0&1&4\end{pmatrix}$ $\begin{pmatrix}0&5&0\\2&0&3\\0&2&3\end{pmatrix}$ $\begin{pmatrix}0&5&0\\2&1&2\\0&1&4\end{pmatrix}$ $\begin{pmatrix}0&5&0\\3&0&2\\0&1&4\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&0&5\\1&2&2\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&0&5\\1&3&1\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&0&5\\1&4&0\end{pmatrix}$
$\begin{pmatrix}1&0&4\\0&0&5\\2&3&0\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&1&4\\1&1&3\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&1&4\\1&2&2\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&1&4\\1&3&1\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&1&4\\1&4&0\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&1&4\\2&2&1\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&1&4\\2&3&0\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&2&3\\1&1&3\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&2&3\\1&2&2\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&2&3\\1&3&1\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&2&3\\2&2&1\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&2&3\\2&3&0\end{pmatrix}$
$\begin{pmatrix}1&0&4\\0&3&2\\1&1&3\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&3&2\\1&2&2\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&3&2\\2&1&2\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&3&2\\2&2&1\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&3&2\\3&2&0\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&4&1\\1&1&3\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&4&1\\2&1&2\end{pmatrix}$ $\begin{pmatrix}1&0&4\\0&4&1\\3&1&1\end{pmatrix}$ $\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}$ $\begin{pmatrix}1&1&3\\1&1&3\\2&2&1\end{pmatrix}$ $\begin{pmatrix}1&1&3\\1&3&1\\3&1&1\end{pmatrix}$ $\begin{pmatrix}1&1&3\\1&4&0\\1&0&4\end{pmatrix}$

$$\begin{pmatrix}1&1&3\\1&4&0\\2&0&3\end{pmatrix} \begin{pmatrix}1&1&3\\1&4&0\\3&0&2\end{pmatrix} \begin{pmatrix}1&2&2\\1&0&4\\1&4&0\end{pmatrix} \begin{pmatrix}1&2&2\\1&1&3\\1&3&1\end{pmatrix} \begin{pmatrix}1&2&2\\1&2&2\\1&2&2\end{pmatrix} \begin{pmatrix}1&2&2\\1&3&1\\1&1&3\end{pmatrix} \begin{pmatrix}1&2&2\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}1&2&2\\2&1&2\\1&1&3\end{pmatrix} \begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix} \begin{pmatrix}1&2&2\\2&2&1\\2&1&2\end{pmatrix} \begin{pmatrix}1&2&2\\2&3&0\\1&0&4\end{pmatrix} \begin{pmatrix}1&2&2\\2&3&0\\2&0&3\end{pmatrix}$$

$$\begin{pmatrix}1&4&0\\1&0&4\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\1&0&4\\0&2&3\end{pmatrix} \begin{pmatrix}1&4&0\\1&0&4\\0&3&2\end{pmatrix} \begin{pmatrix}1&4&0\\1&1&3\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\1&1&3\\0&2&3\end{pmatrix} \begin{pmatrix}1&4&0\\1&2&2\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\2&0&3\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\2&0&3\\0&2&3\end{pmatrix} \begin{pmatrix}1&4&0\\2&1&2\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\3&0&2\\0&1&4\end{pmatrix} \begin{pmatrix}2&0&3\\0&0&5\\1&2&2\end{pmatrix} \begin{pmatrix}2&0&3\\0&0&5\\1&3&1\end{pmatrix}$$

$$\begin{pmatrix}2&0&3\\0&0&5\\1&4&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\1&2&2\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\1&3&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\1&4&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\2&3&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\1&1&3\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\1&2&2\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\1&3&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\2&2&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\2&3&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&3&2\\1&1&3\end{pmatrix} \begin{pmatrix}2&0&3\\0&3&2\\1&2&2\end{pmatrix}$$

$$\begin{pmatrix}2&0&3\\0&3&2\\2&2&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&4&1\\1&1&3\end{pmatrix} \begin{pmatrix}2&0&3\\0&4&1\\2&1&2\end{pmatrix} \begin{pmatrix}2&1&2\\1&2&2\\1&1&3\end{pmatrix} \begin{pmatrix}2&1&2\\1&3&1\\2&1&2\end{pmatrix} \begin{pmatrix}2&1&2\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}2&1&2\\1&4&0\\2&0&3\end{pmatrix} \begin{pmatrix}2&3&0\\1&0&4\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\1&0&4\\0&2&3\end{pmatrix} \begin{pmatrix}2&3&0\\1&0&4\\0&3&2\end{pmatrix} \begin{pmatrix}2&3&0\\1&1&3\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\1&1&3\\0&2&3\end{pmatrix}$$

