ON THE SUBGROUPS OF COXETER GROUPS AND THEIR SUBGROUPS

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ABSTRACT

EDEN DELIGHT B. PROVIDO. ON THE SUBGROUPS OF COXETER GROUPS AND THEIR SUBGROUPS (Under the direction of Ma. Louise Antonette N. De Las Peñas, Ph.D.)

In this work, a methodology is presented based on tools in color symmetry theory that will facilitate the determination of subgroups of Coxeter groups and their subgroups. The goal of this thesis is to extend the results given in [25, 24, 4, 50] on index 2, 3 and 4 subgroups of triangle and tetrahedron groups. In particular, this study aims to develop a framework that will address a wider class of Coxeter groups in higher dimensional hyperbolic space, and to make possible the derivation of higher index subgroups. The method is applied to derive index 5 subgroups of selected Coxeter groups in 3- and 4- dimensional hypebolic space, and index 6 subgroups of the modular group. The method involves the construction of trees that yield a set of generators for the subgroups of Coxeter groups and their subgroups.

The other main focus of the work is to come up with a theory for identifying torsion-free subgroups of Coxeter groups.

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Chapter 1

INTRODUCTION

1.1 Background

One of the important problems in mathematical crystallography is the characterization of the subgroup structure of crystallographic groups. While the subgroups of the crystallographic groups in the Euclidean and spherical cases have been studied extensively in previous works, the subgroups of crystallographic groups in hyperbolic space have not yet been completely classified. In this work, we address the problem on the determination of the subgroups of hyperbolic crystallographic groups, as suggested in 2005 by Prof. Marjorie Senechal, a renowned group theorist and crystallographer. We focus on a huge class of crystallographic groups called Coxeter groups and provide a general method for determining the subgroup structure of these groups. The Coxeter groups we consider here are groups generated by reflections in the hyperplanes containing the sides of a Coxeter polytope in either Euclidean, spherical or hyperbolic space. This work extends the setting given in [25, 24, 50] on the enumeration of index 2, 3, and 4 subgroups of the triangle group and tetrahedron and tetrahedron Kleinian groups. One of the motivations of the study is to arrive at a framework that will provide the means to determine higher index subgroups of Coxeter groups and their subgroups, particularly those that exist in higher dimensional hyperbolic space.

Even though the problem of classification of isomorphic types of crystallographic groups dates back to the nineteenth century, the idea of studying and classifying torsion-free crystallographic groups and torsion-free subgroups of crystallographic groups came much later with the study of flat manifolds. In this work, we also provide a method for determining torsion-free subgroups of Coxeter groups.

1.2 Statement of the Problem

The objectives of this work are the following:

1. to arrive at a general framework for determining the subgroups of the Coxeter group Γ and its subgroups, where Γ is generated by reflections in the hyperplanes containing the sides of a Coxeter polytope lying on either Euclidean, spherical or hyperbolic space,

2. to design the construction of trees that will yield a generating set for each subgroup obtained, and

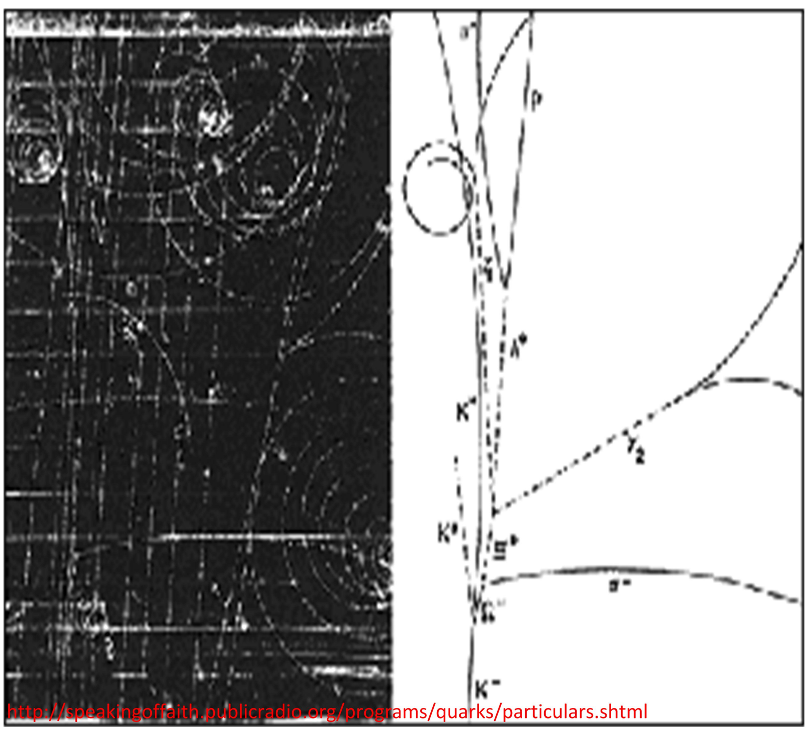
3. to formulate a method for determining torsion-free subgroups of Γ.

1.3 Significance of the Study

The study of the subgroup structure of crystallographic groups is helpful in several branches of mathematics. The modular group and the Picard group, which are well-known examples of subgroups of hyperbolic Coxeter groups, are important in the study of elliptic modular functions and automorphic function theory.

In the first half of the nineteenth century, crystallographers were already busy studying the discrete groups of Euclidean isometries. Primarily, their aim was to give a theoretical classification of the different kinds of symmetry arrangements possible. It is interesting to note that these mathematical explorations were undertaken within 20 years before experimental means were available for analyzing the crystals themselves. In this area, subgroups of crystallographic groups are essential for a number of important studies in chemical research such as providing a concise and powerful tool for describing and classifying crystals. The group-subgroup relation methods are also used to investigate phase transitions and symmetry problems in various types of crystals and liquid crystals.

The symmetry aspects of structural phase transitions make use of group and subgroup transformation matrices which facilitated the formulation of the famous Ladau's theory of second-order phase transition in quantum mechanics. In fact, in the middle of the last century, crystallographic group theory found an important application in quantum mechanics. It was used to study the atom's internal structure. In the 1950s, a new generation of particle accelerators produced a variety of subatomic particles. Scientists used group theory to predict the existence of a tenth nucleon in a tenfold symmetry scheme of nucleons of which nine particles had already been detected. Then, in 1964, the tracks of Omega-Minus, the tenth nucleon, were detected.



On February 11, 1964, a team from the Brookhaven National Laboratory announced the discovery of the omega-minus particle that Murray Gell-Mann had predicted. In the diagram on the left, solid lines show tracks formed by electrically charged particles in the liquid hydrogen in the bubble chamber; dotted lines show where neutral particles have passed (being uncharged they leave no track). A magnetic field bends the tracks of negative particles to the right and positive particles to the left. This discovery helped scientists to understand the underlying structure in the form of quarks. Courtesy: Brookhaven National Laboratory

Figure 1.1: The first Omega-Minus.

The emergence of quasicrystals in the mid-80's influenced mathematicians to rethink notions on periodicity and to establish the role of the higher dimensional crystallographic groups in the study of quasicrystallographic structures.

A problem addressed to in this study is that of finding low index torsion-free subgroups of Coxeter groups. The geometrical interest of such groups is that the associated orbit space is a manifold. If a group Γ, with fundamental domain D has a torsion-free subgroup of index n, then we can construct a manifold by glueing together n copies of D, according to the face pairings determined by Γ, and conversely. In the last quarter of a century, 3-manifold topology has been revolutionized by Thurston and his school. This resulted in a huge literature on hyperbolic 3-manifolds building on the classical 2-dimensional case. On the other hand, for $\mathsf{d} > 3$, there is a relative scarcity of techniques and examples of hyperbolic d-manifolds. This study aims to contribute to this field by providing a method for determining the torsion-free subgroups of Coxeter groups in any dimension.

The subgroup structure and torsion-free subgroups of hyperbolic Coxeter groups have found another application in the study of crystal nets. In [64], Ramsden et al. used the subgroup lattice of the triangle group $*642$ (Figure 1.2) to give an enumeration of crystal nets. In Figure 1.3, the top row and bottom row show the tilings resulting from the subgroup labelled $*2^5$(a) and $*2^5$(b), respectively. The irregularly shaped unit cells in the central column correspond exactly to the highlighted region in the hyperbolic plane in the left. The right column shows the nets derived from tilings by forming the most symmetric Euclidean embeddings of the net topologies. (The reader is referred to [64], for the nets derived from the other subgroups in the lattice.) Torsion-free subgroup Ω^* of $*642$ also plays an essential role in the geometric construction of periodic three-dimensional Euclidean nets by projecting two-dimensional hyperbolic tilings onto a family of triply periodic minimal surfaces.

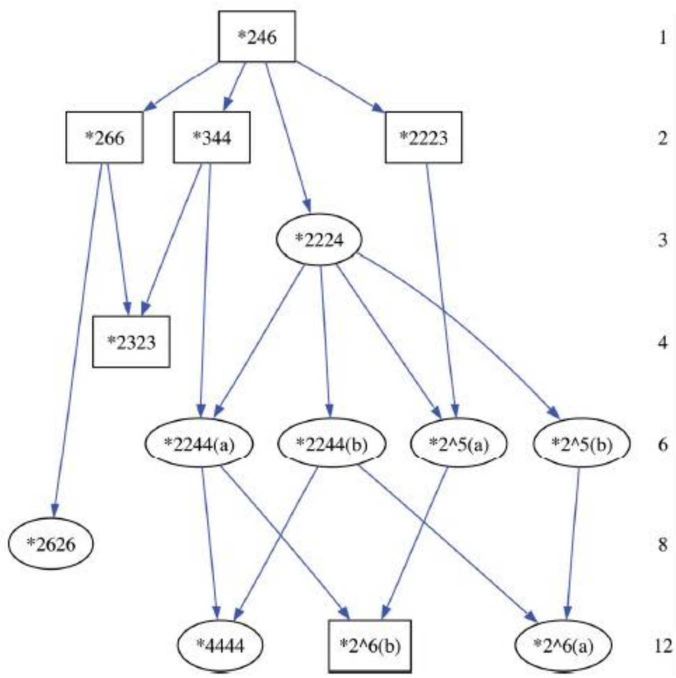


Figure 1.2: The subgroup lattice of $*642$. The groups in rectangular nodes are normal subgroups of $*642$. The subgroups in one conjugation class are labeled (a) and (b) in the ellipsoidal nodes.

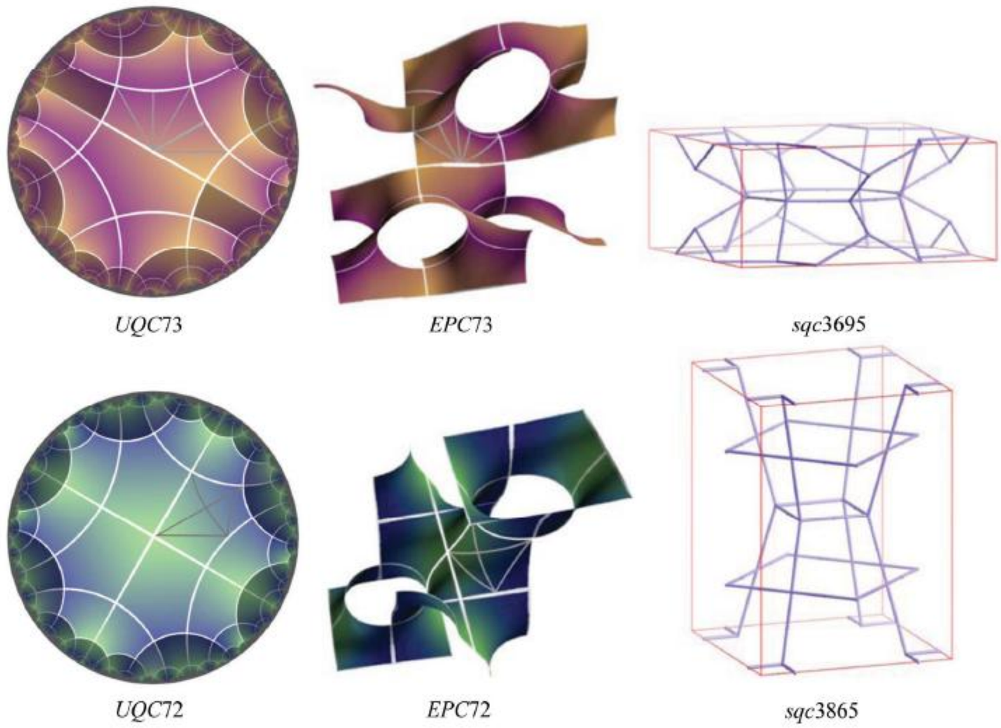


Figure 1.3: The nets derived from the subgroups $*2^5$(a) and $*2^5$(b) of $*642$ [64].

The main results presented in Chapters 4, 5 and 6 are new and reflect original work by the author.

1.4 Scope and Limitation of the Study

The results provided in this work pertain to subgroups of Coxeter groups or subgroups of subgroups of Coxeter groups. The Coxeter groups under consideration are groups generated by reflections in hyperplanes containing the sides of a Coxeter polytope of finite volume in either Euclidean, spherical or hyperbolic space.

Computations have been carried out particularly on the index 5 subgroups of Coxeter groups having Coxeter simplices in $\mathbb{H}^4$ as fundamental polytopes and some examples of singly truncated Coxeter tetrahedron groups. The index 6 subgroups of the modular group were also derived. Moreover, low index torsion-free subgroups have been obtained for the triangle group $*442$, the Hecke groups 32∞, 42∞ and 62∞. In addition, the index 96 torsion-free subgroup of the triangle group $*642$ was also derived.

The calculations were done with the aid of the computer software Groups, Algorithms and Programs version 4 *(GAP4)* [34]. However, due to hardware/software constraints, only groups in at most dimension four were considered.

1.5 Methodology

To come up with a method that will facilitate the determination of the subgroups of Coxeter groups and their subgroups, we first looked at the existing approach given in [25, 24, 50] on the enumeration of the index 2, 3 and 4 subgroups of triangle groups and tetrahedron groups/tetrahedron Kleinian groups in dimensions 2 and 3, respectively. More specifically, we examined the approach and tried applying this to other classes of Coxeter groups. We also referred to the materials in [4] pertaining to the index 5 and 6 subgroups of triangle groups.

Then, an algorithm was formulated to generalize the method presented in those studies to cover other classes of Coxeter groups and their subgroups and to address the derivation of higher index subgroups. The algorithm was then implemented in the routines *Coloring* and *Sieve* using *GAP4*.

For example, to obtain the subgroups of a subgroup Λ of the Coxeter group Γ, we generate a list of color permutation assignments to the generators of Λ that will give rise to the subgroups of Λ distinct up to conjugacy in Λ. The next step involves obtaining a generating set of a subgroup Ω of Λ. The idea is to construct trees from the color permutation assignment giving rise to Ω. We then formulated the branching out rules for the tree construction to come up with paths that will correspond to generators of Ω. To arrive at these rules, we looked at various color permutation assignments that give rise to subgroups of Λ.

The routines were tested and applied to specific examples such as in obtaining index 5 subgroups of selected groups in hyperbolic 3- and 4-space and the index 6 subgroups of the modular group. The results of the generating trees construction were verified vís-a-vís a *GAP4* routine *ListSG* (Appendix D) written to give a listing of the subgroups of Coxeter groups or its subgroups.

To obtain our results in determining torsion-free subgroups of the Coxeter group, a theory was developed which focuses on the color permutation assignments that yield subgroups of the group of orientation preserving isometries in the Coxeter group Γ. To determine if Ω is a torsion-free subgroup of Γ, we used the insights obtained during the construction of trees in arriving at a generating set of Ω. The results were verified by finding some well-known torsion-free subgroups of particular Coxeter groups. In arriving at the index 96 torsion-free subgroup of $*642$, we constructed a 642-transitive 48-coloring of a tiling by triangles with interior angles $\frac{\pi}{4}$, $\frac{\pi}{6}$ and $\frac{\pi}{2}$.

The major results obtained in this work, mainly on the characterization of the subgroup structure of Coxeter groups and their subgroups, use the well-known relation of the group-subgroup problem to the algebraic structure of color symmetry groups. The approach given in this paper demonstrates the connection between group theory and color symmetry theory.

1.6 Review of Related Literature

The problem of determining and characterizing subgroups of Euclidean and spherical crytallographic groups has been addressed in several literatures: [6, 12, 72].

More recently, several studies were also done on the Coxeter groups in the hyperbolic plane. In [37], a classification of the low index subgroups of the hyperbolic triangle group $*832$ according to their symmetry structure was discussed. The method uses right coset colorings of the subgroups. Selected classes of the subgroups of hyperbolic groups were discussed and illustrated in [54]. In particular, the subgroups of $*732$, $*882$ and $*642$ were discussed and their corresponding generators were computed using the Reidemeister-Schreier method. The subgroups of the well-known extended modular group $*32\infty$ were also discussed using numerical methods. The subgroups of the extended modular groups and the modular groups were described using the notion of Farey symbols in [49]. Moreover, in [66] some subgroups of the extended Hecke group $*p2\infty$ were described and a result was presented stating that for any prime $p \geq 3$ every normal subgroup with torsion has index 2, 4 or $2p$. In [67], it was shown that the group $*32\infty$ has no normal subgroups of index 3 while it has exactly three normal subgroups of index 2. The Reidemeister-Schreier method was also used in obtaining presentation of the subgroups. Based on this method, an algorithm using search trees for finding all subgroups in a finitely presented group is described in [14]. In [60], similar results with adaptation of concepts of free groups and representation theory are applied to symmetry groups of regular hyperbolic tessellations.

The color symmetry theory used in this study was first used in [25] where an approach for determining the index 2 subgroups of the triangle group $*pqr$ was presented using black and white colorings of tilings by triangles. This result was then extended to determine the index 3 and 4 subgroups of $*pqr$ in [25]. Recently, in [4], the index 5 and 6 subgroups of the triangle group $*pqr$ were derived. In [50], the index 2, 3, and 4 subgroups of the tetrahedron groups and tetrahedron Kleinian groups were also determined using tools in color symmetry theory.

In the last quarter of the 20th century, the study on the geometry and topology of hyperbolic 3-orbifolds and manifolds became an important field in mathematics mainly due to the works of William Thurston's group in the classification of 3-manifolds which is mainly influenced by the Poincaré conjecture [59]. In connection with the studies in 3-manifolds, studies on the compact tetrahedra arise such as in [13, 55, 52]. A Coxeter tetrahedron is a fundamental polytope of a Coxeter tetrahedron group which forms an important class of groups in $\mathbb{H}^3$. Among its famous subgroups are the extended Eisenstein group and extended Picard group. Picard group occurs as an index 2 subgroup of the extended Picard group, and it has become an interest in number theory and the theory of automorphic functions. In [44], Weiss together with N. Johnson shows how linear fractional transformations over rings of rational and (real or imaginary) quadratic are related to the symmetry groups of regular tilings of the hyperbolic 3-space. They also shed a new light on the properties of the rational modular group $PSL_2(\mathbb{Z})$, the Gaussian modular (Picard) group $PSL_2(\mathbb{Z}[i])$, and the Eisenstein modular group $PSL_2(\mathbb{Z}[\omega])$. The modular group also occurs as a subgroup of the Picard group. In [77], Asia Ivic Weiss explains how certain kinds of subgroups of Coxeter groups can be used to derive regular and semi-regular tessellations. Furthermore, she also shows that matrices whose entries belong to certain rings of algebraic integers are associated with symmetry groups of tessellations of hyperbolic spaces.

It is worth noting that the concept of Coxeter group have been already studied by mathematicians even before Coxeter introduced the concept; then, such groups are called reflection groups. The spherical, Euclidean, and hyperbolic Coxeter groups whose fundamental polygons are triangles were determined by Schwarz in his 1873 paper [68]. The Coxeter groups whose fundamental polyhedra are tetrahedra in $\mathbb{H}^3$ were considered by Dyck in his 1883 paper [23]. The Coxeter groups whose fundamental polyhedra are tetrahedron in $\mathbb{S}^3$ were determined by Goursat in his 1889 paper [35]. The spherical and Euclidean, d-simplex as fundamental polytopes of Coxeter groups were enumerated by Coxeter in his 1931 note *Groups whose fundamental regions are simplexes* [20] and also in his 1934 paper *Discrete groups generated by reflections* [16]. The hyperbolic, tetrahedron Coxeter groups appear in Coxeter and Whitrows 1950 paper *World-structure and non-Euclidean honeycombs* [17]. The hyperbolic d-simplex Coxeter groups were enumerated by Lannér in his 1950 thesis *On complexes with transitive groups of automorphisms* [51]. Meanwhile, the hyperbolic, noncompact d-simplex Coxeter groups were enumerated by Chein in his 1969 paper [10]. A survey of hyperbolic Coxeter groups is given in Vinberg's1985 survey *Hyperbolic reflection groups* [76]. The theory of Coxeter groups and general theory of reflection groups are also found in Bourbaki's 1968 treatise [5], Coxeters 1973 treatise *Regular Polytopes* [19], and the 1990 treatise *Reflection Groups* *and Coxeter Groups* of Humphrey [40].

After the classification of some d-dimensional polytopes such as the simplices, many studies focused on the volume formula of such polytopes and, consequently, the co-volume of their corresponding Coxeter groups. In [43], N.W. Johnson et al. determined the covolumes of all hyperbolic Coxeter simplex reflection groups. They presented volume computations involving several different methods according to the parity of dimension, subgroup relations and arithmeticity properties. In [13], Conder and Martin provided a number of explicit examples of small volume hyperbolic 3-manifolds and 3-orbifolds with various geometric properties. These include a sequence of orbifolds with torsion of order q interpolating between the smallest volume cusped orbifold $(q = 6)$ and the smallest volume limit orbifold $(q \to \infty)$, hyperbolic 3-manifolds with automorphism groups with large orders in relation to volume and in arithmetic progression, and the smallest volume hyperbolic manifolds with totally geodesic surfaces. In [30], the theory of Coxeter groups is used to provide an algebraic construction of finite volume hyperbolic manifolds. Combinatorial properties of finite images of these groups can be used to compute the volumes of the resulting manifolds. Three examples, in 4, 5 and 6-dimensions, are given, each of very small volume, and in one case of smallest possible volume. Daniel Allcock in [1] proved that there are infinitely many finite-covolume (resp. cocompact) Coxeter groups acting on hyperbolic space $\mathbb{H}^{\mathsf{d}}$ for every $\mathsf{d} \leq 19$ (resp.$\mathsf{d} \leq 6$). When $\mathsf{d} = 7$ or 8, they may be taken to be nonarithmetic. Furthermore, for $2 \leq \mathsf{d} \leq 19$, with the possible exceptions $\mathsf{d} = 16$ and 17, the number of essentially distinct Coxeter groups in $\mathbb{H}^{\mathsf{d}}$ with noncompact fundamental domain of volume $\leq$ *Vol* grows at least exponentially with respect to *Vol*. The same result holds for cocompact groups for $\mathsf{d} \leq 6$. The technique is a doubling trick and variations of it; getting the most out of the method requires some work with the Leech lattice.

Another type of studies that have been carried out related to higher dimensional polytopes are on identifying other types of polytopes aside from the simplices that can exist in a particular dimension. For example, Tumarkin, in [74], used methods of combinatorics of polytopes together with geometrical and computational ones to obtain the complete list of compact hyperbolic Coxeter d-polytopes with $\mathsf{d}+3$ facets, $4 \leq \mathsf{d} \leq 7$. Combined with the results of Esselman in [28] this gives the classification of all compact hyperbolic Coxeter d-polytopes with $\mathsf{d}+3$ facets, $\mathsf{d} \geq 4$. In the paper [42], Im Hof classified the polytopes that can be described by Napier cycles. These polytopes have at most $\mathsf{d}+3$ facets. In [75], Tumarkin classified all the hyperbolic noncompact Coxeter polytopes of finite volume whose combinatorial type is either that of a pyramid over a product of two simplices or a product of two simplices of dimension greater than one. Kaplinskaja in [46] and Vinberg in [76] listed simplicial prisms while, in [29], Esselmann classified the remaining compact d-polytopes with $\mathsf{d}+2$ facets. This completes the classification of hyperbolic Coxeter d-polytopes of finite volume with $d + 2$ facets. In dimensions 2 and 3 compact Coxeter polytopes were completely classified by Poincaré [61] and Andreev [2]. Compact polytopes of the simplest combinatorial type, the simplices, were classified by Lannér [51].

In comparison to the studies of volume and identification of polytopes related to Coxeter groups in higher dimension, there is a scarcity of literature on the techniques and examples of determining the subgroup structure of such groups. This study will hopefully become a springboard for more studies to redress the current situation.

Related to the construction and identification of manifolds are the studies on identifying the torsion-free subgroups of Coxeter groups since, as mentioned earlier, the associated orbit space of such groups is a manifold.

The fact that a finitely generated Fuchsian group, i.e. a finitely generated discrete subgroup of orientation-preserving isometries of the hyperbolic plane, contains a torsion-free subgroup of finite index has been known for a long time. The problem was posed by W. Fenchel and was first solved in full by R. Fox in [33] whose proof is by representations in the symmetric groups. Later, in a simplified form, J. Mennicke solved the problem by the method of congruence subgroups in [58]. In [69], A. Selberg used the latter method to extend the result to all finitely generated matrix groups. A consequence of Selberg's result is that all finitely generated Fuchsian and Kleinian groups have torsion-free subgroups of finite index [55].

In the proofs mentioned above, there is no information about the possible indices of the torsion-free subgroups. For the case of the Fuchsian group, this was settled by A. Edmonds, J. Ewing and R. Kulkarni [26] who gave the precise finite indices possible. As described by the authors, the proof is of a geometric-topological nature. Shortly after, R.G. Burns and D. Solitar [8] used elementary algebraic techniques to also obtain the precise possible indices of torsion-free subgroups of finite index of finitely generated Fuchsian groups. However, for Kleinian groups, in general, it remains open [55].

R.D. Feuer in [32], considered certain torsion-free subgroups of various triangle groups. He provides proof of their existence, and in some cases outlined the calculations. In [3], L.A. Best studied a particular family of discrete subgroups of $PSL_2(\mathbb{C})$, namely the groups which have compact orbit space. Any such group which is torsion-free is the fundamental group of its own orbit space, which is a compact 3-manifold. Best exhibits methods of obtaining examples of torsion-free groups and illustrate how to construct the corresponding 3-manifolds. His main tools were the theorem on the existence of discrete subgroups of $PSL_2(\mathbb{C})$ given a hyperbolic polyhedra with appropriate geometric properties, and the Reidemeister-Schreier method. It is known that the torsion-free subgroups of finite index in the Picard group are the fundamental groups of hyperbolic 3-manifolds. In [7], Brunner et al. classified the torsion-free subgroups of the Picard groups of index 12 and 14 wher they obtain 2 and 17 nonisomorphic subgroups, respectively. Then they identified the 3-manifolds by using Dehn surgery.

In [11], the computation and classification of 5- and 6-dimensional torsion-free crystallographic groups, known as *Bierberbach groups* is presented. The basis of the algorithm which decides torsion-freeness of a crystallographic group as well as the triviality of its center. Recently, in [31], Everitt et al. constructed torsion free subgroups of small and explicitly determined index in a large infinite class of Coxeter groups by studying the action of the Weyl group of a simple Lie algebra on its root lattice. One spin-off is the construction of hyperbolic manifolds of very small volume in up to eight dimensions.

1.7 Organization of the Paper

The paper consists of 7 chapters and 4 appendices. The first chapter gives the background and states the problem addressed in the study. The second chapter presents concepts in hyperbolic geometry, polytope and tiling theory. In the third chapter, we introduce Coxeter groups generated by reflections in hyperplanes containing the sides of Coxeter polytopes. We give examples belonging to this family of Coxeter groups. Moreover, we define the group of orientation preserving isometries in the Coxeter group.

In the fourth chapter, we discuss the framework for determining the subgroups of a Coxeter group Γ and its subgroups. In Section 4.1, we present the main theorem which establishes the connection between index n subgroups of a subgroup Λ of Γ and Λ-transitive n-colorings of the Λ-orbit of a fundamental polytope D of Γ. In Section 4.2, we present the step by step procedure for obtaining the subgroups of Λ aided by the *GAP4* routines *TranSub*, *Coloring* and *Seive*. In Sections 4.3 and 4.4, we illustrate the application of the method in obtaining index 5 subgroups of some Coxeter groups and the index 6 subgroups of the modular group.

In the fifth chapter, we discuss the tree construction for determining a set of generators of a subgroup Ω of Λ (called $GenTree(\Omega)$ construction) obtained using the method in the previous chapter. In Section 5.1, we set the assumptions for the $GenTree(\Omega)$ construction. Then in Section 5.2, we define the components of $GenTree(\Omega)$ such as the vertices, edges, paths and trees. In Section 5.3, we present the branching out rules which when applied to the construction of $GenTree(\Omega)$ yield paths that represent generators of Ω. In Section 5.4, we derive a set of generators for each of the subgroups obtained in Sections 4.3 and 4.4.

In the sixth chapter, we present a method for identifying torsion-free subgroups of a Coxeter group.

In the final chapter, we give a summary of the paper and some recommendations for further research on the problems considered in and related to this study.

In Appendix A, we give the *GAP4* routine *TranSub* which checks whether a given transitive subgroup of S_n (symmetric group of order n) can be generated by order 2 elements. The *GAP4* routine *Coloring* is given in Appendix B. It determines all the possible permutation assignments to the generators of Λ which give rise to index n subgroups of Λ distinct up to conjugacy in Λ. In Appendix C we present the *GAP4* routine *Sieve* which is used to verify some order conditions on the permutation assignments to arrive at the final list of permutation assignments that give rise to index n subgroups of Λ distinct up to conjugacy in Λ. Finally in Appendix D, we present the *GAP4* routine *ListSG* which we use to verify the generators of a subgroup Ω obtained from $GenTree(\Omega)$.

Chapter 2

PRELIMINARIES

In this chapter, we present some concepts on hyperbolic geometry, polytope and tiling theory which will be used in the succeeding chapters.

2.1 Standard Hyperbolic Space

There are only three essentially distinct simply connected geometries with constant curvatures in any dimension: a 0, 1 and -1 sectional curvature.

Let $\mathbb{H}^{\mathsf{d}}$ denote the standard hyperbolic d-space. It is, up to isometry, the unique simply connected Riemannian manifold of constant sectional curvature -1. In this paper, our main focus will be groups in $\mathbb{H}^{\mathsf{d}}$ although the material also holds for groups in Euclidean space $\mathbb{E}^{\mathsf{d}}$ and the sphere $\mathbb{S}^{\mathsf{d}}$ of constant sectional curvature 0 and +1, respectively.

We realise $\mathbb{H}^{\mathsf{d}}$ in the upper half space $\mathbb{E}^{\mathsf{d}}{}_{+} = \{x = (x_1, \dots, x_{\mathsf{d}}) \in \mathbb{E}^{\mathsf{d}} | x_{\mathsf{d}} > 0\}$ equipped with the hyperbolic metric $\partial s^2 = |\partial x|^2 / {x_{\mathsf{d}}}^2$. In particular, the distance between two points $x, y \in \mathbb{H}^{\mathsf{d}}$ lying on the d-axis is given by $log(\frac{x}{y})$. The geodesics are either open semi-circles or vertical lines orthogonal to $\mathbb{E}^{\mathsf{d}-1} = \{x_{\mathsf{d}} = 0\}$. The set $\partial\mathbb{H}^{\mathsf{d}} = \widehat{\mathbb{E}}^{\mathsf{d}-1} = \mathbb{E}^{\mathsf{d}-1} \cup \{\infty\}$ is called the ideal boundary of $\mathbb{H}^{\mathsf{d}}$.

Another important frame for hyperbolic geometry is the Lorentz-Minkowski space $\mathbb{E}^{(\mathsf{d},1)}$ equipped with the bilinear form

$$\langle x, y\rangle = x_1y_1 + \cdots + x_{\mathsf{d}}y_{\mathsf{d}} - x_{\mathsf{d}+1}y_{\mathsf{d}+1}$$

of signature $(\mathsf{d}, 1)$. Then, $\mathbb{H}^{\mathsf{d}}$ can be interpreted as vector subset

$$\{x \in \mathbb{E}^{(\mathsf{d},1)} | \langle x, x\rangle = -1, x_{\mathsf{d}+1} > 0\}.$$

A hyperbolic line of $\mathbb{H}^{\mathsf{d}}$ is the intersection of $\mathbb{H}^{\mathsf{d}}$ with a 2-dimensional time-like vector subspace of $\mathbb{E}^{\mathsf{d}+1}$. A vector subspace V in $\mathbb{E}^{\mathsf{d}}$ is said to be time-like if and only if it has a time-like vector. A hyperbolic c-plane of $\mathbb{H}^{\mathsf{d}}$ is the intersection of $\mathbb{H}^{\mathsf{d}}$ with a c-dimensional time-like vector subspace of $\mathbb{E}^{\mathsf{d}+1}$. Note that a hyperbolic 1-plane of $\mathbb{H}^{\mathsf{d}}$ is the same as a hyperbolic line of $\mathbb{H}^{\mathsf{d}}$. A hyperbolic $(\mathsf{d}-1)$-plane of $\mathbb{H}^{\mathsf{d}}$ is called a hyperplane of $\mathbb{H}^{\mathsf{d}}$.

Another useful model of the hyperbolic plane is the *Poincaré circle model*. It is a conformal model which means that the hyperbolic measure of an angle is just its Euclidean measure. Furthermore, it is represented in a bounded region of the Euclidean plane. The points in the model are the interior of a bounding circle (called the fundamental circle). The hyperbolic lines are circular arcs orthogonal to the fundamental circle, including diameters.

2.2 Convex Polytopes and Dihedral Angles

Throughout this study, we let $\mathbb{X}^{\mathsf{d}} = \mathbb{S}^{\mathsf{d}}, \mathbb{E}^{\mathsf{d}}$, or $\mathbb{H}^{\mathsf{d}}$ with $\mathsf{d} > 0$. A pair of points x, y of $\mathbb{X}^{\mathsf{d}}$ is said to be *proper* if and only if x, y are distinct and $y \neq -x$ in $\mathbb{X}^{\mathsf{d}} = \mathbb{S}^{\mathsf{d}}$. If x, y are a proper pair of points of $\mathbb{X}^{\mathsf{d}}$, then there is a unique geodesic segment in $\mathbb{X}^{\mathsf{d}}$ joining x to y. We denote such segment by $[x, y]$.

We define a subset $\mathcal{C}$ of $\mathbb{X}^{\mathsf{d}}$ to be *convex* if and only if for each pair of proper points x, y in $\mathcal{C}$, the geodesic segment $[x, y]$ is contained in $\mathcal{C}$. The *boundary* of a nonempty convex subset $\mathcal{C}$ is denoted by $\partial\mathcal{C}$, and the closure of $\mathcal{C}$ is denoted by $\overline{\mathcal{C}}$. A *side* of a convex subset $\mathcal{C}$ of $\mathbb{X}^{\mathsf{d}}$ is a nonempty, maximal, convex subset of $\partial\mathcal{C}$.

A *convex polytope* in $\mathbb{X}^{\mathsf{d}}$ is a subset of the form $D = \cap H_i^-$ for $i \in I$, where H_i^- is the closed half-space bounded by the hyperplane H_i, under the assumption that (a) D contains a non-empty open subset of $\mathbb{X}^{\mathsf{d}}$; (b) every bounded subset of it intersects only finitely many hyperplanes H_i. In what follows we assume that none of the half-spaces H_i^- contains the intersection of all the others. Under this condition, the half-spaces H_i^- are uniquely determined by D. We say that each of the H_i bounds the polytope D, and we associate it to a side of D.

In this study, the word *polytope* means *convex polytope* unless stated otherwise.

In the setting where we interpret $\mathbb{H}^{\mathsf{d}}$ as

$$\{x \in \mathbb{E}^{(\mathsf{d},1)} | \langle x, x \rangle = -1, x_{\mathsf{d}+1} > 0\},$$

the hyperplanes H in $\mathbb{H}^{\mathsf{d}}$ arise as orthogonal complements of normed space-like vectors ν, that is, $H = \nu^{\perp}$ with $\langle \nu, \nu \rangle = 1$. Let $D \subset \mathbb{H}^{\mathsf{d}}$ denote a convex polytope bounded by finitely many hyperplanes $H_i = \nu_i^{\perp}$, $i \in I$, with ν_i directed inwards to D. Assume that D is acute-angled (all non-right dihedral angles are strictly less that $\frac{\pi}{2}$) of finite volume. If H_i and H_j intersect at D, then the dihedral angle $\alpha_{ij} = \cos^{-1}(-\langle \nu_i, \nu_j \rangle)$. On the other hand, if H_i and H_j are parallel, we let $\alpha_{ij} = 0$.

2.3 Simplices, Fundamental Polytopes and Tesselations

From our definition of polytope in the previous section, it is clear that a polytope is a geometrical figure bounded by portions of lines, planes, or hyperplanes. In two dimensions it is *polygon*, in three a *polyhedron*.

In the space of no dimension the only possible figure is a point which we denote by Ψ_0. In one-dimensional space, we can have any number of points. As we know, any two points bound a line segment Ψ_0 which is a one-dimensional analogue of the polygon Ψ_2 and Ψ_3.

The simplest polytope is called a *simplex* whose construction can be described as follows. By joining Ψ_0 to another point, we construct Ψ_1 which is a simplex in one dimension. On the other hand, joining Ψ_1 to a third point (outside its line) yields a *triangle*, the simplest polygon Ψ_2. Furthermore, by joining the triangle to a fourth point (outside its plane) we construct a *tetrahedron*, the simplest polyhedron. Continuing the process, joining the tetrahedron to a fifth point (outside its 3-space) we construct a 4-simplex which is the simplest polytope in four dimension. It is now evident that any $\mathsf{d}+1$ points which do not lie in an $(\mathsf{d}-1)$-space are vertices of a d-dimensional simplex.

Observe that a line segment is enclosed by two points, a triangle by three lines, a tetrahedron by four planes, and so on. Thus a simplex may alternatively be defined as a finite region of d-space enclosed by $\mathsf{d}+1$ hyperplanes.

If the midpoints of all the edges that emanate from a given vertex O of Ψ_d lie in one hyperplane, then these midpoints are the vertices of an $(\mathsf{d}-1)$-dimensional polytope called the *vertex figure* of Ψ_d at O. A polytope is said to be *regular* if its sides are regular and there is a regular vertex figure at every vertex.

We proceed to describe the *simplicial subdivision* of a regular polytope, beginning with the one-dimensional case. The segment Ψ_1 is divided into two equal parts by its center O_1. The polygon Ψ_2 is divided by its lines of symmetry into $2p$ right-angled triangles, which join the center O_2 to the simplicially subdivided sides. The polyhedron Ψ_3 is divided by its planes of symmetry into g quadrirectangular tetrahedral which join the center O_3 to the simplicially divided faces. Analogously, the general regular polytope Ψ_d is divided into a number of congruent simplexes (of a special kind) which join the center O_d to the simplicially subdivided cells.

A subset F of a metric space $\mathbb{X}^{\mathsf{d}}$ is a *fundamental region* for a group Γ of isometries of $\mathbb{X}^{\mathsf{d}}$ if and only if

(1) the set F is open in $\mathbb{X}^{\mathsf{d}}$;

(2) the members of $\{\gamma F : \gamma \in \Gamma\}$ are mutually disjoint; and

(3) $\mathbb{X}^{\mathsf{d}} = \cup\{\gamma \overline{F} : \gamma \in \Gamma\}$.

A subset E of a metric space $\mathbb{X}^{\mathsf{d}}$ is a *fundamental domain* for a group Γ of isometries of $\mathbb{X}^{\mathsf{d}}$ if and only if E is a connected fundamental region for Γ.

We define a *fundamental polytope* for a discrete group Γ of isometries of $\mathbb{X}^{\mathsf{d}}$ as a polytope in $\mathbb{X}^{\mathsf{d}}$ whose interior is a locally finite fundamental domain for Γ.

A tessellation of $\mathbb{X}^{\mathsf{d}}$ is a collection $\mathcal{T}$ of d-dimensional polytope in $\mathbb{X}^{\mathsf{d}}$ such that the interiors of the polyhedra in $\mathcal{T}$ are mutually disjoint; the union of the polyhedra in $\mathcal{T}$ is $\mathbb{X}^{\mathsf{d}}$; and the collection $\mathcal{T}$ is locally finite.

Chapter 3

COXETER GROUPS

The setting by which we consider Coxeter groups is defined in this chapter. We present examples from two families of Coxeter groups; namely, Coxeter groups with simplices as fundamental polytopes and the singly truncated Coxeter tetrahedron groups. We also define the group of orientation-preserving isometries in the Coxeter group.

3.1 Coxeter Polytopes, Coxeter Groups, Coxeter Diagrams

A convex polytope $D \subset \mathbb{X}^{\mathsf{d}}$ bounded by hyperplanes H_i, $i \in I$, is said to be a *Coxeter polytope* if for all $i, j \in I = \{1, \ldots, m\}$, $i \neq j$, the hyperplanes H_i and H_j are disjoint or form a dihedral angle of $\frac{\pi}{h_{ij}}$, where $h_{ij} \in \mathbb{Z}$ and $h_{ij} \geq 2$.

If D is a Coxeter polytope, then the group Γ of isometries of $\mathbb{X}^{\mathsf{d}}$ generated by the reflections R_i in the hyperplanes H_i is discrete and D is a fundamental polytope for Γ. This means that the polytopes $\gamma D, \gamma \in \Gamma$ do not have pairwise common interior points and cover $\mathbb{X}^{\mathsf{d}}$; that is, they form a tessellation $\mathcal{T}$ for $\mathbb{X}^{\mathsf{d}}$.

Γ is defined by the relations

$$R_i{}^2 = e, (R_iR_j)^{h_{ij}} = e \tag{1}$$

If H_i and H_j are disjoint, the corresponding relation with $h_{ij} = \infty$ may be omitted. The group Γ with generators R_i and defining relations (1) is called a *Coxeter Group*. Γ is also referred to as the group

$$\begin{aligned}\Gamma &= \Gamma(h_{21}, \ldots, h_{m1}, h_{23}, \ldots, h_{2m}, \ldots, h_{m-1,m}) \\ &= \left\langle R_1, R_2, \ldots, R_m \;\middle|\; \begin{array}{c} R_1{}^2 = \cdots = R_m{}^2 = (R_2R_1)^{h_{21}} = \cdots = (R_mR_1)^{h_{m1}} = \\ (R_2R_3)^{h_{23}} = \cdots = (R_2R_m)^{h_{2m}} = \cdots = (R_{m-1}R_m)^{h_{m-1m}} = e \end{array} \right\rangle.\end{aligned}$$

The Coxeter group Γ may be denoted by its *Coxeter diagram*, a graph whose nodes i, j correspond to the generators of Γ. The nodes are connected by an edge labeled by h_{ij} whenever the product of the corresponding reflections is of period h_{ij}. The label is omitted when $h_{ij} = 3$ and the nodes are not connected when $h_{ij} = 2$. If $h_{ij} = \infty$, the nodes i, j are connected by a dashed line. The Coxeter diagram can also be used to denote the corresponding fundamental polytope D of Γ.

Γ is said to be Euclidean, spherical or hyperbolic depending on whether its corresponding fundamental polytope D is in $\mathbb{E}^{\mathrm{d}}$, $\mathbb{S}^{\mathrm{d}}$ or $\mathbb{H}^{\mathrm{d}}$. In the next sections, some examples of Coxeter group Γ are presented.

3.2 Coxeter Groups and Simplices as Fundamental Polytopes

In $\mathbb{X}^2$, a well-known example of a Coxeter group is the triangle group. Consider a triangle $\triangle$ (a two-dimensional Coxeter polytope) with interior angles $\frac{\pi}{p'}$, $\frac{\pi}{q'}$ and $\frac{\pi}{r'}$, where p', q', r' are integers ≥ 2. $\triangle$ lies on $\mathbb{E}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$ according as $\frac{\pi}{p'}+\frac{\pi}{q'}+\frac{\pi}{r'}$ is larger, equal or smaller than one, respectively. Repeatedly reflecting $\triangle$ in its sides results in a tessellation of the appropriate plane by copies of $\triangle$. Let P', Q' and R' denote respectively the reflections in the sides of $\triangle$ opposite the angles $\frac{\pi}{p'}$, $\frac{\pi}{q'}$ and $\frac{\pi}{r'}$. The group generated by these reflections is called a *triangle group* denoted by $*p'q'r'$ or $\Gamma(r',q',p')$ with fundamental triangle $\triangle$. The generators P', Q',R' satisfy the relations

$$P'^{\,2} = Q'^{\,2} = R'^{\,2} = (Q'P')^{r'} = (R'P')^{q'} = (Q'R')^{p'} = e.$$

A special type of hyperbolic triangle group is the *extended Hecke group*, denoted by $*p'2\infty$ or $\Gamma(\infty,2,p')$. In this case, $\triangle$ has zero as one of its interior angles.

To illustrate an example of the extended Hecke group, consider $\triangle$ with interior angles $\frac{\pi}{3}$, $\frac{\pi}{2}$ and 0. Repeatedly reflecting $\triangle$ in its sides results in a tessellation of $\mathbb{H}^2$ by copies of $\triangle$ as shown in Figure 3.1. Let P', Q', R' denote respectively the reflections in the sides opposite the angles $\frac{\pi}{3}$, $\frac{\pi}{2}$ and 0. The group generated by P',Q', R' is the extended Hecke group $*32\infty$ also known as the *extended modular group*. We also refer to it as the group

$$\Gamma(\infty,2,3)=\langle P',Q',R'|P'^{\,2}=Q'^{\,2}=R'^{\,2}=(R'P')^2=(Q'R')^3=e\rangle.$$

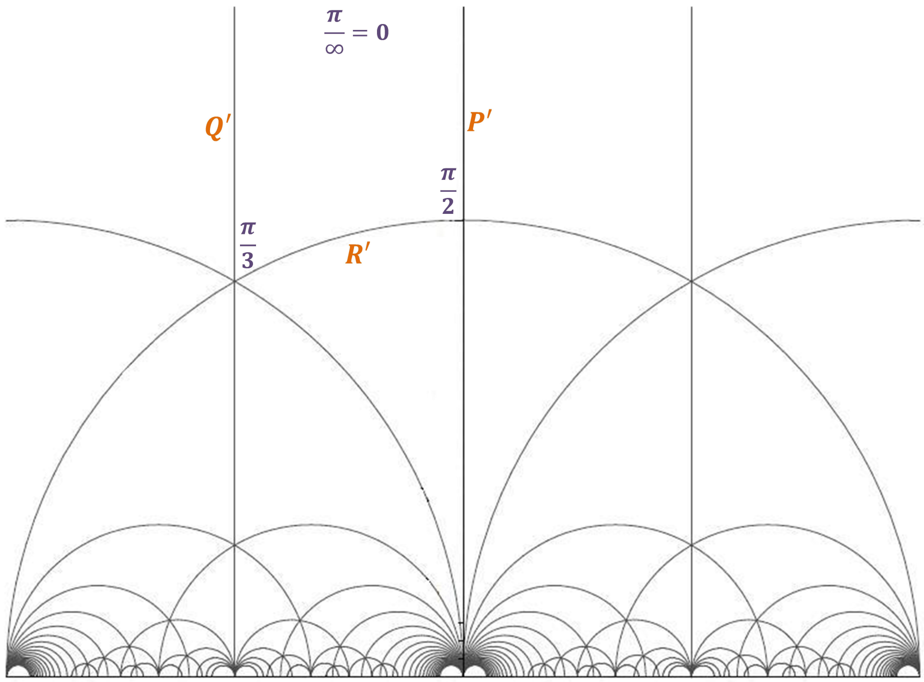


Figure 3.1: Tessellation of the upper halfplane by copies of $\triangle$ with interior angles $\frac{\pi}{3}$, $\frac{\pi}{2}$ and 0.

In $\mathbb{X}^3$, an example of a Coxeter group is the Coxeter tetrahedron group, which is a three-dimensional analogue of a triangle group. Consider a tetrahedron $\mathtt{t}$ of finite volume with dihedral angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, $\frac{\pi}{s}$, $\frac{\pi}{t}$ and $\frac{\pi}{u}$, $p,q,r,s,t,u\geq 2$ as shown in Figure 3.2. $\mathtt{t}$ is denoted by $[p,q,r,s,t,u]$ and is a three-dimensional Coxeter polytope called a *Coxeter tetrahedron*.

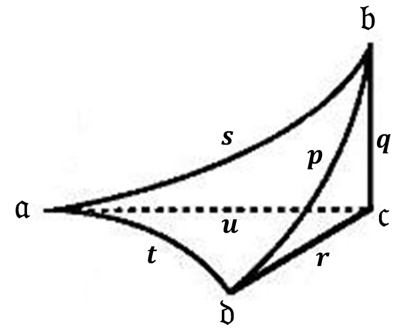


Figure 3.2: A tetrahedron $\mathfrak{t}$ with vertices $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. A label k at an edge of $\mathfrak{t}$ means that the dihedral angle of $\mathfrak{t}$ at this edge is $\frac{\pi}{k}$.