$$\begin{pmatrix}2&3&0\\1&2&2\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\2&0&3\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\2&0&3\\0&2&3\end{pmatrix} \begin{pmatrix}2&3&0\\2&1&2\\0&1&4\end{pmatrix} \begin{pmatrix}3&0&2\\0&0&5\\1&3&1\end{pmatrix} \begin{pmatrix}3&0&2\\0&0&5\\1&4&0\end{pmatrix} \begin{pmatrix}3&0&2\\0&1&4\\1&3&1\end{pmatrix} \begin{pmatrix}3&0&2\\0&1&4\\1&4&0\end{pmatrix} \begin{pmatrix}3&0&2\\0&2&3\\1&2&2\end{pmatrix} \begin{pmatrix}3&0&2\\0&2&3\\1&3&1\end{pmatrix} \begin{pmatrix}3&0&2\\0&3&2\\1&1&3\end{pmatrix} \begin{pmatrix}3&0&2\\0&3&2\\1&2&2\end{pmatrix}$$

$$\begin{pmatrix}3&0&2\\0&4&1\\1&1&3\end{pmatrix} \begin{pmatrix}3&1&1\\1&3&1\\1&1&3\end{pmatrix} \begin{pmatrix}3&1&1\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}3&2&0\\1&0&4\\0&1&4\end{pmatrix} \begin{pmatrix}3&2&0\\1&0&4\\0&2&3\end{pmatrix} \begin{pmatrix}3&2&0\\1&0&4\\0&3&2\end{pmatrix} \begin{pmatrix}3&2&0\\1&1&3\\0&1&4\end{pmatrix} \begin{pmatrix}3&2&0\\1&1&3\\0&2&3\end{pmatrix} \begin{pmatrix}3&2&0\\1&2&2\\0&1&4\end{pmatrix}$$

Appendix B: Perfect colourings of Platonic graphs

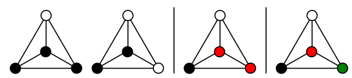

Figure 1. The perfect 2-, 3- and 4-colourings of the tetrahedral graph.

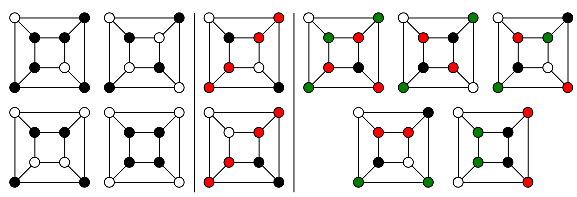

Figure 2. The perfect 2-, 3- and 4-colourings of the cube graph.

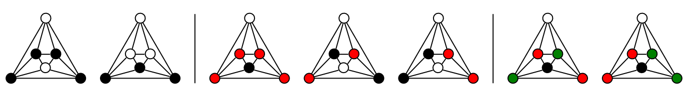

Figure 3. The perfect 2-, 3- and 4-colourings of the octahedral graph.

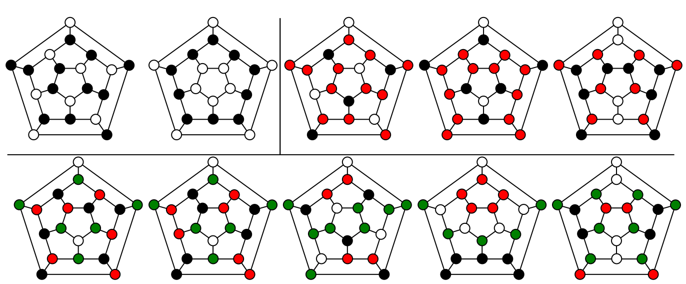

Figure 4. The perfect 2-, 3- and 4-colourings of the dodecahedral graph.

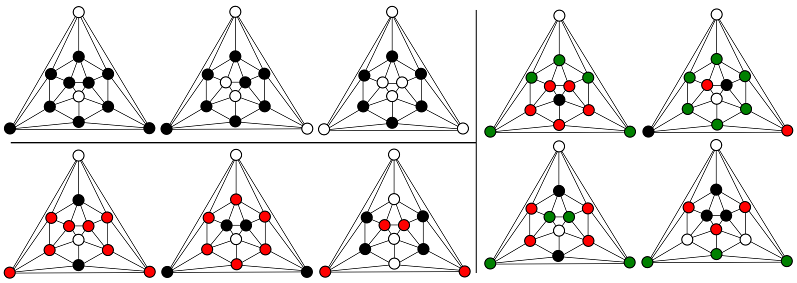

FIGURE 5. The perfect 2-, 3- and 4-colourings of the icosahedral graph.