Reflecting $\mathfrak{t}$ in its faces results in a tessellation in $\mathbb{X}^3$. Let P, Q, R, and S denote reflections along the respective faces $\mathfrak{bcd}$, $\mathfrak{abd}$, $\mathfrak{abc}$, and $\mathfrak{acd}$ of $\mathfrak{t}$. The group generated by the reflections P, Q, R, and S is called the *Coxeter tetrahedron group* with fundamental tetrahedron $\mathfrak{t}$. This group is referred to as

$$\begin{aligned}&\Gamma(p,q,r,s,t,u)\\ &= \left\langle P,Q,R,S \;\middle|\; \begin{array}{c} P^2 = Q^2 = R^2 = S^2 = (QP)^p = (RP)^p = \\ (SP)^r = (QR)^s = (QS)^t = (RS)^p = e \end{array} \right\rangle.\end{aligned}$$

There are five Coxeter tetrahedra in $\mathbb{S}^3$ namely $[3,2,2,3,2,3]$, $[3,2,2,3,2,4]$, $[3,2,2,4,2,3]$ and $[3,2,2,3,3,3]$. On the other hand, the Coxeter tetrahedra in $\mathbb{E}^3$ are $[4,2,2,3,2,4]$, $[4,2,2,3,3,2]$ and $[3,2,3,3,2,3]$. In the hyperbolic case, it is known that there are 9 compact and 23 noncompact Coxeter tetrahedra of finite volume. The noncompact types have one, two, three or four vertices at infinity. The list of Coxeter tetrahedron groups with fundamental tetrahedra of compact type (respectively, non-compact type) is presented in Table 3.1 (respectively, Tables 3.2 and 3.3).

In $\mathbb{H}^{\mathsf{d}}$, $\mathsf{d} > 3$, known examples of Coxeter groups include those having simplices (called *Coxeter simplices*) as fundamental polytopes. Hyperbolic Coxeter simplices exist up to dimension 9. In $\mathbb{H}^4$, there are 5 compact and 9 non-compact Coxeter simplices. There are 12, 3, 4, 4 and 3 hyperbolic Coxeter simplices in $\mathbb{H}^5$, $\mathbb{H}^6$, $\mathbb{H}^7$, $\mathbb{H}^8$ and $\mathbb{H}^9$, respectively, all of which are non-compact. The Coxeter diagrams of these hyperbolic Coxeter groups are listed in Table 3.4.

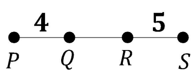
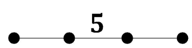
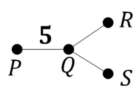
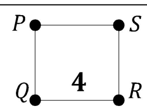
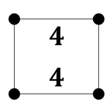
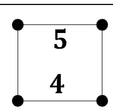
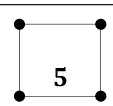
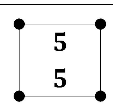
	Coxeter Diagram	Coxeter Tetrahedron Groups
1		$\Gamma(4,2,2,3,2,5) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^2 = (RS)^5 = e \end{matrix} \right\rangle$
2		$\Gamma(3,2,2,5,2,3) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^2 = (QR)^5 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
3		$\Gamma(5,2,2,3,2,5) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^5 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^2 = (RS)^5 = e \end{matrix} \right\rangle$
4		$\Gamma(5,2,2,3,3,2) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^5 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^3 = (RS)^2 = e \end{matrix} \right\rangle$
5		$\Gamma(3,2,3,4,2,3) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^3 = (QR)^4 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
6		$\Gamma(3,2,4,4,2,3) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^4 = (QR)^4 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
7		$\Gamma(3,2,5,4,2,3) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^5 = (QR)^4 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
8		$\Gamma(3,2,3,5,2,3) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^3 = (QR)^5 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
9		$\Gamma(3,2,5,5,2,3) = \left\langle P, Q, R, S \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^5 = (QR)^5 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$

Table 3.1: Coxeter tetrahedron groups with fundamental tetrahedra of compact type together with their respective Coxeter diagrams.

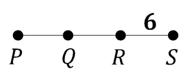
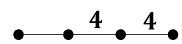
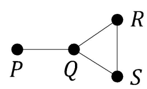
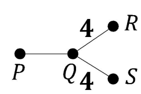
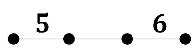
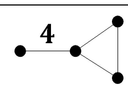
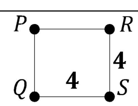
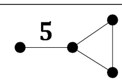
	Coxeter Diagram	Coxeter Tetraheron Groups
1		$\Gamma(3,2,2,3,2,6) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^2 = (RS)^6 = e \end{array} \right\rangle$
2		$\Gamma(3,2,2,4,2,4) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^2 = (QR)^4 = (QS)^2 = (RS)^4 = e \end{array} \right\rangle$
3		$\Gamma(3,2,2,3,3,3) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^3 = (RS)^3 = e \end{array} \right\rangle$
4		$\Gamma(4,2,2,3,2,6) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^2 = (RS)^6 = e \end{array} \right\rangle$
5		$\Gamma(3,2,2,4,4,2) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^2 = (QR)^4 = (QS)^4 = (RS)^2 = e \end{array} \right\rangle$
6		$\Gamma(5,2,2,3,2,6) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^5 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^2 = (RS)^6 = e \end{array} \right\rangle$
7		$\Gamma(4,2,2,3,3,3) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^3 = (RS)^3 = e \end{array} \right\rangle$
8		$\Gamma(3,2,3,4,2,4) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^3 = (QR)^4 = (QS)^2 = (RS)^4 = e \end{array} \right\rangle$
9		$\Gamma(5,2,2,3,3,3) = \left\langle P,Q,R,S \middle\vert \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = (QP)^5 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^3 = (RS)^3 = e \end{array} \right\rangle$

Table 3.2: Coxeter tetrahedron groups with fundamental tetrahedra of non-compact type (having one vertex at infinity) together with their respective Coxeter diagrams.

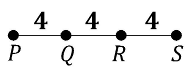
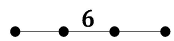
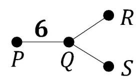
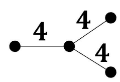
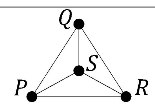
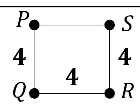
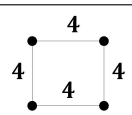
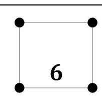
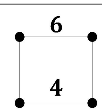
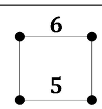
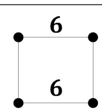
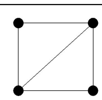
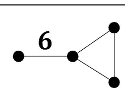
	Coxeter Diagram	Coxeter Tetrahedron Groups
1		$\Gamma(4,2,2,4,2,4) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^2 = (QR)^4 = (QS)^2 = (RS)^4 = e \end{matrix} \right\rangle$
2		$\Gamma(3,2,2,6,2,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^2 = (QR)^6 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
3		$\Gamma(6,2,2,3,2,6) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^6 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^2 = (RS)^6 = e \end{matrix} \right\rangle$
4		$\Gamma(6,2,2,3,3,2) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^6 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^3 = (RS)^2 = e \end{matrix} \right\rangle$
5		$\Gamma(4,2,2,4,4,2) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^2 = (QR)^4 = (QS)^4 = (RS)^2 = e \end{matrix} \right\rangle$
6		$\Gamma(3,3,3,3,3,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^3 \\ = (SP)^3 = (QR)^3 = (QS)^3 = (RS)^3 = e \end{matrix} \right\rangle$
7		$\Gamma(4,2,3,4,2,4) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^3 = (QR)^4 = (QS)^2 = (RS)^4 = e \end{matrix} \right\rangle$
8		$\Gamma(4,2,4,4,2,4) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^4 = (RP)^2 \\ = (SP)^4 = (QR)^4 = (QS)^2 = (RS)^4 = e \end{matrix} \right\rangle$
9		$\Gamma(3,2,3,6,2,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^3 = (QR)^6 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
10		$\Gamma(3,2,6,4,2,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^6 = (QR)^4 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
11		$\Gamma(3,2,6,5,2,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^6 = (QR)^5 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
12		$\Gamma(3,2,6,6,2,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^6 = (QR)^6 = (QS)^2 = (RS)^3 = e \end{matrix} \right\rangle$
13		$\Gamma(3,2,3,3,3,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^3 = (RP)^2 \\ = (SP)^3 = (QR)^3 = (QS)^3 = (RS)^3 = e \end{matrix} \right\rangle$
14		$\Gamma(6,2,2,3,3,3) = \left\langle P,Q,R,S \,\middle|\, \begin{matrix} P^2 = Q^2 = R^2 = S^2 = (QP)^6 = (RP)^2 \\ = (SP)^2 = (QR)^3 = (QS)^3 = (RS)^3 = e \end{matrix} \right\rangle$

Table 3.3: Coxeter tetrahedron groups with fundamental tetrahedra of non-compact type (with two or more vertices at infinity) together with their respective Coxeter diagrams.

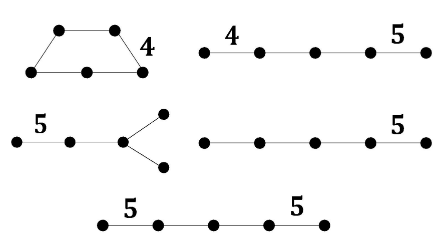
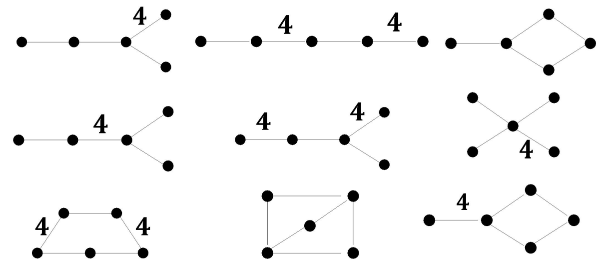
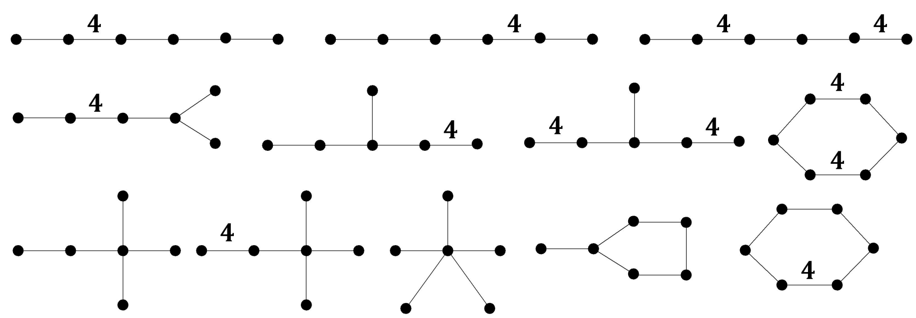
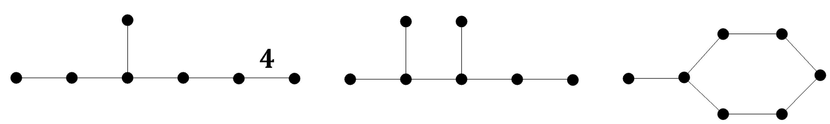
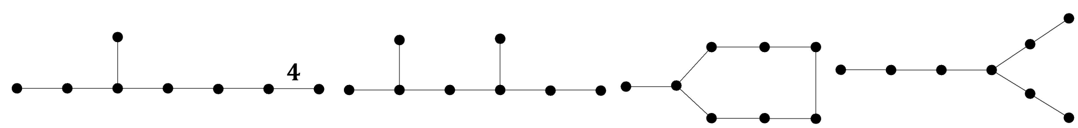
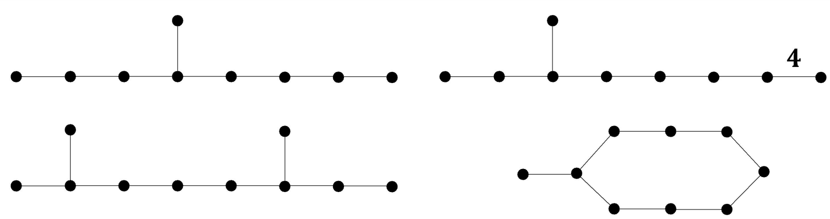
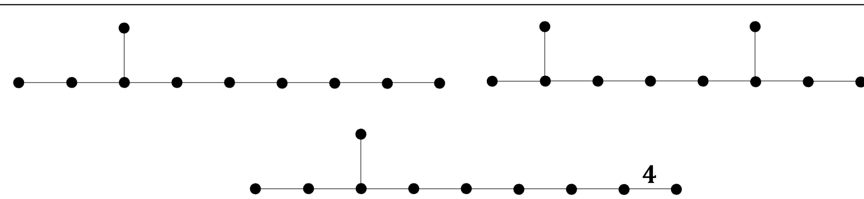
d = 4	 compact	 non-compact
d = 5		
d = 6		
d = 7		
d = 8		
d = 9		

Table 3.4: The Coxeter diagrams of the Coxeter groups in $\mathbb{H}^d$, $d > 3$.

3.4 Singly Truncated Coxeter Tetrahedron Groups

Previously, we were introduced to Coxeter tetrahedron groups arising from Coxeter tetrahedra with one vertex at infinity (cusp). In this section, we will look at examples of Coxeter groups whose fundamental polytopes are truncated tetrahedra, which are obtainable from such Coxeter tetrahedra. These groups belong to the family referred to as singly truncated Coxeter tetrahedron groups.

In [13], Conder and Martin discussed a procedure for truncating a tetrahedron $\mathfrak{t}$ with a vertex $\mathfrak{a}$ at infinity (see Figure 3.3). This process is achieved by opening up a cusp of a hyperbolic tetrahedron, which can be done in three different ways.

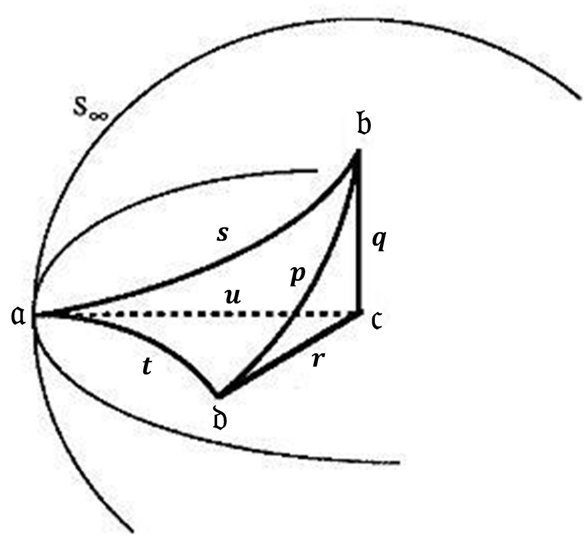


Figure 3.3: A singly cusped tetrahedron with vertices $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$; S_∞ denotes the sphere at infinity.

The general idea is to choose a pair of faces in $\mathfrak{t}$ having common vertex $\mathfrak{a}$. For example, consider the faces $\mathfrak{acd}$ and $\mathfrak{abc}$ with dihedral angle $\frac{\pi}{u}$ (one may also consider the faces with dihedral angles $\frac{\pi}{s}$ or $\frac{\pi}{t}$). Then continuously decrease the angle $\frac{\pi}{u}$ to $\theta = \frac{\pi}{v}$, $0 \leq \theta \leq \frac{\pi}{u}$ while fixing all the other dihedral angles. By decreasing the angle $\frac{\pi}{u}$, the sum of the dihedral angles around the cusp decreases to less than π, opening up the cusp. This amounts to moving the ideal vertex $\mathfrak{a}$ beyond the sphere at infinity (see Figure 3.4).

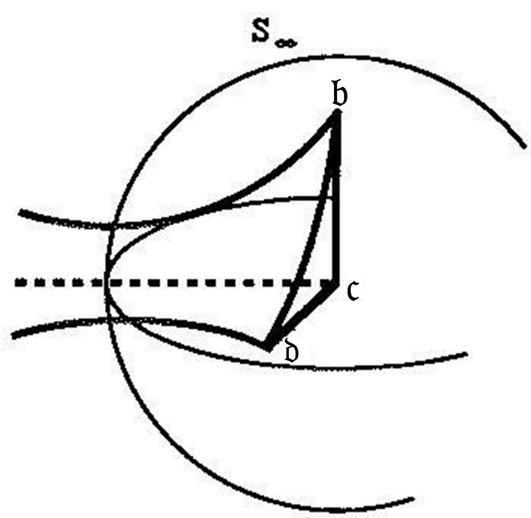


Figure 3.4: Opening up a cusp of a tetrahedron.

Furthermore, Conder and Martin showed that such a tetrahedron with an *open* cusp can be truncated by a hyperbolic plane which is perpendicular to each of the unbounded faces. The resulting polytope is a hyperbolic prism and we refer to it as a truncated tetrahedron $\mathfrak{t}'$ (see Figure 3.5). For more details on this procedure, the reader is referred to [13]. The group generated by the reflections about the faces of this prism is an example of a *singly truncated Coxeter tetrahedron group*. We introduce some notations to clearly denote this particular Coxeter group.

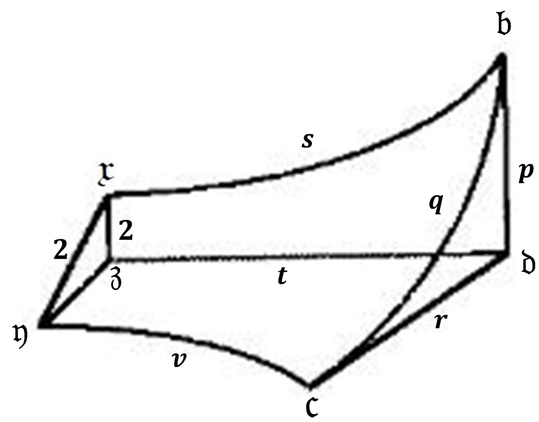


Figure 3.5: A truncated tetrahedron $\mathfrak{t}'$ which is a hyperbolic prism. The additional three vertices of the polytope are $\mathfrak{x}$, $\mathfrak{y}$ and $\mathfrak{z}$.

For $i = 1, 2, \ldots, 9$, $u \in \{2, 3, 4, 6\}$, and $0 \leq \frac{\pi}{v} \leq \frac{\pi}{u}$, we denote by $\Gamma_{i,u}(v)$ the singly truncated Coxeter tetrahedron group generated by reflections in the faces of the hyperbolic prism obtained by opening up the cusp of the tetrahedron with the i^{th} Coxeter diagram in Table 3.2 along the edge going out to the cusp with dihedral angle $\frac{\pi}{u}$ by an angle $\frac{\pi}{v}$. The group $\Gamma_{i,u}(v)$ has five reflections P, Q, R, S and T. Referring to Figure 3.5, P, Q, R, S and T are the reflections along the faces $\mathfrak{bcd}$, $\mathfrak{bdzx}$, $\mathfrak{bcyx}$, $\mathfrak{cdzy}$ and $\mathfrak{xyz}$, respectively. $\Gamma_{i,u}(v)$ is also referred to as the group

$$\Gamma(p,q,r,\infty,s,t,2,v,2,2)$$
$$= \left\langle P,Q,R,S,T \;\middle|\; \begin{array}{l} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^p = (RP)^q = (SP)^r = \\ (QR)^s = (QS)^t = (QT)^2 = (RS)^v = (RT)^2 = (ST)^2 = e \end{array} \right\rangle.$$

We have the following result on $\Gamma_{i,u}(v)$ from [13].

Theorem 3.1 *The Coxeter group* $\Gamma_{i,u}(v)$ *is a Coxeter group with the truncated tetrahedron* t′ *as fundamental polytope for the following triples* (i,u,v)*:*

(i) $(i,6,v)$ *for* $i = 1,4,6$ *and* $v \geq 6$

(ii) $(i,3,v)$ *for* $i = 3,7,9$ *and* $v \geq 3$

(iii) $(2,4,v)$ *for* $v \geq 4$

(iv) $(5,2,v)$ *for* $v \geq 2$

(v) $(8,4,4)$

In Table 3.5, we present examples of $\Gamma_{i,u}(v)$ for $i = 1,2,\ldots,9$ together with their Coxeter diagrams. In arriving at these groups, we looked at particular instances of truncated tetrahedra realizable from Coxeter tetrahedra with one cusp as given in Theorem 3.1.

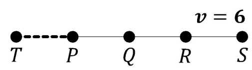
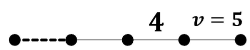
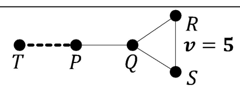
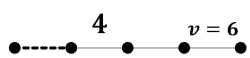
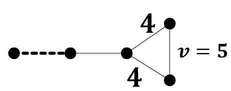
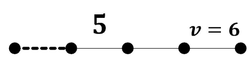
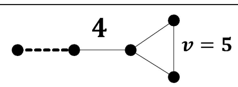
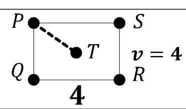
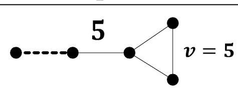
Coxeter Diagram	Truncated Tetrahedron Groups
	$\Gamma_{1,6}(6) = \Gamma(3,2,2,0,3,2,2,6,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^3 = (RP)^2 = (SP)^2 = \\ (QR)^3 = (QS)^2 = (QT)^2 = (RS)^6 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{2,4}(5) = \Gamma(3,2,2,0,4,2,2,5,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^3 = (RP)^2 = (SP)^2 = \\ (QR)^4 = (QS)^2 = (QT)^2 = (RS)^5 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{3,3}(5) = \Gamma(3,2,2,0,3,3,2,5,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^3 = (RP)^2 = (SP)^2 = \\ (QR)^3 = (QS)^3 = (QT)^2 = (RS)^5 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{4,6}(6) = \Gamma(4,2,2,0,3,2,2,6,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^4 = (RP)^2 = (SP)^2 = \\ (QR)^3 = (QS)^2 = (QT)^2 = (RS)^6 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{5,2}(5) = \Gamma(3,2,2,0,4,4,2,5,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^3 = (RP)^2 = (SP)^2 = \\ (QR)^4 = (QS)^4 = (QT)^2 = (RS)^5 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{6,6}(6) = \Gamma(5,2,2,0,3,2,2,6,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^5 = (RP)^2 = (SP)^2 = \\ (QR)^3 = (QS)^2 = (QT)^2 = (RS)^6 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{7,3}(5) = \Gamma(4,2,2,0,3,3,2,5,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^4 = (RP)^2 = (SP)^2 = \\ (QR)^3 = (QS)^3 = (QT)^2 = (RS)^5 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{8,4}(4) = \Gamma(3,2,3,0,4,2,2,4,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^3 = (RP)^2 = (SP)^3 = \\ (QR)^4 = (QS)^2 = (QT)^2 = (RS)^4 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$
	$\Gamma_{9,3}(5) = \Gamma(5,2,2,0,3,3,2,5,2,2)$ $= \left\langle P,Q,R,S,T \middle\vert \begin{matrix} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^5 = (RP)^2 = (SP)^2 = \\ (QR)^3 = (QS)^3 = (QT)^2 = (RS)^5 = (RT)^2 = (ST)^2 = e \end{matrix} \right\rangle$

Table 3.5: Examples of truncated Coxeter tetrahedron groups $\Gamma_{i,u}(v)$, for $i = 1, 2, \ldots, 9$.

3.4 Group of Orientation-Preserving Isometries

One of the important subgroups of the Coxeter group

$$\Gamma(h_{21},\ldots,h_{m1},h_{23},\ldots,h_{2m},\ldots,h_{m-1,m}) = \left\langle R_1,R_2,\ldots,R_m \;\middle|\; \begin{array}{c} {R_1}^2=\cdots={R_m}^2=(R_2R_1)^{h_{21}}=\cdots=(R_mR_1)^{h_{m1}}= \\ (R_2R_3)^{h_{23}}=\cdots=(R_2R_m)^{h_{2m}}=\cdots=(R_{m-1}R_m)^{h_{m-1m}}=e \end{array} \right\rangle$$

is its index 2 subgroup Γ^0 consisting of orientation preserving isometries. The group is given by

$$\Gamma^0 = \Gamma^0(h_{21},\ldots,h_{m1},h_{23},\ldots,h_{2m},\ldots,h_{m-1,m}) = \left\langle R_2R_1,R_3R_1,\ldots,R_mR_1 \;\middle|\; \begin{array}{c} (R_2R_1)^{h_{21}}=\cdots=(R_mR_1)^{h_{m1}}=(R_2R_3)^{h_{23}} \\ =\cdots=(R_2R_m)^{h_{2m}}=\cdots=(R_{m-1}R_m)^{h_{m-1m}}=e \end{array} \right\rangle.$$

Γ^0 is generated by $R_2R_1, R_3R_1, \ldots, R_mR_1$. For any pair $i,j \in \{1,\ldots,m\}$, observe that $R_iR_j = (R_iR_1)(R_jR_1)^{-1}$. If the order of R_iR_j is finite, it is called a *relator* of Γ^0.

For instance, the orientation preserving subgroup of the triangle group $*p'q'r'$ given by

$$\Gamma(p',q',r') = \langle P',Q',R'|P'^{\,2}=Q'^{\,2}=R'^{\,2}=(Q'P')^{r'}=(R'P')^{q'}=(Q'R')^{p'}=e\rangle$$

is the group

$$\Gamma^0(p',q',r') = \langle Q'P',R'P'|(Q'P')^{r'}=(R'P')^{q'}=(Q'R')^{p'}=e\rangle$$

which is also called $p'q'r'$.

A special case of $p'q'r'$ is the orientation preserving subgroup of the extended modular group

$$\Gamma(\infty,2,3) = \langle P', Q', R' | P'^{\,2} = Q'^{\,2} = R'^{\,2} = (R'P')^2 = (Q'R')^3 = e \rangle$$

which can be generated by $Q'P'$, $R'P'$ and, consequently, it is defined as

$$\Gamma^0(\infty,2,3) = \langle Q'P', R'P' | (R'P')^2 = (Q'R')^3 = e \rangle.$$

This group is called the *modular group* and is also denoted by 32∞. Take note that $R'P'$ and $Q'R'$ are of orders 2 and 3, respectively whereas $Q'P'$ is of infinite order. Hence, $R'P'$ and $Q'R'$ are relators of 32∞.

The index 2 subgroup of the Coxeter tetrahedron group

$$\Gamma(p,q,r,s,t,u) = \left\langle P,Q,R,S \;\middle|\; \begin{array}{c} P^2 = Q^2 = R^2 = S^2 = T^2 = (QP)^p = (RP)^q = \\ (SP)^r = (QR)^s = (QS)^t = (RS)^u = e \end{array} \right\rangle.$$

consisting of orientation preserving isometries is called the *tetrahedron Kleinian group* and is referred to as

$$\Gamma^0(p,q,r,s,t,u) = \langle QP, RP, SP \,|(QP)^p = (RP)^q = (SP)^r = (QR)^s = (QS)^t = (RS)^u = e \rangle\,.$$

Chapter 4

DETERMINING THE SUBGROUPS OF COXETER GROUPS AND THEIR SUBGROUP

In this chapter, we address the problem on determining the subgroups of Coxeter groups and their subgroups. We present the setting and framework for the problem which exhibit a connection between group theory and color symmetry theory. The results are applied in obtaining the index 5 subgroups of hyperbolic Coxeter groups having Coxeter simplices in $\mathbb{H}^4$ as fundamental polytopes and some examples of singly truncated Coxeter tetrahedron groups. The index 6 subgroups of the groups of orientation preserving isometries of the triangle group will also be derived using the method.

4.1 Colorings of n-dimensional Tessellations by Coxeter Polytopes

In the results that follow, our primary assumption is that we start with a Coxeter group

$$\Gamma = \Gamma(h_{21}, \ldots, h_{m1}, h_{23}, \ldots, h_{2m}, \ldots, h_{m-1,m})$$
$$= \left\langle R_1, R_2, \ldots, R_m \;\middle|\; \begin{array}{c} {R_1}^2 = \cdots = {R_m}^2 = (R_2R_1)^{h_{21}} = \cdots = (R_mR_1)^{h_{m1}} = \\ (R_2R_3)^{h_{23}} = \cdots = (R_2R_m)^{h_{2m}} = \cdots = (R_{m-1}R_m)^{h_{m-1m}} = e \end{array} \right\rangle.$$

on either $\mathbb{X}^{\mathsf{d}} = \mathbb{E}^{\mathsf{d}}, \mathbb{S}^{\mathsf{d}}$ or $\mathbb{H}^{\mathsf{d}}$, given the conditions discussed and notations developed in Section 3.1.

The following result from [76] holds for the Coxeter group Γ.

Lemma 4.1 *The Coxeter polytope D forms a fundamental polytope for Γ and the tessellation $\mathcal{T}$ is the Γ-orbit of D; that is, $\mathcal{T} = \{\gamma D | \gamma \in \Gamma\}$. Moreover, $Stab_{\Gamma}(D) = \{\gamma \in \Gamma | \gamma D = D\}$ is the trivial group $\{e\}$ and Γ acts transitively on $\mathcal{T}$.*

Consequently, as explained in Section 3.1, if $\gamma \in \Gamma$ and γD is the image of D under γ, then the union of these images as γ varies over the elements of Γ is $\mathbb{X}^{\mathsf{d}}$. Moreover, if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, the respective interiors of $\gamma_1 D$ and $\gamma_2 D$ are disjoint. Each polytope in $\mathcal{T}$ is the image of D under a uniquely determined $\gamma \in \Gamma$.

Now, consider a subgroup Λ of Γ and $\mathcal{O} = \Lambda D = \{\lambda D | \lambda \in \Lambda\}$ the Λ-orbit of D. Then $Stab_{\Lambda}(D) = \{e\}$, and Λ acts transitively on $\mathcal{O}$. Hence, there is a one-to-one correspondence between Λ and $\mathcal{O}$ given by $\lambda \mapsto \lambda D$. Take note that $\lambda' \in \Lambda$ acts on $\lambda D \in \mathcal{O}$ by sending it to its image under λ'.

In this work, the methodology for determining the subgroups of the Coxeter groups and their subgroups is based on concepts in color symmetry theory. In particular, we look at colorings of $\mathcal{T}$ and ΛD when determining the subgroups of Γ and Λ, respectively.

We now give the definition of a coloring of the Λ-orbit of D for a subgroup Λ of Γ as follows.

Let $C = \{c_1, c_2, \ldots, c_n\}$ be a set of n colors and $\mathcal{O} = \Lambda D = \{\lambda D | \lambda \in \Lambda\}$, where Λ is a subgroup of Γ. An onto function $\iota : \mathcal{O} \to C$ is called an *n-coloring* of $\mathcal{O}$ if each $\lambda D \in \mathcal{O}$ is assigned a color in C.

The coloring determines a partition $\mathfrak{P} = \{\iota^{-1}(c_i) | c_i \in C\}$ where $\iota^{-1}(c_i)$ is the set of elements of $\mathcal{O}$ assigned a color c_i. Equivalently, we may think of the coloring as a partition of $\mathcal{O}$.

We look at n-colorings of $\mathcal{O}$ where for every pair of colors $c_i, c_j \in C$ there is a $\lambda \in \Lambda$ such that $c_j = \lambda c_i$. We call such colorings Λ-transitive n-colorings of $\mathcal{O}$. To arrive at such a coloring consider the following result:

Lemma 4.2 *Let $\mathfrak{P} = \{X_1, X_2, \ldots, X_n\}$ be a partition of $\mathcal{O}$ into n sets. Suppose Λ acts transitively on $\mathfrak{P}$, then $\mathfrak{P} = \{\lambda \Omega D | \lambda \in \Lambda\}$ for some index n subgroup Ω of Λ.*

Proof Assume that $D \in X_1$ and let $\Omega = Stab_{\Lambda}(X_1)$. Since Λ acts transitively on $\mathcal{O}$ and Λ acts transitively on $\mathfrak{P}$, then $X_1 = \Omega D$ and each set in $\mathfrak{P}$ is equal to λX_1 for some $\lambda \in \Lambda$. Thus, $\mathfrak{P} = \{\lambda \Omega D | \lambda \in \Lambda\}$. Moreover, by the orbit-stabilizer theorem, $n = |\mathfrak{P}| = [\Lambda : Stab_{\Lambda}(X_1)] = [\Lambda : \Omega]$. ∎

Suppose Ω is an index n subgroup of Λ with $\{\lambda_1', \lambda_2', \ldots, \lambda_n'\}$ a complete set of left coset representatives of Ω in Λ. We can define a coloring $\iota : \mathcal{O} = \Lambda D \to C = \{c_1, c_2, \ldots, c_n\}$ by $\iota(\overline{x}) = c_i$ if $\overline{x} \in \lambda_i' \Omega D$. Clearly, $\iota^{-1}(c_i) = \lambda_i' \Omega D$ and $\mathfrak{P} =$

$\{\iota^{-1}(c_1), \ldots, \iota^{-1}(c_n)\} = \{\lambda_1'\Omega D, \lambda_2'\Omega D, \ldots, \lambda_n'\Omega D\}$. The group Λ acts transitively on $\mathfrak{P}$ with $\lambda \in \Lambda$ sending $\lambda_i'\Omega D$ to its image $\lambda\lambda_i'\Omega D$. Consequently, we get a transitive action of Λ on C by defining for $\lambda \in \Lambda$, $\lambda c_i = c_j$ if and only if $\lambda\iota^{-1}(c_i) = \iota^{-1}(c_j)$. The n-coloring of $\mathcal{O}$ determined by ι is Λ-transitive.

As a consequence of the above discussions, we have the following result which will be the basis of our methodology in studying the subgroup structure of Coxeter groups and their subgroups.

Theorem 4.3 *Let Λ be a subgroup of Γ and $\mathcal{O} = \{\lambda D | \lambda \in \Lambda\}$ the Λ-orbit of D.*

(i) Suppose Ω is an index n subgroup of Λ. Let $\{\lambda_1', \lambda_2', \ldots, \lambda_n'\}$ be a complete set of left coset representatives of Ω in Λ and $\{c_1, c_2, \ldots, c_n\}$ a set of n colors. Then the assignment $\lambda_i'\Omega D \mapsto c_i$ defines an n-coloring of $\mathcal{O}$ which is Λ-transitive.

(ii) In a Λ-transitive n-coloring of $\mathcal{O}$, the elements of Λ which fix a specific color in the colored set $\mathcal{O}$ form an index n subgroup of Λ.

Remark: Note that given a subgroup Ω of Λ of index n and a set of n colors $\{c_1, c_2, \ldots, c_n\}$, there corresponds $(n-1)!$ Λ-transitive n-colorings of $\mathcal{O}$ with Ω fixing c_1. In a Λ-transitive n-coloring of $\mathcal{O}$ with Ω fixing c_1, the set of Coxeter polytopes ΩD is assigned c_1 and the remaining $n-1$ colors are distributed among the $\lambda_i'\Omega D$, $\lambda_i' \in \Lambda$.

4.2 Index n Subgroups of Coxeter Groups and their Subgroups

Let Λ be a subgroup of the Coxeter group Γ where Λ is generated by $\lambda_1, \lambda_2, \ldots, \lambda_l$ and $\mathcal{O}$ the Λ-orbit of D. In this section, Theorem 4.3 is applied to formulate a method for determining the index n subgroups of Λ distinct up to conjugacy in Λ. In obtaining these subgroups, we construct all Λ-transitive n-colorings of $\mathcal{O}$ using the set $C = \{c_1, c_2, \ldots, c_n\}$ of colors where all elements of Λ effect permutations of C and Λ acts transitively on C. Denote the colors $c_1, c_2, \ldots, c_n$ respectively by $1, 2, \ldots, n$. For each such coloring of $\mathcal{O}$, a homomorphism $\pi : \Lambda \to S_n$ is defined where for each $\lambda \in \Lambda$, $\pi(\lambda)$ is the permutation of the colors in $\mathcal{O}$ effected by λ. Since $\Lambda = \langle \lambda_1, \lambda_2, \cdots, \lambda_l \rangle$, π is completely determined when $\pi(\lambda_1), \pi(\lambda_2), \ldots, \pi(\lambda_l)$ are specified. To construct the coloring, a fundamental region D of the tessellation $\mathcal{T}$ is considered and assigned color c_1.

We come up with a set $\mathbb{P}$ of permutation assignments to $\lambda_1, \lambda_2, \cdots, \lambda_l$ corresponding to the Λ-transitive n-colorings of $\mathcal{O}$ that will give rise to the index n subgroups of Λ distinct up to conjugacy in Λ. An element τ of the set $\mathbb{P}$ is such that $\tau = \{\hat{\tau}_1, \ldots, \hat{\tau}_l\}$ where $\hat{\tau}_i = \pi(\lambda_i), i = 1, \ldots, l$.

To arrive at the set $\mathbb{P}$ aided by the software *GAP4* (Groups, Algorithms, Programming) [34], the process consists of the following steps:

(1) We refer to [15] for a list $\mathbb{L}$ of transitive subgroups of S_n. From $\mathbb{L}$, the set $\mathbb{T} = \{\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_k\}$ is formed consisting of groups in $\mathbb{L}$ that can be

generated by elements of order $\vartheta_1, \vartheta_2, \ldots, \vartheta_r$ or their factors where ϑ_i is the order of λ_i, $i = 1, \ldots, r$. In particular, if $\Lambda = \Gamma$, a *GAP4* routine called *TranSub* (Appendix A) is used to help determine the list $\mathbb{T}$ consisting of groups in $\mathbb{L}$ that can be generated by order 2 elements.

(2) For each of the groups $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_k \in \mathbb{T}$, the *GAP4* routine *Coloring* (Appendix B) determines respectively the sets $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_k$. For $\mathbb{T}_j \in \mathbb{T}$, $j = 1, \ldots, k$, the set $\mathbb{P}_j$ consists of permutation assignments to $\lambda_1, \ldots, \lambda_l$ such that $\pi(\Lambda) \cong \mathbb{T}_j$.

(3) Now by the remark in the previous section, corresponding to an index n subgroup Ω of Λ, there are $(n-1)!$ Λ-transitive n-colorings of $\mathcal{O}$ with Ω fixing c_1. That is, $(n-1)!$ permutation assignments will yield the same subgroup. The *GAP4* routine *Coloring* eliminates in each $\mathbb{P}_j$, $j = 1, \ldots, k$, permutation assignments that will yield the same subgroup. To arrive at a list of possible permutation assignments in each $\mathbb{P}_j$ that will yield distinct subgroups of Λ, we consider the set $\mathbb{J} = \{\mathrm{j} \in S_n | \mathrm{j}(1) = 1\}$. Note that for a particular permutation assignment $\tau \in \mathbb{P}$, $\mathrm{j}\tau\mathrm{j}^{-1}$ will yield the same subgroup for all $\mathrm{j} \in \mathbb{J}$.

(4) After step (3), our sets $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_k$ now consist of possible permutation assignments that will yield distinct subgroups of Λ. The next step would be to narrow down the elements of each set $\mathbb{P}_j$, $j = 1, \ldots, k$ to arrive at permu-

tation assignments that will possibly give rise to subgroups of Λ distinct up to conjugacy in Λ. Still using the *GAP4* routine *Coloring*, a set $\mathbb{P}'_j$ from each $\mathbb{P}_j$ is obtained consisting of these permutation assignments. Then the set $\mathbb{P}' = \cup\mathbb{P}'_j$ is obtained.

(5) As a last step, we use the *GAP4* routine *Sieve* (Appendix C). This routine checks each entry τ' in $\mathbb{P}'$ to ensure that it satisfies the divisibility conditions relative to the defining relations of Λ. For example, when $\Lambda = \Gamma$, we check whether for each τ' in $\mathbb{P}'$ the order of the permutation $\pi(\lambda_i\lambda_j)$ is a divisor of the order of $\lambda_i\lambda_j$, $i, j = 1, \ldots, l$. All permutation assignments satisfying the order conditions form the set $\mathbb{P}$.

The diagram in Figure 4.1 summarizes the process discussed above for constructing the colorings that will give rise to the index n subgroups of Λ distinct up to conjugacy in Λ.

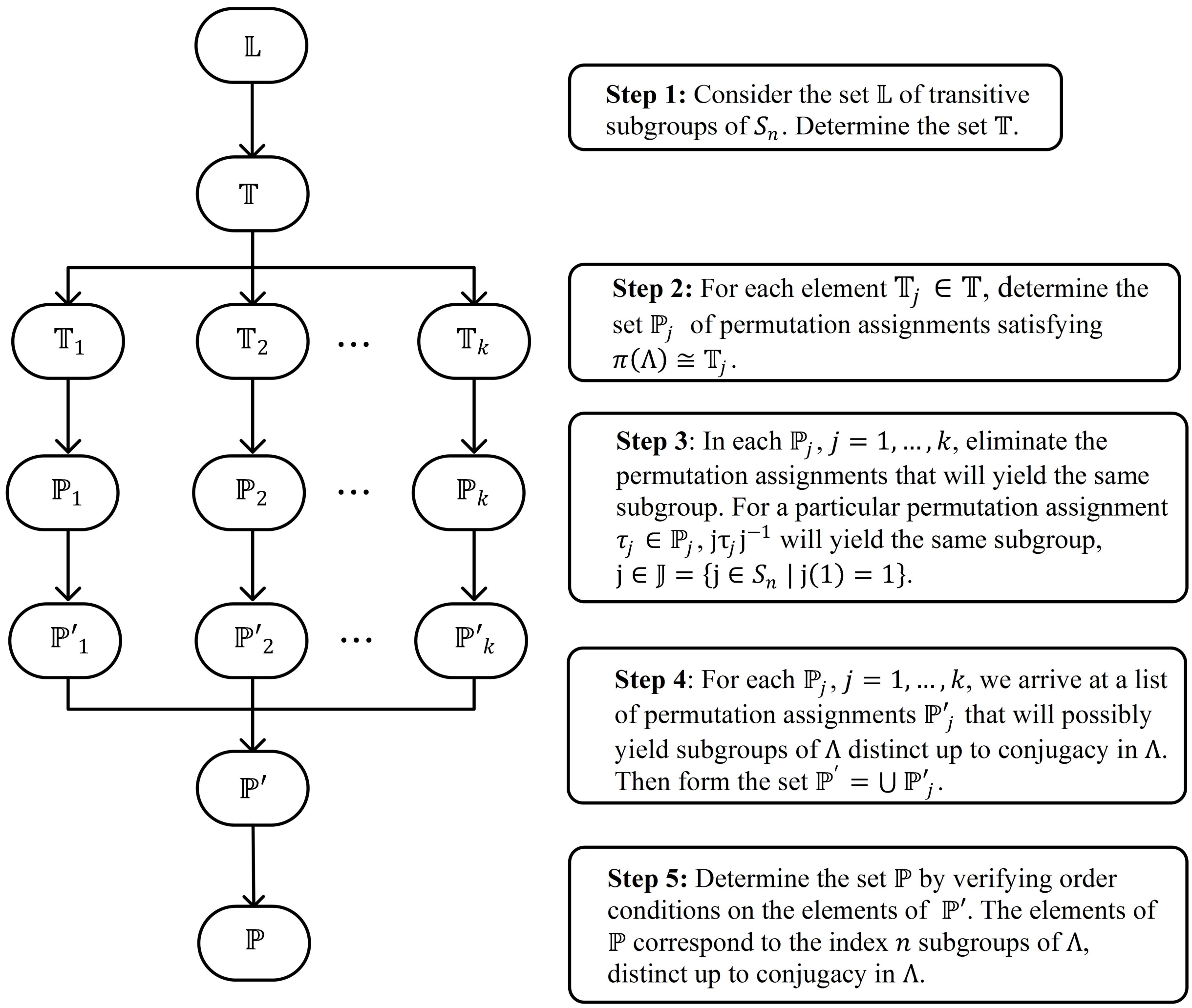


Figure 4.1: Outline of steps in constructing Λ-transitive n-colorings of $\mathcal{O}$.

4.3 Some Examples on Index 5 Subgroups of Coxeter Groups

To illustrate the process of obtaining the subgroups of a Coxeter group, let us determine the index 5 subgroups of a Coxeter group generated by five reflections. Consider

$$\Gamma^{(5)} = \Gamma(h_{21}, h_{31}, h_{41}, h_{51}, h_{23}, h_{24}, h_{25}, h_{34}, h_{35}, h_{45}) =$$
$$\left\langle R_1, R_2, R_3, R_4, R_5 \;\middle|\; \begin{array}{c} {R_1}^2 = {R_2}^2 = {R_3}^2 = {R_4}^2 = {R_5}^2 = (R_2R_1)^{h_{21}} = (R_3R_1)^{h_{31}} \\ = (R_4R_1)^{h_{41}} = (R_5R_1)^{h_{51}} = (R_2R_3)^{h_{23}} = (R_2R_4)^{h_{24}} = \\ (R_2R_5)^{h_{25}} = (R_3R_4)^{h_{34}} = (R_3R_5)^{h_{35}} = (R_4R_5)^{h_{45}} = e \end{array} \right\rangle.$$

The first step is to obtain the set $\mathbb{T}$ consisting of the transitive subgroups of S_5 that may be generated by order 2 elements. The set $\mathbb{L}$ consists of the following transitive subgroups of S_5; namely, S_5 itself, the cyclic group C_5, the alternating group A_5, the dihedral group D_5, and the group F_5, a group of order 20 isomorphic to $AGL_1(5)$ a 1-dimensional affine group over a field of order 5. Note that C_5 and F_5 cannot be generated by elements of order 2. Thus, $\mathbb{T} = \{\mathbb{T}_1 = S_5, \mathbb{T}_2 = A_5, \mathbb{T}_3 = D_5\}$. (See Table 4.1)

Next, using the $GAP4$ routine $Coloring$, we determine the sets $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ consisting of permutation assignments to P, Q, R, S and T such that $\pi(\Gamma^{(5)}) \cong S_5, A_5$ and D_5, respectively. We obtain a total of 10068840, 988440, and 45720 permutation assignments in $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively.

	Transitive subgroups of S_5	Order	Generators
$\mathbb{T}_1$	S_5	120	$\{(45), (34), (13)(24)\}$
$\mathbb{T}_2$	A_5	60	$\{(23)(45), (24)(35), (12)(45)\}$
$\mathbb{T}_3$	D_5	10	$\{(23)(45), (12)(34)\}$

Table 4.1: The elements of $\mathbb{T}$ in the routine to arrive at index 5 subgroups of $\Gamma^{(5)}$. For each group, a set of generators is given.

In this example, $\mathbb{J} = \{(25), (45), (35), (23), (34), (24), (23)(45), (253), (254), (345), (354), (234), (24)(35), (235), (243), (245), (2354), (2453), (2435), (2543), (2534), (2345), (25)(34)\}$. There are $(5-1)! = 24$ $\Gamma^{(5)}$-transitive 5-colorings of $\mathcal{T}$ giving rise to an index 5 subgroup of $\Gamma^{(5)}$; that is, there are 24 permutation assignments that will yield the same index 5 subgroup of $\Gamma^{(5)}$. After the *GAP4* routine *Coloring* eliminates the permutation assignments that will yield the same subgroups, there are a total of 419535, 41185, and 1905 permutation assignments in $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively.

The next step will be to arrive at the sets $\mathbb{P}_1'$, $\mathbb{P}_2'$ and $\mathbb{P}_3'$ consisting of permutation assignments that will possibly yield subgroups distinct up to conjugacy in $\Gamma^{(5)}$.

Let us first consider the set $\mathbb{P}_1$ consisting of permutation assignments such that $\pi(\Gamma^{(5)}) \cong S_5$. Suppose $\tau_1 \in \mathbb{P}_1$ gives rise to an index 5 subgroup Ω_1 of $\Gamma^{(5)}$.

Then the permutation assignments

$$(12345)\tau_1(12345)^{-1}, (13524)\tau_1(13524)^{-1},$$

$$(14253)\tau_1(14253)^{-1}, (15432)\tau_1(15432)^{-1}$$

will yield conjugate subgroups of Ω_1 in $\Gamma^{(5)}$. Hence there are $10068840 \div (5 \times 24) = 83907$ possible distinct subgroups of index 5 in $\Gamma^{(5)}$ up to conjugacy.

Next, in the set $\mathbb{P}_2$ consisting of permutation assignments such that $\pi(\Gamma^{(5)}) \cong A_5$, assume $\tau_2 \in \mathbb{P}_2$ gives rise to an index 5 subgroup Ω_2 of $\Gamma^{(5)}$. The permutation assignments

$$(14253)\tau_2(14253)^{-1}, (13524)\tau_2(13524)^{-1},$$

$$(12345)\tau_2(12345)^{-1}, (15432)\tau_2(15432)^{-1}$$

yield conjugate subgroups of Ω_2 in $\Gamma^{(5)}$ and there are $988440 \div (5 \times 24) = 8237$ possible distinct subgroups of index 5 in $\Gamma^{(5)}$ up to conjugacy.

Finally, in the set $\mathbb{P}_3$ consisting of permutation assignments such that $\pi(\Gamma^{(5)}) \cong D_5$, consider $\tau_3 \in \mathbb{P}_3$ which gives rise to an index 5 subgroup Ω_3 of $\Gamma^{(5)}$. The permutation assignments

$$(13542)\tau_3(13542)^{-1}, (15234)\tau_3(15234)^{-1},$$

$$(14325)\tau_3(14325)^{-1}, (12453)\tau_3(12453)^{-1}$$

yield conjugate subgroups of Ω_3 in $\Gamma^{(5)}$. Thus, there are $45720 \div (5 \times 24) = 381$ possible distinct subgroups of index 5 in $\Gamma^{(5)}$ up to conjugacy.

Hence, $\mathbb{P}'$ has $83907 + 8237 + 381 = 93525$ elements. This implies that if h_{21}, h_{31}, h_{41}, h_{51}, h_{23}, h_{24}, h_{25}, h_{34}, h_{35}, h_{45} are all divisible by 2, 3, 4, and 5, then $\Gamma^{(5)} = \Gamma(h_{21}, h_{31}, h_{41}, h_{51}, h_{23}, h_{24}, h_{25}, h_{34}, h_{35}, h_{45})$ has 93525 index 5 subgroups; 83907 subgroups such that $\pi(\Gamma^{(5)}) \cong S_5$, 8237 such that $\pi(\Gamma^{(5)}) \cong A_5$ and 381 such that $\pi(\Gamma^{(5)}) \cong D_5$. Let us state this result formally in the following theorem.

Theorem 4.4 *The Coxeter group* $\Gamma(h_{21}, h_{31}, h_{41}, h_{51}, h_{23}, h_{24}, h_{25}, h_{34}, h_{35}, h_{45})$ *generated by five reflections* R_1*,* R_2*,* R_3*,* R_4*,* R_5 *with defining relations*

$$R_1{}^2 = R_2{}^2 = R_3{}^2 = R_4{}^2 = R_5{}^2 = (R_2R_1)^{h_{21}} = (R_3R_1)^{h_{31}} = (R_4R_1)^{h_{41}} = (R_5R_1)^{h_{51}}$$

$$= (R_2R_3)^{h_{23}} = (R_2R_4)^{h_{24}} = (R_2R_5)^{h_{25}} = (R_3R_4)^{h_{34}} = (R_3R_5)^{h_{35}} = (R_4R_5)^{h_{45}} = e$$

has 93525 *index 5 subgroups if* h_{21}*,* h_{31}*,* h_{41}*,* h_{51}*,* h_{23}*,* h_{24}*,* h_{25}*,* h_{34}*,* h_{35}*,* h_{45} *are all divisible by 2, 3, 4, and 5.*

We now look at two special cases of $\Gamma(h_{21}, h_{31}, h_{41}, h_{51}, h_{23}, h_{24}, h_{25}, h_{34}, h_{35}, h_{45})$. In the first case, we consider hyperbolic Coxeter groups having compact and noncompact hyperbolic Coxeter simplices in $\mathbb{H}^4$ as fundamental polytopes. In the second case, we consider examples of hyperbolic Coxeter groups with singly truncated tetrahedra as fundamental polytopes.

We first determine the index 5 subgroups of the Coxeter group

$$\Gamma(3,2,2,2,3,2,2,3,4,2) = \left\langle R_1, R_2, R_3, R_4, R_5 \;\middle|\; \begin{array}{c} {R_1}^2 = {R_2}^2 = {R_3}^2 = {R_4}^2 = {R_5}^2 = (R_2R_1)^3 = (R_3R_1)^2 \\ = (R_4R_1)^2 = (R_5R_1)^2 = (R_2R_3)^3 = (R_2R_4)^2 = \\ (R_2R_5)^2 = (R_3R_4)^3 = (R_3R_5)^4 = (R_4R_5)^2 = e \end{array} \right\rangle.$$

This group has a noncompact Coxeter simplex in $\mathbb{H}^4$ as fundamental polytope.

Given the set $\mathbb{P}'$ consisting of 93525 possible index 5 subgroups of $\Gamma(3,2,2,2,3,2,2,3,4,2)$, we use the *GAP4* routine *Sieve* to obtain the set $\mathbb{P}$.

We obtain the permutation assignment $\tau = \{(45),(34),(23),(12),(1)\} \in \mathbb{P}$. In this case $\pi(R_1) = (45)$, $\pi(R_2) = (34)$, $\pi(R_3) = (23)$, $\pi(R_4) = (12)$, $\pi(R_5) = (1)$, which implies $\pi(R_2R_1) = (354)$, $\pi(R_3R_1) = (23)(45)$, $\pi(R_4R_1) = (12)(45)$, $\pi(R_5R_1) = (45)$, $\pi(R_2R_3) = (234)$, $\pi(R_2R_4) = (12)(34)$, $\pi(R_2R_5) = (34)$,$\pi(R_3R_4) = (123)$, $\pi(R_3R_5) = (23)$, and $\pi(R_4R_5) = (12)$. Note that order 2 elements of S_5 are assigned to the images of R_1, R_2, R_3, R_4, R_5 under π. Moreover, observe that the order of $\pi(R_2R_1)$ is 3, $\pi(R_3R_1)$ is 2, $\pi(R_4R_1)$ is 2, $\pi(R_5R_1)$ is 2, $\pi(R_2R_3)$ is 3, $\pi(R_2R_4)$ is 2, $\pi(R_2R_5)$ is 2 , $\pi(R_3R_4)$ is 3, $\pi(R_3R_5))$ is 2 and $\pi(R_4R_5)$ is 2 which, respectively, divides the orders of R_2R_1, R_3R_1, R_4R_1, R_5R_1, R_2R_3, R_2R_4, R_2R_5, R_3R_4, R_3R_5 and R_4R_5. Thus, the permutation assignment τ corresponds to an index 5 subgroup of $\Gamma(3,2,2,2,3,2,2,3,4,2)$. In fact this is its only index 5 subgroup.

From our calculations, the only other hyperbolic Coxeter group (with a simplex in $\mathbb{H}^4$ as fundamental polytope) having an index 5 subgroup is $\Gamma(3,2,2,2,$

$3,2,3,3,2,3)$. This index 5 subgroup corresponds to the permutation assignment $\{(45),(34),(23),(12),(23)\}$. This result has also been obtained using the *GAP4* routine *Sieve* on the set of 93525 permutation assignments $\mathbb{P}'$.

Truncated Tetrahedron Groups	$\pi(\Gamma_{i,u}(v))$	$\pi(R_1)$	$\pi(R_2)$	$\pi(R_3)$	$\pi(R_4)$	$\pi(R_5)$
$\Gamma_{1,6}(6) = \Gamma(3,2,2,0,3,2,2,6,2,2)$	S_5	(45)	(34)	(23)	(1)	(15)
	S_5	(45)	(34)	(23)	(12)	(1)
$\Gamma_{2,4}(5) = \Gamma(3,2,2,0,4,2,2,5,2,2)$	A_5	(1)	(1)	(23)(45)	(12)(34)	(1)
	S_5	(23)(45)	(12)(45)	(1)	(1)	(14)(25)
$\Gamma_{3,3}(5) = \Gamma(3,2,2,0,3,3,2,5,2,2)$	S_5	(45)	(34)	(23)	(23)	(15)
$\Gamma_{4,6}(6) = \Gamma(4,2,2,0,3,2,2,6,2,2)$	D_5	(23)(45)	(1)	(1)	(1)	(12)(34)
	S_5	(23)(45)	(34)	(45)	(1)	(12)
$\Gamma_{5,2}(5) = \Gamma(3,2,2,0,4,4,2,5,2,2)$	D_5	(1)	(1)	(23)(45)	(12)(34)	(1)
	S_5	(23)(45)	(12)(45)	(1)	(1)	(14)(25)
$\Gamma_{6,6}(6) = \Gamma(5,2,2,0,3,2,2,6,2,2)$	A_5	(23)(45)	(12)(34)	(25)(34)	(1)	(1)
	S_5	(23)(45)	(12)(34)	(25)(34)	(1)	(34)
$\Gamma_{7,3}(5) = \Gamma(4,2,2,0,3,3,2,5,2,2)$	D_5	(23)(45)	(1)	(1)	(1)	(12)(34)
	A_5	(1)	(23)(45)	(12)(45)	(14)(23)	(1)
	S_5	(23)(45)	(34)	(45)	(45)	(12)
$\Gamma_{8,4}(4) = \Gamma(3,2,3,0,4,2,2,4,2,2)$	A_5	(23)(45)	(12)(45)	(1)	(12)(45)	(14)(25)
$\Gamma_{9,3}(5) = \Gamma(5,2,2,0,3,3,2,5,2,2)$	A_5	(23)(45)	(12)(34)	(25)(34)	(25)(34)	(1)
	S_5	(23)(45)	(12)(34)	(25)(34)	(25)(34)	(34)

Table 4.2: The permutation assignments corresponding to the index 5 subgroups of the truncated Coxeter tetrahedron groups $\Gamma_{i,v}(v)$, $i = 1, 2, \ldots, 9$, listed in Table 3.5.

For the next case, we look at the singly truncated tetrahedron groups given in Table 3.5. Using the *GAP4* routine *Sieve*, the permutation assignments that give rise to index 5 subgroups of each of the groups are given in Table 4.2. From the set of 93525 permutation assignments in $\mathbb{P}'$, we arrive at two permutation assignments that give rise to index 5 subgroups of the groups $\Gamma_{1,6}(6)$, $\Gamma_{2,4}(5)$, $\Gamma_{4,6}(6)$, $\Gamma_{5,2}(5)$, $\Gamma_{6,6}(6)$ and $\Gamma_{9,3}(5)$, while we obtain one permutation assignment for the groups $\Gamma_{8,4}(4)$ and $\Gamma_{3,3}(5)$, and three permutation assignments for the group $\Gamma_{7,3}(5)$.

4.4 Index 6 Subgroups of $\Gamma^0(r', q', p')$

Let us now explain the process involved in obtaining index 6 subgroups of the orientation preserving subgroup $\Gamma^0(r', q', p')$ of a triangle group $\Gamma(r', q', p')$. In particular, we are going to derive the index 6 subgroups of the modular group 32∞.

Our first step is to obtain the set $\mathbb{L}$ consisting of the transitive subgroups of S_6. From [15], the set $\mathbb{L}$ consists of 16 groups: the symmetric group S_6, the dihedral group D_6, the alternating group A_6, two groups of order 6, C_6 and $S_3(6)$ (isomorphic to Z_6 and S_3, respectively), the order 12 group $A_4(6)$ (isomorphic to $A_6 \cap S_4$), three order 24 groups $2A_4(6)$, $S_4(6d)$ and $S_4(6c)$ (isomorphic to $C_2\ wr_3\ C_3$, $A_6 \cap (S_2\ wr_3\ S_3)$ and S_4, respectively), the order 18 group $F_{18}(6)$ (isomorphic to $C_3\ wr_2\ C_2$), two order 36 groups $F_{18}(6) : 2$ and $F_{36}(6)$ (isomorphic to $A_6 \cap (S_3\ wr_2\ C_2)$), the order 48 group $2S_4(6)$ (isomorphic to $S_2\ wr_3\ S_3$), the order 60 group $A_5(6)$ (isomorphic to $PSL_2(5)$, a 2-dimensional projective special linear group over a field of order 5), the order 72 group $F_{36}(6) : 2$ (isomorphic to $S_3\ wr_2\ S_3$) and an order 120 group $S_5(6)$ (isomorphic to $PGL_2(5)$, a 2-dimensional projective general linear group over a field of order 5). All of these subgroups may be generated by a permutation of order 2 and another one of any order. Hence, the collection $\mathbb{T}$ of transitive subgroups that we will use in the computation consists of all 16 transitive subgroups of S_6. We let $\mathbb{T}_1 = C_6$, $\mathbb{T}_2 = S_3(6)$, $\mathbb{T}_3 = D_6$, $\mathbb{T}_4 = A_4(6)$, $\mathbb{T}_5 = F_{18}(6)$, $\mathbb{T}_6 = 2A_4(6)$, $\mathbb{T}_7 = S_4(6d)$, $\mathbb{T}_8 = S_4(6c)$, $\mathbb{T}_9 = F_{18}(6) : 2$, $\mathbb{T}_{10} = F_{36}(6)$, $\mathbb{T}_{11} = 2S_4(6)$, $\mathbb{T}_{12} = A_5(6)$ $\mathbb{T}_{13} = F_{36}(6) : 2$, $\mathbb{T}_{14} = S_5(6)$, $\mathbb{T}_{15} = A_6$, and $\mathbb{T}_{16} = S_6$. (See Table 4.3 for the generators of these subgroups.)

Using the *GAP* routine *Coloring*, we determine the sets $\mathbb{P}_1$, $\mathbb{P}_2, \ldots,$ $\mathbb{P}_{16}$ where each $\mathbb{P}_j$ consists of permutation assignments to $Q'P'$ and $R'P'$ such that $\pi(\Gamma^0(r', q', p')) \cong \mathbb{T}_j$, for $j = 1, \ldots, 16$. The number of permutation assignments in each set $\mathbb{P}_j$ is also given in Table 4.3.

There are $(6-1)! = 120$ permutation assignments that will yield the same subgroup of $\Gamma^0(r', q', p')$. (The elements of the set $\mathbb{J}$ are listed in Table 4.4.) We use the *GAP4* routine *Coloring* to eliminate the permutation assignments which will yield the same subgroup.

The next step is to arrive at the sets $\mathbb{P}'_1$, $\mathbb{P}'_2, \ldots,$ $\mathbb{P}'_{16}$ consisting of permutation assignments that will possibly yield subgroups distinct up to conjugacy in $\Gamma^0(r', q', p')$ still with the aid of *GAP4*. For each set $\mathbb{P}_j$, $j = 1, \ldots, 16$, we consider a set $\mathbb{J}'_j$ consisting of *conjugating elements*. Let $\mathrm{j} \in \mathbb{J}'_j$. If $\tau' \in \mathbb{P}_j$ is a permutation assignment that gives rise to a subgroup Ω' of $\Gamma(r', q', p')$, then $\mathrm{j}\tau'\mathrm{j}^{-1}$ is a permutation assignment that gives rise to a conjugate subgroup of Ω'. The sets of conjugating elements to be used in this step are listed in Table 4.5.

We also give in Table 4.5 the number of permutation assignments in each $\mathbb{P}'_j$ which will possibly give rise to the subgroups of $\Gamma^0(r', q', p')$ distinct up to conjugacy in $\Gamma^0(r', q', p')$. We obtain the set $\mathbb{P}'_1$ with 12 permutation assignments, $\mathbb{P}'_2$ with 3, $\mathbb{P}'_3$ with 6, $\mathbb{P}'_4$ with 4, $\mathbb{P}'_5$ with 12, $\mathbb{P}'_6$ with 12, $\mathbb{P}'_7$ with 9, $\mathbb{P}'_8$ with 9, $\mathbb{P}'_9$ with 3, $\mathbb{P}'_{10}$ with 12, $\mathbb{P}'_{11}$ with 18, $\mathbb{P}'_{12}$ with 19, $\mathbb{P}'_{13}$ with 24, $\mathbb{P}'_{14}$ with 57, $\mathbb{P}'_{15}$ with 106, and $\mathbb{P}'_{16}$ with 381. Thus, $\mathbb{P}'$ has a total of 675 elements. This gives us the following theorem for the index 6 subgroups of the orientation preserving subgroup $\Gamma^0(r', q', p')$ of the triangle group $\Gamma(r', q', p')$.

Theorem 4.5 *The group $\Gamma^0(r', q', p')$ generated by $Q'P'$, $R'P'$ with defining relation*

$$(Q'P')^{r'} = (R'P')^{q'} = (Q'R')^{p'} = e$$

has 675 index 6 subgroups if r', q', p' are all divisible by 2, 3, 4, 5 and 6.

Finally, we now proceed to obtain the index 6 subgroups of the modular group $\Gamma^0(\infty, 2, 3)$. Using the *GAP4* routine *Sieve*, we verify the order conditions. Specifically, we collect the permutation assignments where the order of $\pi(R'P')$ divides 2 and the order of $\pi(Q'R')$ divides 3. Of the 675 permutation assignments in $\mathbb{P}'$, we have 8 such permutation assignments and they are such that $\pi(\Gamma^0(r', q', p'))$ is isomorphic to either C_6, S_3, $A_4(6)$, $F_{18}(6)$, $2A_4(6)$, $S_4(6c)$, $S_4(6d)$ or $A_5(6)$. Hence, we have 8 permutation assignments in $\mathbb{P}$ that give rise to index 6 subgroups of the modular group $\Gamma^0(3, 2, \infty)$ which we present in Table 4.6.

	Transitive subgroups of S_6	Order	Generators	Cardinality of set $\mathbb{P}_j$
$\mathbb{T}_1$	C_6	6	$\{(123456)\}$	1440
$\mathbb{T}_2$	$S_3(6)$	6	$\{(135)(246),(16)(25)(34)\}$	360
$\mathbb{T}_3$	D_6	12	$\{(123456),(16)(25)(34)\}$	2160
$\mathbb{T}_4$	$A_4(6)$	12	$\{(25)(36),(135)(246)\}$	1440
$\mathbb{T}_5$	$F_{18}(6)$	18	$\{(135),(14)(25)(36)\}$	2880
$\mathbb{T}_6$	$2A_4(6)$	24	$\{(14),(135)(246)\}$	4320
$\mathbb{T}_7$	$S_4(6d)$	24	$\{(25)(36),(135)(246),(26)(35)\}$	3240
$\mathbb{T}_8$	$S_4(6c)$	24	$\{(25)(36),(135)(246),(14)(26)(35)\}$	3240
$\mathbb{T}_9$	$F_{18}(6):2$	36	$\{(135),(26)(35),(14)(25)(36)\}$	2160
$\mathbb{T}_{10}$	$F_{36}(6)$	36	$\{(135),(26)(35),(14)(2563)\}$	8640
$\mathbb{T}_{11}$	$2S_4(6)$	48	$\{(14),(135)(246),(26)(35)\}$	6480
$\mathbb{T}_{12}$	$A_5(6)$	60	$\{(12345),(16)(25)\}$	13680
$\mathbb{T}_{13}$	$F_{36}(6):2$	72	$\{(135),(35),(14)(25)(36)\}$	17280
$\mathbb{T}_{14}$	$S_5(6)$	120	$\{(12345),(16)(23)(45)\}$	41040
$\mathbb{T}_{15}$	A_6	360	$\{(123),(134),(145),(156)\}$	76320
$\mathbb{T}_{16}$	S_6	720	$\{(12),(13),(14),(15),(16)\}$	228960

Table 4.3: Column 1-2. The elements of $\mathbb{T}$ in the routine to arrive at index 6 subgroups of $\Gamma^0(r', q', p')$; Column 3. A set of generators for each group; Column 4. The number of elements in $\mathbb{P}_j$.

$$\{(56), (45), (456), (465), (46), (34), (34)(56), (345), (3456), (3465), (346), (354), (3564),$$
$$(35)(46), (3546), (3654), (364), (365), (36), (3645), (36)(45), (23), (23)(56), (23)(45),$$
$$(23)(46), (234), (234)(56), (2345), (23456), (23465), (2346), (2354), (23564), (235), (263),$$
$$(23546), (23654), (2364), (2365), (236), (23645), (236)(45), (243), (243)(56), (2453), (24563),$$
$$(2463), (24), (24)(56), (245), (2456), (2465), (246), (24)(35), (24)(356), (2435), (24356),$$
$$(246)(35), (24)(365), (24)(36), (24365), (2436), (26)(35), (245)(36), (24536), (2543), (25643),$$
$$(253)(46), (25463), (254), (2564), (25), (256), (25)(46), (2546), (2534), (25634), (25)(34),$$
$$(25346), (25364), (254)(36), (25)(364), (25436), (25)(36), (2536), (26543), (2643), (2653),$$
$$(26453), (263)(45), (2654), (264), (265), (26), (2645), (26)(45), (26534), (2634), (265)(34),$$
$$(26)(345), (264)(35), (26354), (26435), (26)(354), (35), (356), (23)(456), (23)(465), (26)(34)$$
$$(2356), (235)(46), (24653), (24635), (2635), (253), (2563), (25)(346), (26345), (256)(34)\}$$

Table 4.4: The set $\mathbb{J}$ in the routine to arrive at index 6 subgroups of $\Gamma^0(r', q', p')$

$\pi(\Gamma^0(r', q', p'))$	Order	The set $\mathbb{J}'_j$	No. of elements in $\mathbb{P}'_j$
C_6	**6**	$\mathbb{J}'_1 = \{\}$	$12 = 1440 \div (1 \times 120)$
$S_3(6)$	**6**	$\mathbb{J}'_2 = \{\}$	$3 = 360 \div (1 \times 120)$
D_6	**12**	$\mathbb{J}'_3 = \{(153)(264), (135)(246)\}$	$6 = 2160 \div (3 \times 120)$
$A_4(6)$	**12**	$\mathbb{J}'_4 = \{(153)(264), (135)(246)\}$	$4 = 1440 \div (3 \times 120)$
$F_{18}(6)$	**18**	$\mathbb{J}'_5 = \{(14)(25)(36)\}$	$12 = 2880 \div (2 \times 120)$
$2A_4(6)$	**24**	$\mathbb{J}'_6 = \{(135)(246), (153)(264)\}$	$12 = 4320 \div (3 \times 120)$
$S_4(6d)$	**24**	$\mathbb{J}'_7 = \{(153)(264), (135)(246)\}$	$9 = 3240 \div (3 \times 120)$
$S_4(6c)$	**24**	$\mathbb{J}'_8 = \{(153)(264), (135)(246)\}$	$9 = 3240 \div (3 \times 120)$
$F_{18}(6) : 2$	**36**	$\mathbb{J}'_9 = \{(163452), (135)(264),$ $(14)(23)(56), (153)(246), (125436)\}$	$3 = 2160 \div (6 \times 120)$
$F_{36}(6)$	**36**	$\mathbb{J}'_{10} = \{(1452)(36), (15)(24),$ $(1254)(36), (1654)(23), (13)(26)\}$	$12 = 8640 \div (6 \times 120)$
$2S_4(6)$	**48**	$\mathbb{J}'_{11} = \{(135)(246), (153)(264)\}$	$18 = 6480 \div (3 \times 120)$
$A_5(6)$	**60**	$\mathbb{J}'_{12} = \{(156342), (164)(253),$ $(13)(26)(45), (146)(235), (124365)\}$	$19 = 13680 \div (6 \times 120)$
$F_{36}(6) : 2$	**72**	$\mathbb{J}'_{13} = \{(152436), (123)(465),$ $(14)(26)(35), (132)(456), (163425)\}$	$24 = 17280 \div (6 \times 120)$
$S_5(6)$	**120**	$\mathbb{J}'_{14} = \{(123564), (136)(254),$ $(15)(26)(34), (163)(245), (146532)\}$	$57 = 41040 \div (6 \times 120)$
A_6	**360**	$\mathbb{J}'_{15} = \{(164325), (142)(356),$ $(13)(26)(45), (124)(365), (152346)\}$	$106 = 76320 \div (6 \times 120)$
S_6	**720**	$\mathbb{J}'_{16} = \{(146325), (162)(354),$ $(13)(24)(56), (126)(345), (152364)\}$	$318 = 228960 \div (6 \times 120)$

Table 4.5: Column 1-2. The elements of $\mathbb{T}$ in the routine to arrive at index 6 subgroups of $\Gamma^0(r', q', p')$; Column 3. The set $\mathbb{J}'_j$ of conjugating elements. Column 4. The number of elements in $\mathbb{P}'_j$.

$\pi(\Gamma^0(r', q', p'))$	$\pi(Q'P')$	$\pi(R'P')$	$\pi(Q'R')$
C_6	(123456)	(14)(25)(36)	(153)(264)
$S_3(6)$	(12)(34)(56)	(13)(25)(46)	(154)(236)
$A_4(6)$	(123)(456)	(25)(36)	(153)(264)
$F_{18}(6)$	(123456)	(12)(34)(56)	(246)
$2A_4(6)$	(12)(3546)	(13)(24)	(146)(235)
$S_4(6c)$	(2356)	(12)(36)(45)	(126)(345)
$S_4(6d)$	(3546)	(13)(24)(56)	(136)(245)
$A_5(6)$	(23564)	(12)(34)	(124)(356)

Table 4.6: The permutation assignments which give rise to the index 6 subgroups of $\Gamma^0(\infty, 2, 3)$.

Chapter 5

EXTRACTING THE GENERATORS OF THE SUBGROUPS

This chapter presents a method for obtaining a set of generators of the subgroups of Coxeter groups or the subgroups of subgroups of Coxeter groups. The process which we call the $GenTree(\Omega)$ construction starts with a permutation assignment corresponding to a subgroup Ω of Λ, and arrives at a generating set for Ω. We first discuss the $GenTree(\Omega)$ construction, then apply this to determine a set of generators for the subgroups derived in Sections 4.3 and 4.4.

5.1 Assumptions for the $GenTree(\Omega)$ Construction

In the previous section, we discussed a method for determining the index n subgroups of $\Lambda = \langle \lambda_1, \lambda_2, \ldots, \lambda_l \rangle$, a subgroup of the Coxeter group Γ. The method produced a set $\mathbb{P}$ which consists of the permutation assignments corresponding to the Λ-transitive n-colorings of the Λ-orbit of D that will give rise to index n subgroups of Λ, distinct up to conjugacy in Λ. For such colorings, a homomorphism $\pi : \Lambda \to S_n$ is defined. Now, let $\tau = \{\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_l\} \in \mathbb{P}$, where $\hat{\tau}_i = \pi(\lambda_i)$, for $i = 1, 2, \ldots, l$. The goal of this chapter is to obtain a set of generators for a subgroup Ω of Λ corrresponding to τ in terms of the generators of Λ. To come up with this set, the method used involves constructing trees which consist of paths that represent generators of Ω. The collection of trees with such paths will be referred to as the $GenTree$ corresponding to Ω, denoted by $GenTree(\Omega)$.

In the discussions throughout this chapter, we assume the conditions and notations given in the previous paragraph.

5.2 Vertices, Edges, Paths and Trees in the $GenTree(\Omega)$ Construction

Let $\mathcal{S} = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ and $\mathcal{S}^{-1} = \{{\lambda_1}^{-1}, {\lambda_2}^{-1}, \ldots, {\lambda_l}^{-1}\}$. Define the *set of vertices* $\mathcal{V}$ as

$$\mathcal{V} = \left\{ (\mathsf{x}, \mathsf{y}) \;\middle|\; \begin{array}{c} \mathsf{x} \in \mathcal{S} \cup \mathcal{S}^{-1} \text{ and } \mathsf{y} = (\bar{\mathsf{y}}_1, \bar{\mathsf{y}}_2) \\ \text{where } \bar{\mathsf{y}}_2 = \pi(\mathsf{x})(\bar{\mathsf{y}}_1) \text{ for } \bar{\mathsf{y}}_1 \in \{1, \ldots, n\} \end{array} \right\}.$$

Given a vertex $\mathsf{v} = (\mathsf{x}, (\bar{\mathsf{y}}_1, \bar{\mathsf{y}}_2))$, x is referred to as the *generator of* v and $(\bar{\mathsf{y}}_1, \bar{\mathsf{y}}_2)$ as the *ordered pair* of v. We define the *inverse of the vertex* v, denoted by v^{-1}, as $(\mathsf{x}^{-1}, (\bar{\mathsf{y}}_2, \bar{\mathsf{y}}_1))$. Note that $\pi(\mathsf{x}^{-1})(\bar{\mathsf{y}}_2) = [\pi(\mathsf{x})]^{-1}(\bar{\mathsf{y}}_2) = \bar{\mathsf{y}}_1$.

Define the *set of roots* $\mathcal{W}$ as the set

$$\mathcal{W} = \{(\mathsf{x}_1, \mathsf{y}_1) \in \mathcal{V} \mid \mathsf{y}_1 = (1, \bar{\mathsf{y}}_2)\}.$$

An *edge* is defined by two vertices $(\mathsf{x}_1, (\bar{\mathsf{y}}_1, \bar{\mathsf{y}}_2))$ and $(\mathsf{x}_2, (\bar{\mathsf{y}}_2, \bar{\mathsf{y}}_3))$ provided that $\mathsf{x}_1\mathsf{x}_2 \neq e$ (identity in Γ) and $\pi(\mathsf{x}_1\mathsf{x}_2) \neq (1)$ (identity permutation).

For every root $\mathsf{w}_k \in \mathcal{W}$, a tree denoted by $\mathfrak{t}_k$ is constructed consisting of paths $\mathfrak{p}_{kj}$ with w_k as the *initial node*. A vertex in a path is considered a *terminal node* if its ordered pair is $(\bar{\mathsf{y}}_1, 1)$ for some $\bar{\mathsf{y}}_1 \in \{1, 2, \ldots, n\}$.

A path $\mathfrak{p}_{kj}$ is a sequence of vertices $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\mathsf{z}$ denoted by $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{z}$ where $\mathsf{v}_1 = (\mathsf{x}_1, (1, \bar{\mathsf{y}}_2))$ and $\mathsf{v}_\mathsf{z} = (\mathsf{x}_\mathsf{z}, (\bar{\mathsf{y}}_1, 1))$ for $\bar{\mathsf{y}}_1, \bar{\mathsf{y}}_2 \in \{2, 3, \ldots, n\}$. The inverse of the path $\mathfrak{p}_{kj}$ is denoted by ${\mathfrak{p}_{kj}}^{-1}$ and is defined as the sequence ${\mathsf{v}_\mathsf{z}}^{-1}, {\mathsf{v}_{\mathsf{z}-1}}^{-1}, \ldots, {\mathsf{v}_1}^{-1}$ and denoted by ${\mathsf{v}_\mathsf{z}}^{-1} \to {\mathsf{v}_{\mathsf{z}-1}}^{-1} \to \cdots \to {\mathsf{v}_1}^{-1}$. A subpath of $\mathfrak{p}_{kj}$ is a subsequence of the sequence $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\mathsf{z}$ of vertices in $\mathfrak{p}_{kj}$.

Note that a path is constructed such that $\pi(\mathsf{x}_1)\pi(\mathsf{x}_2)\cdots\pi(\mathsf{x}_\mathsf{z}) = 1$. In which case, we can think of a given path as corresponding to an element $\omega = \mathsf{x}_1\mathsf{x}_2\cdots\mathsf{x}_\mathsf{z} \in \Omega$.

To illustrate the above ideas, let us consider the following example.

Consider the regular hexagon consisting of copies of a triangle $\triangle$ shown in Figure 5.1 (a). The symmetry group G of the hexagon is the dihedral group D_6 of order 12 generated by the 60°-counterclockwise rotation a about the center of the hexagon and the reflection b with axis along the horizontal line through the center satisfying the relations $a^6 = b^2 = (ab)^2 = e$.

A G-transitive 3-coloring of the hexagon corresponding to an index 3 subgroup of G is shown in Figure 5.1 (b). Consider the permutation assignment $\tau_1 = \{(123), (13)\}$ corresponding to this coloring where a sends color 1 to 2, 2 to 3 and 3 to 1 while b interchanges colors 1 and 3 (colors 1, 2, 3 correspond respectively to white, green, yellow in Figure 5.1 (b)). If $\pi : G \to S_3$ is the homomorphism defined by the G-transitive coloring of the hexagon, we have $\pi(a) = (123)$ and $\pi(b) = (13)$. Let Ω_1 be the index 3 subgroup of G arising from τ_1. The set of vertices $\mathcal{V}_1$ in the construction of $GenTree(\Omega_1)$ is given by $\{(a,(1,2)), (a^{-1},(2,1)), (a,(2,3)), (a^{-1},(3,2)), (a,(3,1)), (a^{-1},(1,3)), (b,(1,3)), (b,(2,2)), (b,(3,1))\}$. Moreover, the set of roots $\mathcal{W}_1$ is $\{(a,(1,2)), (b,(1,3))\}$. The trees t_1 and t_2 are constructed with paths having as initial nodes the roots $\mathsf{w}_1 = (a,(1,2))$ and $\mathsf{w}_2 = (b,(1,3))$, respectively.

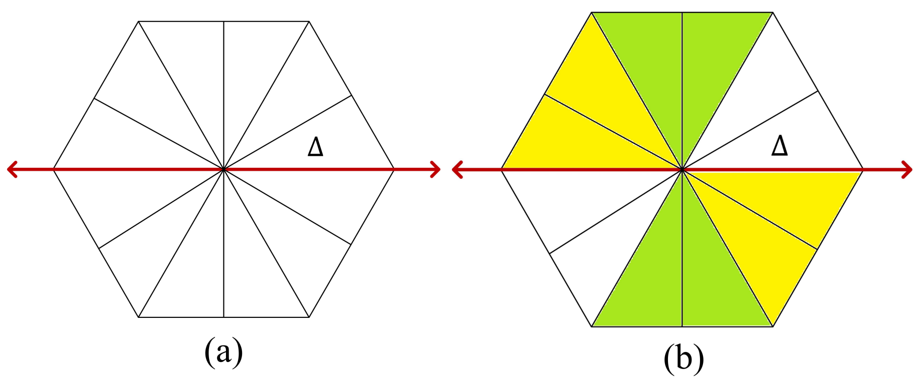


Figure 5.1: (a) A regular hexagon with symmetry group $G = D_6$; (b) A G-transitive 3-coloring of the hexagon arising from the permutation assignment $\tau_1 = \{(123), (13)\}$.

To branch out from a root w to a vertex: connect w to vertex $\mathsf{v} \in \mathcal{V}$ whenever $\mathsf{wv} \neq e$ and $\pi(\mathsf{wv}) \neq (1)$. Therefore, we connect $\mathsf{w}_1 = (a,(1,2))$ to the vertices $(a,(2,3))$ and $(b,(2,2))$. Similarly, we connect $\mathsf{w}_2 = (b,(1,3))$ to the vertices $(a,(3,1))$ and $(a^{-1},(3,2))$. Then, we branch out to the next set of vertices using the same criteria. We continue branching out until all the paths in t_1 and t_2 have reached terminal nodes. Both trees have five paths each as shown in Figure 5.2.

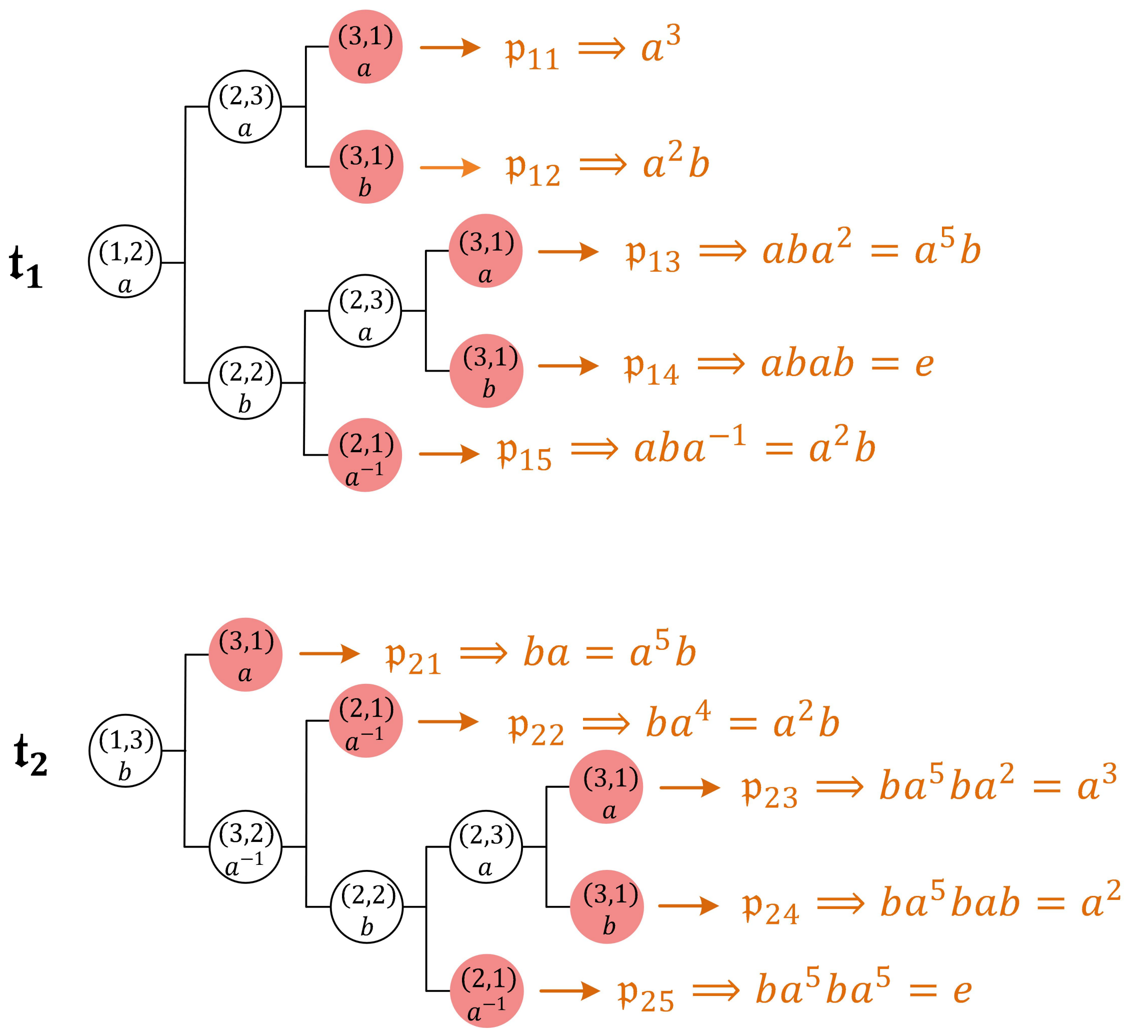


Figure 5.2: The trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ given the permutation assignment $\tau_1 = \{(1,2,3),(1,3)\}$ corresponding to the subgroup Ω_1 of $G = D_6$.

By the nature of the construction of the trees given the permutation assignment τ_1, we can verify using Figure 5.2 that we have considered all sequences $\mathsf{v}_1 = (\mathsf{x}_1,\mathsf{y}_1) \to \mathsf{v}_2 = (\mathsf{x}_2,\mathsf{y}_2) \to \cdots \to \mathsf{v}_z = (\mathsf{x}_z,\mathsf{y}_z)$ such that $\pi(\mathsf{x}_1)\pi(\mathsf{x}_2)\cdots\pi(\mathsf{x}_z) = (1)$ using the vertices of $\mathcal{V}_1$. This means that the paths in Figure 5.2 correspond to the elements of Ω_1. That is, $\Omega_1 = \{e, a^3, a^2b, a^5b\}$.

In the next section, we discuss the branching out rules of $GenTree(\Omega)$ which when applied will result in the construction of paths corresponding to a set of generators of Ω.

5.3 Branching Out Rules for $GenTree(\Omega)$

Before we present the branching out rules of $GenTree(\Omega)$, some definitions are in order.

Consider the paths $\mathfrak{p} : \mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{z}$ and $\mathfrak{p}' : \mathsf{v'}_1 \to \mathsf{v'}_2 \to \cdots \to \mathsf{v'}_{\mathsf{z'}}$ in the construction of $GenTree(\Omega)$. We define the *product* $\mathfrak{p} \times \mathfrak{p}'$ of the paths $\mathfrak{p}$ and $\mathfrak{p}'$ as the sequence $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\mathsf{z}, \mathsf{v'}_1, \mathsf{v'}_2, \ldots, \mathsf{v'}_{\mathsf{z'}}$, where $\mathfrak{p} \times \mathfrak{p}'$ is denoted by the path $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{z} \to \mathsf{v'}_1 \to \mathsf{v'}_2 \to \cdots \to \mathsf{v'}_{\mathsf{z'}}$. In the resulting path, we eliminate the subpath $\mathsf{v}_\mathsf{j} = (\mathsf{x}_\mathsf{j}, \mathsf{y}_\mathsf{j}) \to \cdots \to \mathsf{v}'_{\mathsf{j'}} = (\mathsf{x}'_{\mathsf{j'}}, \mathsf{y}'_{\mathsf{j'}})$ where $\mathsf{x}_\mathsf{j} \cdots \mathsf{x}'_{\mathsf{j'}} = \mathsf{e}$ or $\pi(\mathsf{x}_\mathsf{j} \cdots \mathsf{x}'_{\mathsf{j'}}) = (1)$. If such subpath arises, $\mathfrak{p} \times \mathfrak{p}'$ is given by $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_{\mathsf{j}-1} \to \mathsf{v'}_{\mathsf{j'}+1} \to \mathsf{v'}_{\mathsf{j'}+2} \to \cdots \to \mathsf{v'}_{\mathsf{z'}}$. The product of three or more paths is defined similarly. Note that if $\omega_\mathfrak{p}, \omega_{\mathfrak{p}'} \in \Omega$ correspond to the paths $\mathfrak{p}$ and $\mathfrak{p}'$, respectively, then the path $\mathfrak{p} \times \mathfrak{p}'$ corresponds to $\omega_\mathfrak{p} \cdot \omega_{\mathfrak{p}'} \in \Omega$.

A path which is a product of two or more paths is said to be a *redundant path*. A node or vertex v is considered a *dead node* if every path that passes through v is a redundant path.

In the previous example, we showed how to construct trees given a permutation assignment τ_1 giving rise to an index 3 subgroup Ω_1. The trees have paths which represent elements of Ω_1, and some paths are redundant. In the discussion that follows, we are going to present our first set of rules for branching out from vertex to vertex so that in the construction of the trees, we eliminate redundant paths, and eventually arrive at a generating set for Ω. To illustrate our ideas, we use the trees given in Figure 5.2, which appear in Figure 5.3 with the redundant paths shown in blue.

Suppose in the construction of paths we obtain a subpath $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ where the ordered pair of v_i is of the form (r, r), $\mathsf{r} \in \{1, 2, \ldots, n\}$. In this case, we branch from v_i using the inverse of the subpath from the root v_1 to $\mathsf{v}_{\mathsf{i}-1}$. We state this as our first braching out rule.

Consider the vertex $\mathsf{v}_2 = (b, (2, 2))$ of the tree $\mathfrak{t}_1$ in Figure 5.3. There are three paths containing the subpath $(a, (1, 2)) \to (b, (2, 2))$ namely $\mathfrak{p}_{13}$, $\mathfrak{p}_{14}$ and $\mathfrak{p}_{15}$. Now, the product $\mathfrak{p}_{15} \times \mathfrak{p}_{12} \times \mathfrak{p}_{21}$ yields the path $(a, (1, 2)) \to (b, (2, 2)) \to (a^{-1}, (2, 1)) \to (a, (1, 2)) \to (a, (2, 3)) \to (b, (3, 1)) \to (b, (1, 3)) \to (a, (3, 1))$ which is exactly the path $\mathfrak{p}_{13}$ given by $(a, (1, 2)) \to (b, (2, 2)) \to (a, (2, 3)) \to (a, (3, 1))$. Thus, we can say that $\mathfrak{p}_{13}$ is a redundant path. Similarly, the path $\mathfrak{p}_{14}$ is redundant since the product $\mathfrak{p}_{15} \times \mathfrak{p}_{12}$ yields $\mathfrak{p}_{14}$. Hence, it is enough to consider the path $\mathfrak{p}_{15}$. Note that in $\mathfrak{p}_{15}$, we branch out from $\mathsf{v}_2 = (b, (2, 2))$ to $(a^{-1}, (2, 1))$.

Let us now investigate what happens when we have two vertices v_i and $\mathsf{v}'_{\mathsf{i}'}$

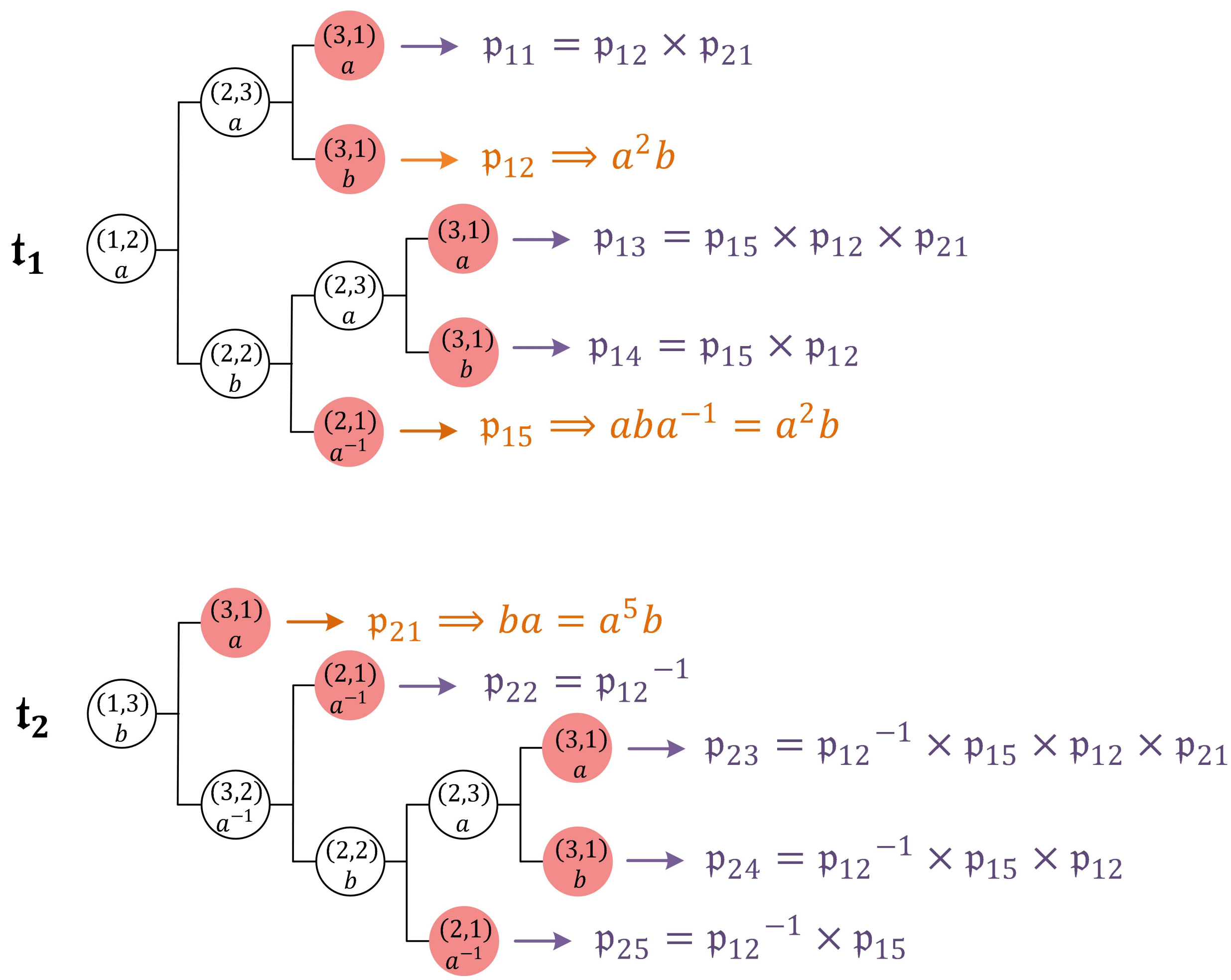


Figure 5.3: The trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ given τ_1 corresponding to the subgroup Ω_1 of $G = D_6$. The redundant paths are labeled blue.

on different subpaths $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ and $\mathsf{v}'_1 \to \mathsf{v}'_2 \to \cdots \to \mathsf{v}'_{\mathsf{i}'}$ where $\mathsf{v}'_{\mathsf{i}'} = {\mathsf{v}_\mathsf{i}}^{-1}$. In Figure 5.3, consider the vertices $\mathsf{v}_2 = (a, (2,3))$ of tree $\mathfrak{t}_1$ and $\mathsf{v}'_2 = (a^{-1}, (3,2))$ of tree $\mathfrak{t}_2$ where $v'_2 = {v_2}^{-1}$ ($(a, (2,3))^{-1} = (a^{-1}, (3,2))$). We are going to show that the vertex $\mathsf{v}'_2 = (a^{-1}, (3,2))$ is a dead node; that is, all the paths containing the subpath $\mathsf{v}'_1 \to \mathsf{v}'_2$, namely $\mathfrak{p}_{22}$, $\mathfrak{p}_{23}$, $\mathfrak{p}_{24}$ and $\mathfrak{p}_{25}$, are redundant.

First, note that the path ${\mathfrak{p}_{12}}^{-1}$ is given by $(b, (1,3)) \to (a^{-1}, (3,2)) \to (a^{-1}, (2, 1))$ which is exactly the path $\mathfrak{p}_{22}$. Moreover, the product ${\mathfrak{p}_{12}}^{-1} \times \mathfrak{p}_{15} \times \mathfrak{p}_{12} \times \mathfrak{p}_{21}$ yields the path $(b, (1,3)) \to (a^{-1}, (3,2)) \to (b, (2,2)) \to (a, (2,3)) \to (a, (3,1))$ which is the same as the path $\mathfrak{p}_{23}$. Likewise, it is easy to show that $\mathfrak{p}_{24} = {\mathfrak{p}_{12}}^{-1} \times \mathfrak{p}_{15} \times \mathfrak{p}_{12}$ and $\mathfrak{p}_{25} = {\mathfrak{p}_{12}}^{-1} \times \mathfrak{p}_{15}$. Indeed, $(a^{-1}, (3,2))$ is a dead node since the paths $\mathfrak{p}_{22}$, $\mathfrak{p}_{23}$, $\mathfrak{p}_{24}$ and $\mathfrak{p}_{25}$ are redundant.

Now, consider the paths passing through $\mathsf{v}_2 = (a, (2,3))$, namely $\mathfrak{p}_{11}$ and $\mathfrak{p}_{12}$. It can be shown that $\mathfrak{p}_{11}$ is redundant since it is exactly the same as the product $\mathfrak{p}_{12} \times \mathfrak{p}_{21}$. In this case, we only consider the path $\mathfrak{p}_{12}$.

In general, when we have two nodes v_i and $\mathsf{v}'_{\mathsf{i}'}$ on different subpaths $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ and $\mathsf{v}'_1 \to \mathsf{v}'_2 \to \cdots \to \mathsf{v}'_{\mathsf{i}'}$ where $\mathsf{v}'_{\mathsf{i}'} = {\mathsf{v}_\mathsf{i}}^{-1}$, one of them, say $\mathsf{v}'_{\mathsf{i}'}$ can be shown to be a dead node. Meanwhile, all paths containing the subpath $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ are redundant except the path where we branch out from v_i using the inverse of the subpath from v'_1 to $\mathsf{v}'_{\mathsf{i}'-1}$. This is our second branching out rule.

In Figure 5.3, we see that the paths $\mathfrak{p}_{12}$, $\mathfrak{p}_{15}$ and $\mathfrak{p}_{21}$ (labeled orange) remain after all the redundant paths are eliminated. We have shown that we can express all redundant paths as products of these paths or their inverses. In order to come up with a generating set for Ω, the redundant paths are removed.

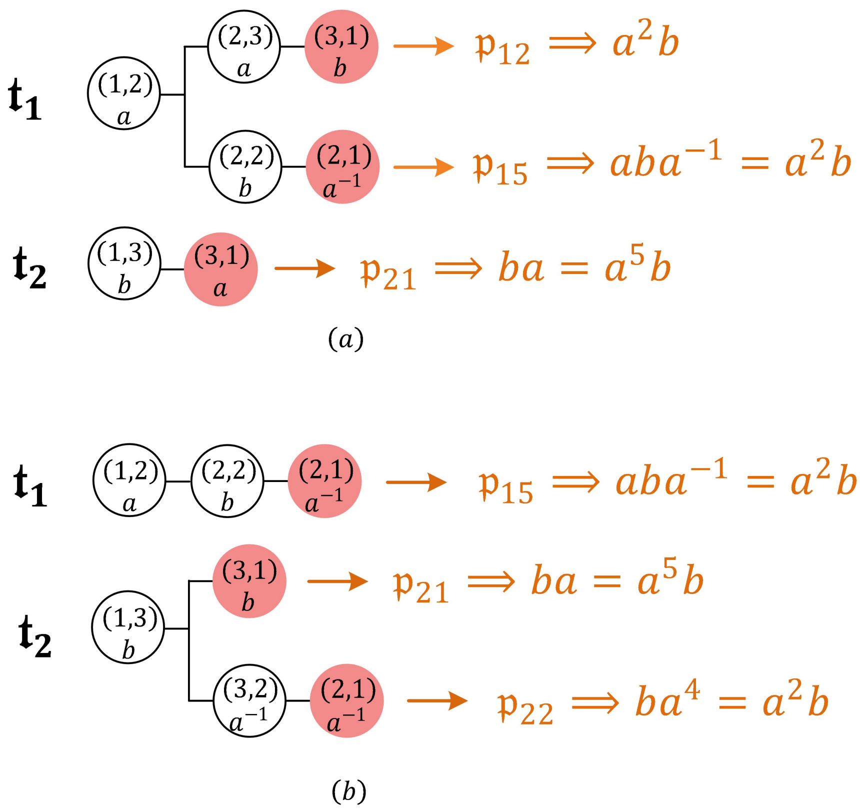


Figure 5.4: Two possible sets of the trees in the construction of $GenTree(\Omega_1)$.

Let us now illustrate how to construct $GenTree(\Omega_1)$ using Rules 1 and 2. We start with the roots $(a,(1,2))$ and $(b,(1,3))$. Then connect the root $(a,(1,2))$ to the vertices $(a,(2,3))$ and $(b,(2,2))$. By Rule 1, branch out from the vertex $(b,(2,2))$ to $(a^{-1},(2,1))$. Next connect the root $(b,(1,3))$ to the vertices $(a,(3,1))$ (which is a terminal node) and $(a^{-1},(3,2))$. Using Rule 2, we consider $(a^{-1},(3,2))$

a dead node since its inverse $(a,(2,3))$ already appeared in $\mathfrak{t}_1$. Thus, branch out from $(a,(2,3))$ to $(b^{-1},(3,1))$. In Figure 5.4 (a), we present $GenTree(\Omega_1)$. The generating set obtained for Ω_1 is $\{a^2b, a^5b\}$.

Now, suppose that in applying Rule 2 we consider the node $(a,(2,3))$ as the dead node, instead of the node $(a^{-1},(3,2))$. Then we obtain the trees given in Figure 5.4 (b). In this case, we get the same set of generators for Ω_1.

To illustrate the next set of branching out rules, let us look at the following example. Consider the permutation assignment $\tau_2 = \{(123),(23)\}$ corresponding to the G-transitive 3-coloring given in Figure 5.5 where a sends colors 1 to 2, 2 to 3 and 3 to 1 while b interchanges colors 2 and 3 (colors 1, 2, 3 correspond to white, green, yellow, respectively). We still refer to G as the symmetry group of the regular hexagon.

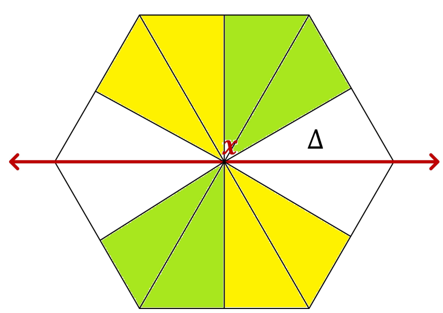


Figure 5.5: A G-transitive 3-coloring of the hexagon arising from the permutation assignment $\tau_2 = \{(123),(23)\}$.

In the construction of $GenTree(\Omega_2)$, where τ_2 gives rise to the subgroup Ω_2 of G, the set of vertices $\mathcal{V}_2$ is $\{(a,(1,2)),(a^{-1},(2,1)),(a,(2,3)),(a^{-1},(3,2)),(a,(3,1)),$

$(a^{-1},(1,3)),(b,(2,3)),(b,(3,2))\}$ and we have the root $(a,(1,2))$. The tree constructed is given in Figure 5.6.

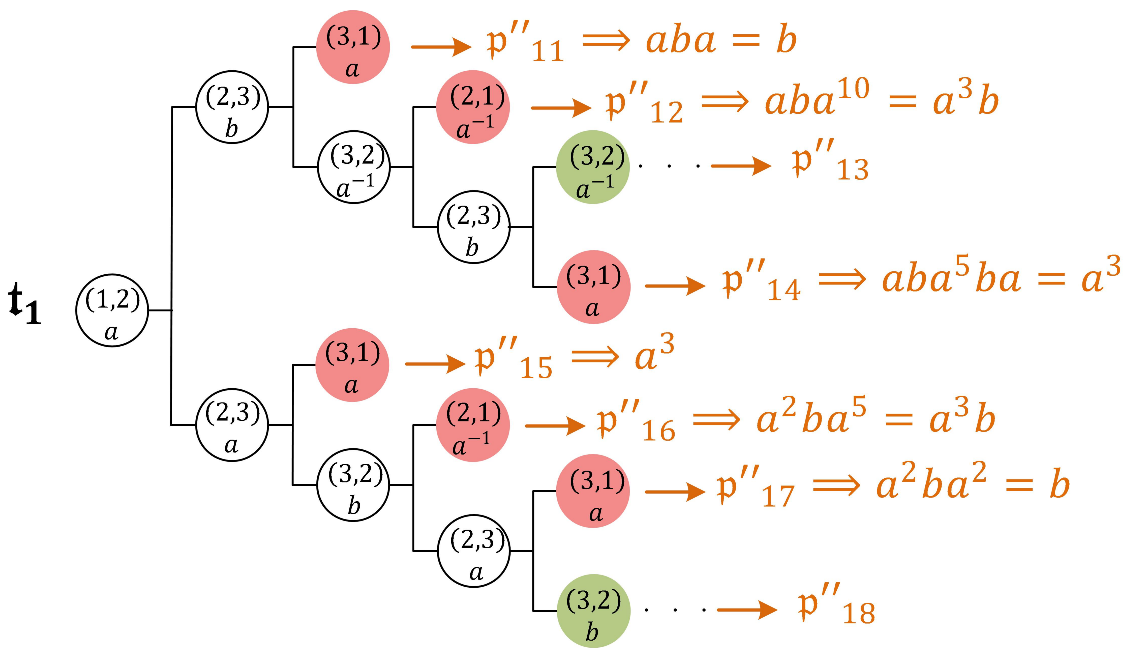


Figure 5.6: The tree $\mathfrak{t}_1$ given the permutation assignment $\tau_2 = \{(123),(23)\}$ corresponding to the subgroup Ω_2 of $G = D_6$.

Observe that if one continues to branch out from the green nodes, one arrives at the situation depicted in Figure 5.7. In Figure 5.7 (a), the vertex $(b,(3,2))$ reappears if branching out is continued; a sequence of vertices $(b,(3,2))$, $(a,(2,3)),(b,(3,2))$, $(a,(2,3)),\ldots$ is obtained. Similarly, in Figure 5.7 (b), the vertex $(a^{-1},(3,2))$ reappears and results in a sequence of vertices $(a^{-1},(3,2))$, $(b,(2,3))$, $(a^{-1},(3,2))$, $(b,(2,3)),\ldots$.

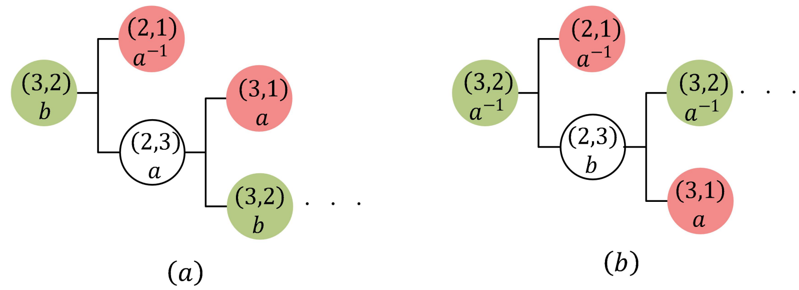


Figure 5.7: The sequence of vertices arising from the green nodes in $\mathfrak{t}_1$ given in Figure 5.6

In the discussion that follows, we give some steps on how these scenarios are avoided. These will constitute the remaining branching out rules. To facilitate the discussion, we use the tree in Figure 5.6 which is redrawn in Figure 5.8 with the redundant paths labeled blue.

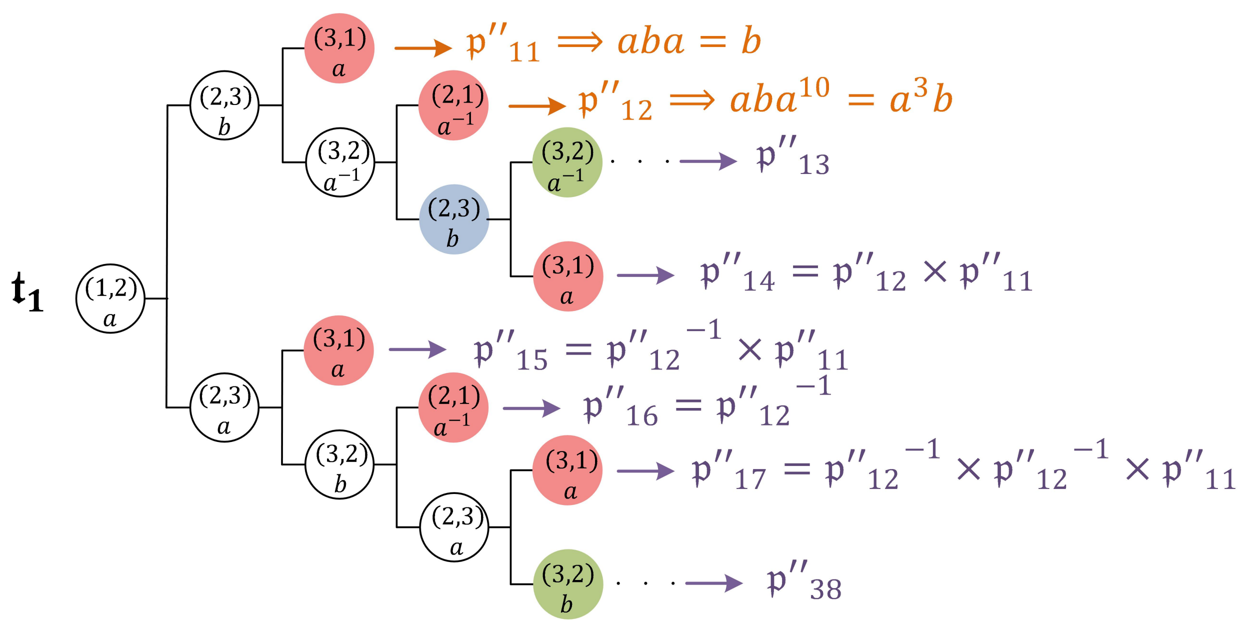


Figure 5.8: The tree $\mathfrak{t}_1$ given τ_2 corresponding to the subgroup Ω_2 of $G = D_6$. The redundant paths are labeled in blue.

Suppose in the construction of the paths, two vertices have the same ordered pair, we can choose to branch out from only one of them and consider the other as a dead node. This will be our third branching out rule.

For instance in Figure 5.8, we have the vertices $(b,(2,3))$ and $(a,(2,3))$. It can be shown that $(a,(2,3))$ is a dead node. Observe that the paths $\mathfrak{p}''_{15}$, $\mathfrak{p}''_{16}$, and $\mathfrak{p}''_{17}$ can be expressed respectively as ${\mathfrak{p}''_{12}}^{-1} \times \mathfrak{p}''_{11}$, ${\mathfrak{p}''_{12}}^{-1}$ and ${\mathfrak{p}''_{12}}^{-2} \times \mathfrak{p}''_{11}$.

Note that $(b,(2,3))$ appears twice in Figure 5.8. If we continue branching out from the vertex $(b,(2,3))$ (colored blue in Figure 5.8) more paths will arise which can be shown to be products of $\mathfrak{p}''_{11}$ and $\mathfrak{p}''_{12}$. Hence, in this case, we consider this a dead node and we choose to branch out only from the first occurence of the vertex $(b,(2,3))$ in the tree. Note that it is also possible to branch out from $(a,(2,3))$ and consider $(b,(2,3))$ a dead node.

Now suppose we have two consecutive vertices $\mathsf{v_i}$ and $\mathsf{v_{i+1}}$ in a subpath $\mathsf{v_1} \to \mathsf{v_2} \to \cdots \to \mathsf{v_i} \to \mathsf{v_{i+1}}$ where $\mathsf{v_i} = (\mathsf{x_i},(\bar{\mathsf{y}}_1,\bar{\mathsf{y}}_2))$ and $\mathsf{v_{i+1}} = (\mathsf{x_{i+1}},(\bar{\mathsf{y}}_2,\bar{\mathsf{y}}_1))$. Then branch out from $\mathsf{v_{i+1}}$ using the inverse of the subpath from the root $\mathsf{v_1}$ to $\mathsf{v_{i-1}}$. This is our fourth branching out rule.

To illustrate this rule, let us consider the consecutive vertices $(b,(2,3))$ and $(a^{-1},(3,2))$ in Figure 5.8. We branch out from vertex $(a^{-1},(3,2))$ to $(a^{-1},(2,1))$ which results in a path $\mathfrak{p}''_{12}$. Note that the path $\mathfrak{p}''_{14}$ is redundant since it can be expressed as the product $\mathfrak{p}''_{12} \times \mathfrak{p}''_{11}$. Moreover, if we continue branching out from the vertex $(a^{-1},(3,2))$ (colored green), then more paths arise which can be shown to be products of $\mathfrak{p}''_{12}$ and $\mathfrak{p}''_{11}$.

Now, let us construct $GenTree(\Omega_2)$ using the above rules. We start with the root $(a,(1,2))$. It is possible to connect this vertex to $(b,(2,3))$ and $(a,(2,3))$. By Rule 3, one of the vertices $(b,(2,3))$ and $(a,(2,3))$ may be considered a dead node, say $(a,(2,3))$. From the vertex $(b,(2,3))$, we reach the terminal node $(a,(3,1))$ giving rise to the path $\mathfrak{p}''_{11}$. Moreover, we can also connect $(b,(2,3))$ to $(a^{-1},(3,2))$, and applying Rule 4 connect $(a^{-1},(3,2))$ to $(a^{-1},(2,1))$ resulting in the path $\mathfrak{p}''_{12}$. Observe that the paths $\mathfrak{p}''_{11}$ and $\mathfrak{p}''_{12}$ correspond to the elements b and a^3b which generate Ω_2. In Figure 5.9, we present $GenTree(\Omega_2)$ after getting rid of the redundant paths.

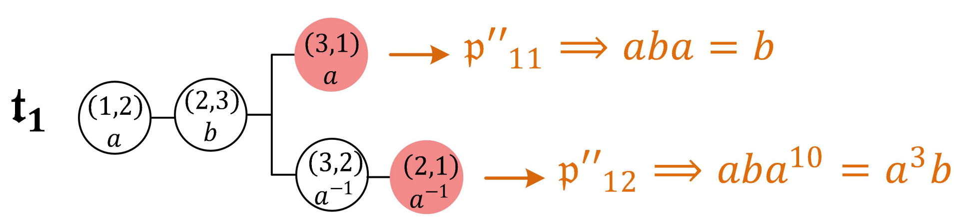


Figure 5.9: $GenTree(\Omega_2)$.

Taking into consideration the situations in the previous discussion, we have the following branching out rules for $GenTree(\Omega)$.

Theorem 5.1 (The Branching Out Rules for GenTree(Ω))

The following rules in the construction of $GenTree(\Omega)$ eliminate the redundant paths.

Rule 1. In the subpath $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ *where* v_i *has ordered pair* (r,r)*,* $\mathsf{r} \in \{1, 2, \ldots, n\}$*, branch out from* v_i *using the inverse of the subpath from* v_1 *to* $\mathsf{v}_{\mathsf{i}-1}$*.*

Rule 2. If two nodes v_i *and* $\mathsf{v}'_{\mathsf{i}'}$ *are on two different subpaths* $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ *and* $\mathsf{v}'_1 \to \mathsf{v}'_2 \to \cdots \to \mathsf{v}'_{\mathsf{i}'}$*, repectively, such that* $\mathsf{v}'_{\mathsf{i}'} = {\mathsf{v}_\mathsf{i}}^{-1}$*, then one of* v_i *and* $\mathsf{v}'_{\mathsf{i}'}$ *can be considered a dead node, say* $\mathsf{v}'_{\mathsf{i}'}$*. Then branch out from* v_i *using the inverse of the subpath from* v'_1 *to* $\mathsf{v}'_{\mathsf{i}'-1}$*.*

Rule 3. If two (or more) nodes have the same ordered pairs (r, t) *where* $r \neq t$*, we can choose to branch out from only one of them and consider the other a dead node.*

Rule 4. If two consecutive nodes v_i *and* $\mathsf{v}_{\mathsf{i}+1}$ *of a subpath* $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i} \to \mathsf{v}_{\mathsf{i}+1}$ *are such that* $\mathsf{v}_\mathsf{i} = (\mathsf{x}_\mathsf{i}, (\mathsf{r},\mathsf{t}))$ *and* $\mathsf{v}_{\mathsf{i}+1} = (\mathsf{x}_{\mathsf{i}+1}, (\mathsf{t},\mathsf{r}))$*, where* $\mathsf{r},\mathsf{t} \in \{1, 2, \ldots, n\}$*, then branch out from* $\mathsf{v}_{\mathsf{i}+1}$ *using the inverse of the subpath from* v_1 *to* $\mathsf{v}_{\mathsf{i}-1}$*.*

Proof

Rule 1. Let $\mathsf{v}_\mathsf{i} = (\mathsf{x}_\mathsf{i}, (\mathsf{r},\mathsf{r}))$ and $\mathfrak{p} : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}_\mathsf{i} \to \mathsf{v}_{\mathsf{i}+1} \to \cdots \to \mathsf{v}_\mathsf{z}$ be the path containing the subpath $\mathsf{v}_1 \to \mathsf{v}_2 \to \cdots \to \mathsf{v}_\mathsf{i}$ such that $\mathsf{v}_{\mathsf{i}+1} = {\mathsf{v}_{\mathsf{i}-1}}^{-1}, \mathsf{v}_{\mathsf{i}+2} = {\mathsf{v}_{\mathsf{i}-2}}^{-1}, \ldots, \mathsf{v}_{\mathsf{z}-1} = {\mathsf{v}_2}^{-1}, \mathsf{v}_\mathsf{z} = {\mathsf{v}_1}^{-1}$.

Assume that there exists another path $\mathfrak{p}'$, distinct from $\mathfrak{p}$ containing the subpath $v_1 \to v_2 \to \cdots \to v_i$. That is, $\mathfrak{p}' : v_1 \to v_2 \to \cdots \to v_i \to v'_{i+1} \to v'_{i+2} \to \cdots \to v'_{z'}$ with $v'_{i+1} \neq v_{i+1}$. We will show that $\mathfrak{p}'$ is redundant.

Consider the path $\mathfrak{p}'' : v_1 \to \cdots \to v_{i-1} \to v'_{i+1} \to \cdots \to v'_{z'}$ containing the subpaths $v_1 \to \cdots \to v_{i-1}$ of $\mathfrak{p}$ and $v'_{i+1} \to v'_{i+2} \to \cdots \to v'_{z'}$ of $\mathfrak{p}'$.

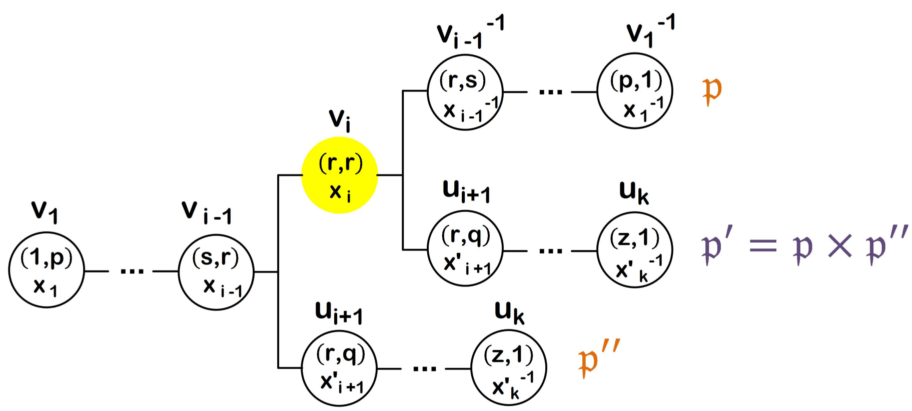


Figure 5.10: Diagram for Rule 1.

Note that $\mathfrak{p}\times\mathfrak{p}''$ is given by $v_1 \to \cdots \to v_{i-1} \to v_i \to {v_{i-1}}^{-1} \to \cdots \to {v_1}^{-1} \to v_1 \to \cdots \to v'_{i-1} \to v'_{i+1} \to \cdots \to v'_{z'}$ and will yield $v_1 \to v_2 \to \cdots \to v_i \to v'_{i+1} \to v'_{i+2} \to \cdots \to v'_{z'}$ which is exactly the path $\mathfrak{p}'$. Thus, $\mathfrak{p}'$ is a redundant path.

Rule 2. Let $\mathfrak{q} : v_1 \to \cdots \to v_{i-1} \to v_i = {v'_{i'}}^{-1} \to {v'_{i'-1}}^{-1} \to \cdots \to {v'_1}^{-1}$ be the path containing the subpath $v_1 \to \cdots \to v_{i-1} \to v_i$ and the inverse of the subpath $v'_1 \to v'_2 \to \cdots \to v'_{i'}$. Then, the path $\mathfrak{q}^{-1} : v'_1 \to \cdots \to v'_{i'-1} \to v'_{i'} \to {v_{i-1}}^{-1} \to \cdots \to {v_1}^{-1}$ contains the subpath $v'_1 \to v'_2 \to \cdots \to v'_{i'}$.

Suppose there is another path also containing the subpath $v'_1 \to v'_2 \to \cdots \to v'_{i'}$

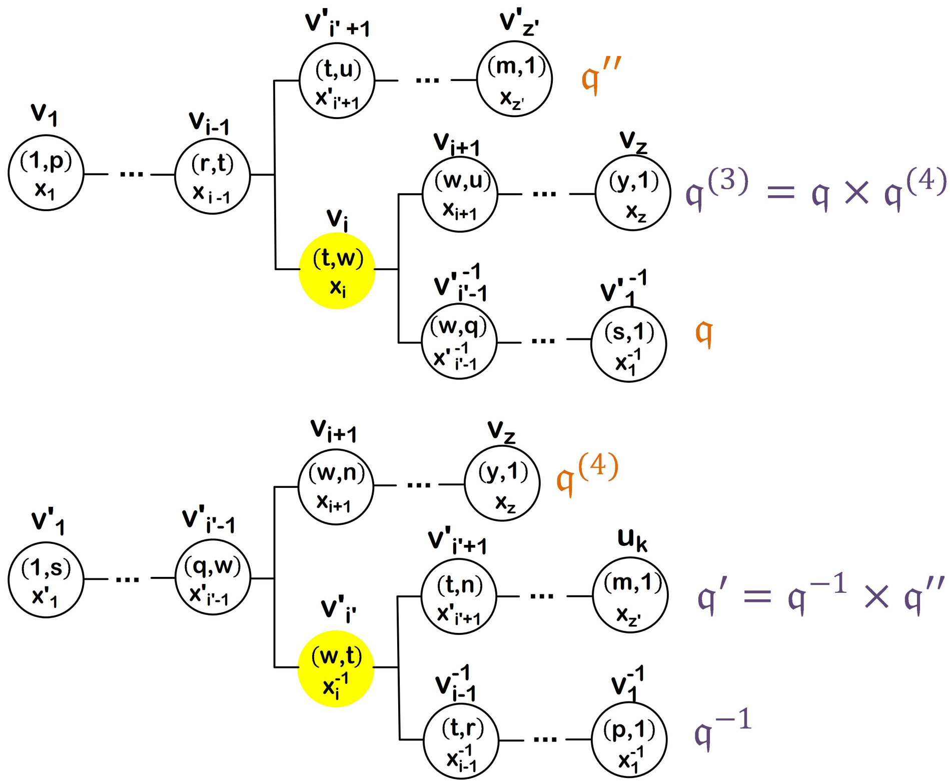


Figure 5.11: Diagram for Rule 2.

but different from $\mathfrak{q}^{-1}$; say, $\mathfrak{q}' : \mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{\mathsf{i}'-1} \to \mathsf{v}'_{\mathsf{i}'} \to \mathsf{v}'_{\mathsf{i}'+1} \to \cdots \to \mathsf{v}'_{\mathsf{z}'}$ with $\mathsf{v}'_{\mathsf{i}'+1} \neq \mathsf{v}_{\mathsf{i}-1}{}^{-1}$.

Consider another path $\mathfrak{q}'' : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}'_{\mathsf{i}'+1} \to \cdots \to \mathsf{v}'_{\mathsf{z}'}$, where $\mathfrak{q}'' = \mathfrak{q} \times \mathfrak{q}'$. Observe that $\mathfrak{q}^{-1} \times \mathfrak{q}''$ yields the path $\mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{\mathsf{i}'-1} \to \mathsf{v}'_{\mathsf{i}'} \to \mathsf{v}'_{\mathsf{i}'+1} \to \cdots \to \mathsf{v}'_{\mathsf{z}'}$ which is exactly the path $\mathfrak{q}'$. Thus, $\mathfrak{q}'$ is redundant.

Now, we want to show that any path containing the subpath $\mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}_\mathsf{i}$ different from $\mathfrak{q}$ is a redundant path. Let $\mathfrak{q}^{(3)} : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}_\mathsf{i} \to \mathsf{v}_{\mathsf{i}+1} \to \cdots \to \mathsf{v}_\mathsf{z}$ with $\mathsf{v}_{\mathsf{i}+1} \neq \mathsf{v}'_{\mathsf{i}'-1}{}^{-1}$.

Consider the path $\mathfrak{q}^{(4)} : \mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{\mathsf{i}'-1} \to \mathsf{v}_{\mathsf{i}+1} \to \cdots \to \mathsf{v}_\mathsf{l}$. It can be verified that $\mathfrak{q}^{(3)} = \mathfrak{q} \times \mathfrak{q}^{(4)}$.

Rule 3. Suppose we have two nodes v_i and $\mathsf{v}'_{\mathsf{i}'}$ in the subpaths $\mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}_\mathsf{i}$ and $\mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{\mathsf{i}'-1} \to \mathsf{v}'_{\mathsf{i}'}$, respectively, such that $\mathsf{v}_\mathsf{i} = (\mathsf{x}_\mathsf{i}, (\mathsf{r}, \mathsf{t}))$ and $\mathsf{v}'_{\mathsf{i}'} = (\mathsf{x}'_\mathsf{j}, ((\mathsf{t}, \mathsf{r})))$, $\mathsf{r}, \mathsf{t} \in \{1, 2, \ldots, n\}$.

Let $\mathfrak{r} : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}_\mathsf{i} \to \mathsf{v}'_{\mathsf{i}'}{}^{-1} \to \mathsf{v}'_{\mathsf{i}'-1}{}^{-1} \to \cdots \to \mathsf{v}'_1{}^{-1}$ contain the subpath $\mathsf{v}_1 \to \cdots \to \mathsf{v}_{\mathsf{i}-1} \to \mathsf{v}_\mathsf{i}$ and the inverse of the subpath $\mathsf{v}'_1 \to \mathsf{v}'_2 \to \cdots \to \mathsf{v}'_{\mathsf{i}'}$. Then, the path $\mathfrak{r}^{-1} : \mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{\mathsf{i}'-1} \to \mathsf{v}'_{\mathsf{i}'} \to \mathsf{v}_\mathsf{i}{}^{-1} \to \mathsf{v}_{\mathsf{i}-1}{}^{-1} \to \cdots \to \mathsf{v}_1{}^{-1}$ contains the subpath $\mathsf{v}'_1 \to \mathsf{v}'_2 \to \cdots \to \mathsf{v}'_{\mathsf{i}'}$.

Suppose there is another path $\mathfrak{r}'$ also containing the subpath $\mathsf{v}'_1 \to \mathsf{v}'_2 \to \cdots \to \mathsf{v}'_{\mathsf{i}'}$ but different from $\mathfrak{r}^{-1}$; say, $\mathfrak{r}' : \mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{\mathsf{i}'-1} \to \mathsf{v}'_{\mathsf{i}'} \to \mathsf{v}'_{\mathsf{i}'+1} \to \cdots \to \mathsf{v}'_{\mathsf{z}'}$ with $\mathsf{v}'_{\mathsf{i}'+1} \neq \mathsf{v}_\mathsf{i}{}^{-1}$.

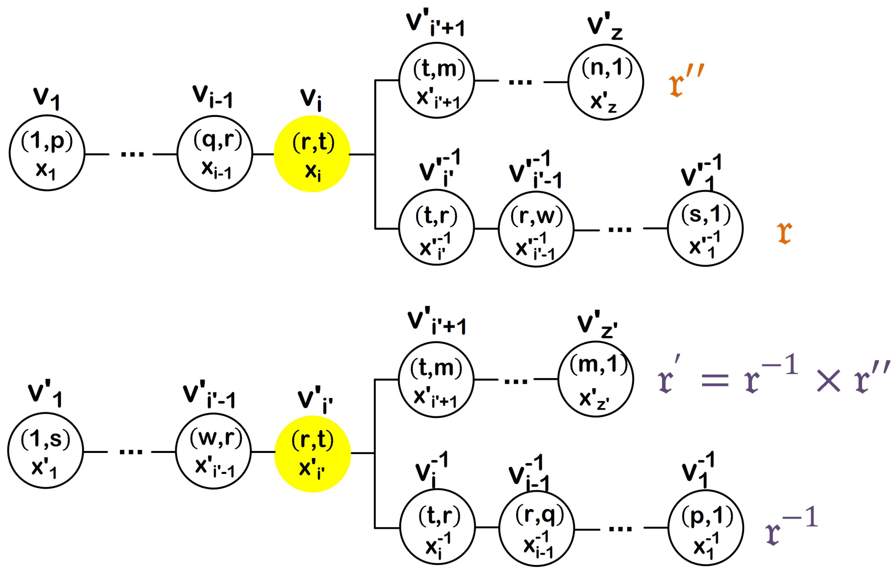


Figure 5.12: Diagram for Rule 3.

Consider another path $\mathfrak{r}'' : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_i \to \mathsf{v}'_{i'+1} \to \cdots \to \mathsf{v}'_{z'}$. Observe that $\mathfrak{r}^{-1} \times \mathfrak{r}''$ yields the path $\mathsf{v}'_1 \to \cdots \to \mathsf{v}'_{i'-1} \to \mathsf{v}'_{i'} \to \mathsf{v}'_{i'+1} \to \cdots \to \mathsf{v}'_{z'}$ which is exactly the path $\mathfrak{r}'$. Thus, $\mathfrak{r}'$ is redundant.

Rule 4. Suppose we have two consecutive nodes v_i and v_{i+1} in the subpath $\mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_i \to \mathsf{v}_{i+1}$ such that $\mathsf{v}_i = (\mathsf{x}_i, (\mathsf{r}, \mathsf{t}))$ and $\mathsf{v}_{i+1} = (\mathsf{x}_{i+1}, (\mathsf{t}, \mathsf{r}))$, $\mathsf{r}, \mathsf{t} \in \{1, 2, \ldots, n\}$. Let the path $\mathfrak{s} : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_i \to \mathsf{v}_{i+1} \to {\mathsf{v}_{i-1}}^{-1} \cdots \to {\mathsf{v}_1}^{-1}$ containing the subpath $\mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_i \to \mathsf{v}_{i+1}$.

Assume there is another path $\mathfrak{s}'$ containing the subpath $\mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_i \to \mathsf{v}_{i+1}$ but different from $\mathfrak{s}$. That is, $\mathfrak{s}' : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_i \to \mathsf{v}_{i+1} \to \mathsf{v}_{i+2} \to \cdots \to \mathsf{v}_z$ such that $\mathsf{v}_{i+2} \neq {\mathsf{v}_{i-1}}^{-1}$. We will show that $\mathfrak{s}'$ is a redundant path.

Consider the path $\mathfrak{s}'' : \mathsf{v}_1 \to \cdots \to \mathsf{v}_{i-1} \to \mathsf{v}_{i+2} \to \cdots \to \mathsf{v}_z$. Observe that $\mathfrak{s}' =$

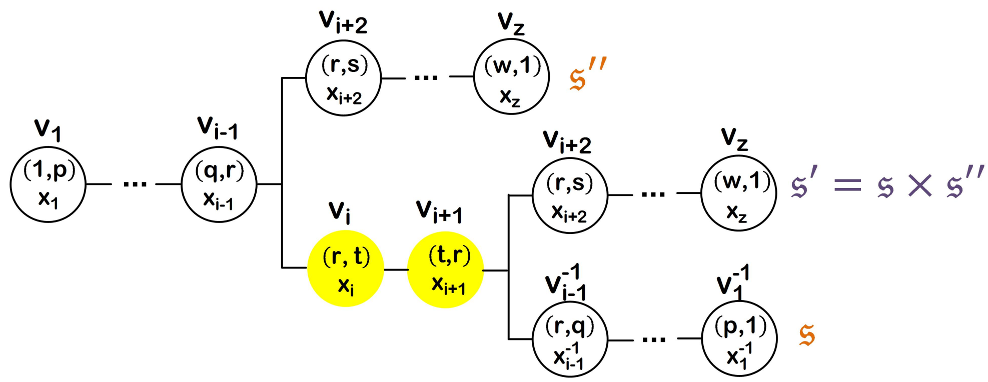


Figure 5.13: Diagram for Rule 4.

$\mathfrak{s} \times \mathfrak{s}''$. Thus, $\mathfrak{s}'$ is redundant.

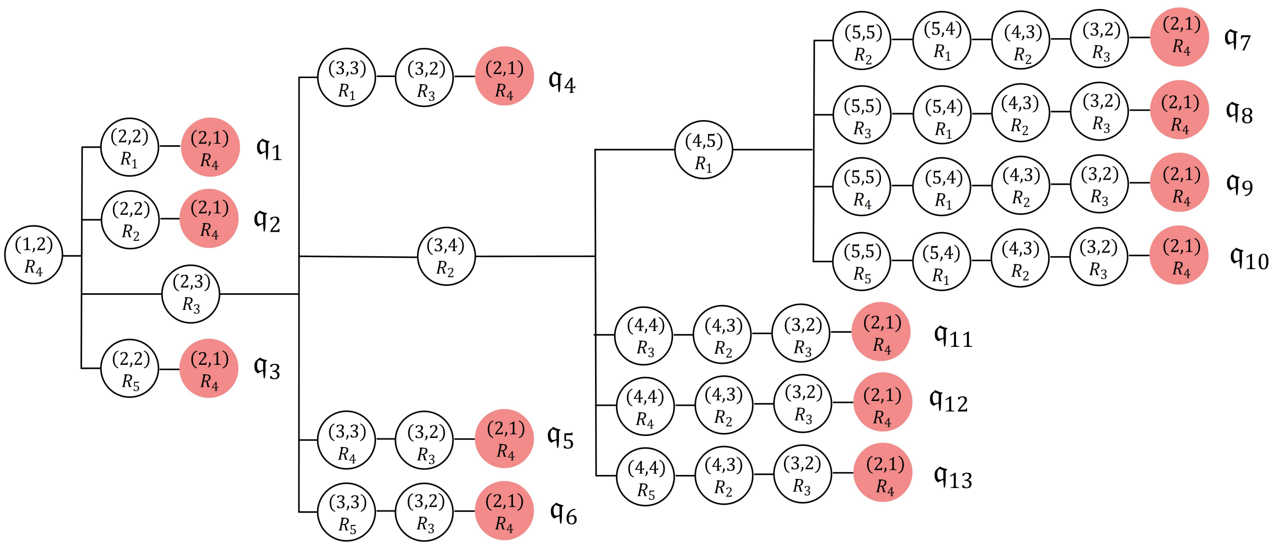
∎

5.3 Some $GenTree(\Omega)$ Construction Examples

In this section, we use $GenTree(\Omega)$ to get a set of generators for the subgroups derived in the previous chapter.

First, we illustrate the derivation of a generating set for the index 5 subgroup Ω' of the Coxeter group $\Gamma(3,2,2,2,3,2,2,3,4,2)$ generated by the reflections R_1, R_2, R_3, R_4 and R_5 which are reflections in the hyperplanes containing the sides of a noncompact Coxeter simplex in $\mathbb{H}^4$.

Consider the permutation assignment $\tau' = \{(45),(34),(23),(12),(1)\} \in \mathbb{P}$ corresponding to the index 5 subgroup Ω' of $\Gamma(3,2,2,2,3,2,2,3,4,2)$ obtained from the $GAP4$ routine $Coloring$ as discussed in Section 3.2. From τ', we obtain the following set of roots $\{(R_4,(1,2)),(R_1,(1,1)),(R_2,(1,1)),(R_3,(1,1)),(R_5,(1,1))\}$. Observe that all roots except $(R_4,(1,2))$ have ordered pairs $(1,1)$ which implies that their corresponding generators are also generators of Ω'.



$\mathfrak{q}_1 = R_4R_1R_4 = R_1$	$\mathfrak{q}_5 = R_4R_3R_4R_3R_4 = R_3$	$\mathfrak{q}_9 = R_4R_3R_2R_1R_4R_1R_2R_3R_4 = R_3$
$\mathfrak{q}_2 = R_4R_2R_4 = R_2$	$\mathfrak{q}_6 = R_4R_3R_5R_3R_4$	$\mathfrak{q}_{10} = R_4R_3R_2R_1R_5R_1R_2R_3R_4$ $= R_4R_3R_5R_3R_4$
$\mathfrak{q}_3 = R_4R_5R_4 = R_5$	$\mathfrak{q}_7 = R_4R_3R_2R_1R_2R_1R_2R_3R_4 = R_1$	$\mathfrak{q}_{11} = R_4R_3R_2R_3R_2R_3R_4 = R_2$
$\mathfrak{q}_4 = R_4R_3R_1R_3R_4 = R_1$	$\mathfrak{q}_8 = R_4R_3R_2R_1R_3R_1R_2R_3R_4 = R_2$	$\mathfrak{q}_{12} = R_4R_3R_2R_4R_2R_3R_4 = R_3$
$\mathfrak{q}_{13} = R_4R_3R_2R_5R_2R_3R_4 = R_4R_3R_5R_3R_4$		

Figure 5.14: The $GenTree(\Omega')$ where $\tau' = \{(45),(34),(23),(12),(1)\}$.

Meanwhile, we connect $(R_4,(1,2))$ to the vertices $(R_1,(2,2))$, $(R_2,(2,2))$,$(R_3,(2,3))$ and $(R_5,(2,2))$. Using Rule 1, we connect each of $(R_1,(2,2))$, $(R_2,(2,2))$, and $(R_5,(2,2))$ to $(R_4,(2,1))$ resulting respectively in the paths $\mathfrak{q}_1$, $\mathfrak{q}_2$, and $\mathfrak{q}_3$. We connect the vertex $(R_3,(2,3))$ to the vertices $(R_1,(3,3))$, $(R_2,(3,4))$, $(R_4,(3,3))$

and $(R_5,(3,3))$. Then we connect each of the vertices $(R_1,(3,3))$, $(R_4,(3,3))$ and $(R_5,(3,3))$ to the subpath $(R_3,(3,2))$, $(R_4,(2,1))$ which respectively yields the paths $\mathfrak{q}_4$, $\mathfrak{q}_5$, and $\mathfrak{q}_6$. We connect the vertex $(R_2,(3,4))$ to the vertices $(R_1,\ (4,5))$, $(R_3,\ (4,4))$, $(R_4,\ (4,4))$ and $(R_5,\ (4,4))$. We connect each of the vertices $(R_3,(4,4))$, $(R_4,(4,4))$ and $(R_5,(4,4))$ to the subpath $(R_2,(4,3))$, $(R_3,(3,2))$, $(R_4,(2,1))$ by applying Rule 1 which respectively yields the paths $\mathfrak{q}_{11}$, $\mathfrak{q}_{12}$, and $\mathfrak{q}_{13}$. We connect the vertex $(R_1,(4,5))$ to the vertices $(R_2,(5,5))$, $(R_3,(5,5))$, $(R_4,(5,5))$ and $(R_5,(5,5))$. By Rule 1, we connect each of the vertices $(R_2,(5,5))$, $(R_3,(5,5))$, $(R_4,(5,5))$ and $(R_5,(5,5))$ to the subpath $(R_1,(5,4))$, $(R_2,(4,3))$, $(R_3,(3,2))$, $(R_4,(2,1))$ resulting in the paths $\mathfrak{q}_7$, $\mathfrak{q}_8$, $\mathfrak{q}_9$, and $\mathfrak{q}_{10}$. We give in Figure 5.14 $GenTree(\Omega')$. Hence, Ω' has a generating set $\{R_1, R_2, R_3, R_5,\ R_4R_3R_5R_3R_4\}$.

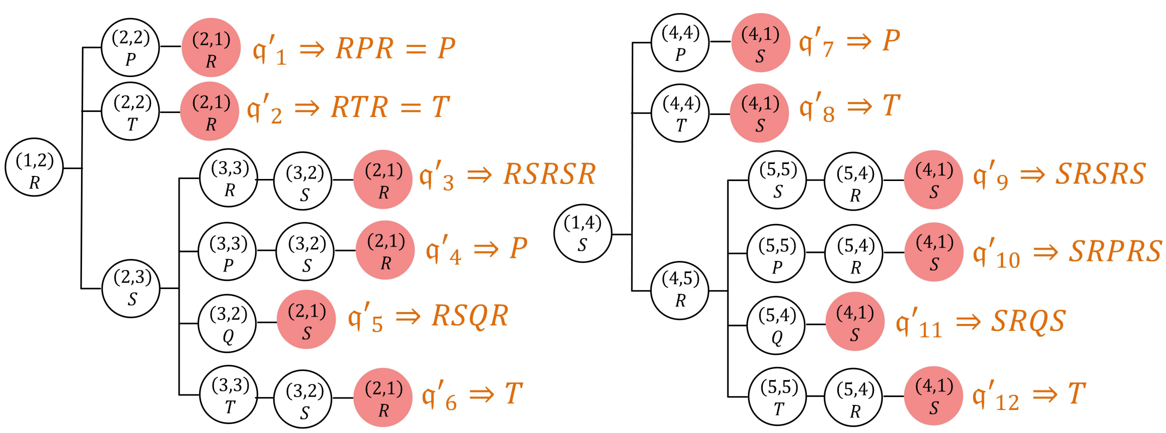


Figure 5.15: The $GenTree(\Omega'')$ where $\tau'' = \{(1), (23)(45), (12)(45), (14)(23), (1)\}$.

We now consider the permutation assignment $\tau'' = \{(1), (23)(45), (12)(45), (14)(23), (1)\}$ which gives rise to an index 5 subgroup Ω'' of the truncated tetrahedron group $\Gamma_{7,3}(5) = \Gamma(4,2,2,0,3,3,2,5,2,2)$ and construct $GenTree(\Omega'')$. Note that $\pi(\Gamma_{7,3}(5)) \cong A_5$. The roots of $GenTree(\Omega'')$ are the vertices $(P,(1,1))$, $(Q,(1,1))$, $(R,(1,2))$, $(S,(1,4))$ and $(T,(1,1))$. The generators of roots the $(P,(1,1))$, $(Q,(1,1))$ and $(T,(1,1))$ are generators of Ω''. We connect the root $(R,(1,2))$ to the vertices $(P,(2,2))$, $(Q,(2,3))$, $(S,(2,3))$ and $(T,(2,2))$. We branch out from each of $(P,(2,2))$ and $(T,(2,2))$, by Rule 1, to the vertex $(R,(2,1))$ resulting respectively in the paths $\mathfrak{q}'_1$ and $\mathfrak{q}'_2$. By Rule 3, we branch out from only one of the vertices $(Q,(2,3))$ and $(S,(2,3))$, say $(S,(2,3))$. We connect it to the vertices $(P,(3,3))$, $(Q,(3,2))$, $(R,(3,3))$, and $(T,(3,3))$. We branch out from each of $(R,(3,3))$, $(P,(3,3))$, and $(T,(3,3))$, by Rule 1, to the subpath $(S,(3,2))$, $(R,(2,1))$ which yields respectively in the paths $\mathfrak{q}'_3$, $\mathfrak{q}'_4$, and $\mathfrak{q}'_6$. On the other hand, we connect the vertex $(Q,(3,2))$ to the vertex $(S,(2,1))$ by Rule 4 giving rise to the path $\mathfrak{q}'_5$.

Similarly, we connect the root $(S,(1,4))$ to the vertices $(P,(4,4))$, $(Q,(4,5))$, $(R,(4,5))$ and $(T,(4,4))$. We branch out from each of $(P,(4,4))$ and $(T,(4,4))$ to $(S,(4,1))$ by Rule 1 giving rise respectively to the paths $\mathfrak{q}'_7$ and $\mathfrak{q}'_8$. By Rule 3, we branch out from one of $(Q,(4,5))$ and $(R,(4,5))$. We branch out from $(R,(4,5))$, and connect it to the vertices $(P,(5,5))$, $(Q,(5,4))$, $(S,(5,5))$ and $(T,(5,5))$. We connect each of $(S,(5,5))$, $(P,(5,5))$ and $(T,(5,5))$ to the subpath $(R,(5,4))$, $(S,(4,1))$ resulting respectively in the paths $\mathfrak{q}'_9$, $\mathfrak{q}'_{10}$, and $\mathfrak{q}'_{12}$ by Rule 1. By Rule 4, we connect $(Q,(5,4))$ to the vertex $(S,(4,1))$ which yields the path $\mathfrak{q}'_{11}$. In Figure 5.15 we present $GenTree(\Omega'')$ which gives the generating set $\{P, Q, T, RSRSR, RSQR,$

$SRQS\}$ of Ω''.

Group	$\pi(\Gamma_{i,u}(v))$	π(P)	π(Q)	π(R)	π(S)	π(T)	Generators/GenTree
$\Gamma_{1,6}(6)$= Γ(3,2,2,0,3,2,2,6,2,2)	S_5	(4,5)	(3,4)	(2,3)	(1)	(1,5)	*P, Q, R, S, TPTPT, TPQRSRQPT* / (Figure 5.16)
	S_5	(4,5)	(3,4)	(2,3)	(1,2)	(1)	*P, Q, R, T, SRSRS, SRQPTPQRS* / (Figure 5.17)
$\Gamma_{2,4}(5)$= Γ(3,2,2,0,4,2,2,5,2,2)	D_5	(1)	(1)	(23)(45)	(12)(34)	(1)	*P, Q, R, T, SRQRS, SRSRQRSRS* / (Figure 5.18)
	A_5	(23)(45)	(12)(45)	(1)	(1)	(14)(25)	*P, R, S, QRQ, QTPT, QPTPQ* / (Figure 5.19)
$\Gamma_{3,3}(5)$= Γ(3,2,2,0,3,3,2,5,2,2)	S_5	(4,5)	(3,4)	(2,3)	(2,3)	(1,5)	*P, Q, R, S, TPTPT, TPQSRQPT* / (Figure 5.20)
$\Gamma_{4,6}(6)$= Γ(4,2,2,0,3,2,2,6,2,2)	D_5	(2,3)(4,5)	(1)	(1)	(1)	(1,2)(3,4)	*P, Q, R, S, TPQPT, TPTPQPTPT, TPTPTPTPT* / (Figure 5.21)
	S_5	(2,3)(4,5)	(3,4)	(4,5)	(1)	(1,2)	*P, Q, R, S, TPTPT, TPQRPQPT* / (Figure 5.22)
$\Gamma_{5,2}(5)$= Γ(3,2,2,0,4,4,2,5,2,2)	D_5	(1)	(1)	(2,3)(4,5)	(1,2)(3,4)	(1)	*P, Q, R, T, SQS, SRQRS, SRSQSRS, SRSRQRSRS* (Figure 5.23)
	S_5	(2,3)(4,5)	(1,2)(4,5)	(1)	(1)	(1,4)(2,5)	*P, R, S, QRQ, QSQ, TPTQ, QPTPQ* / (Figure 5.24)
$\Gamma_{6,6}(6)$= Γ(5,2,2,0,3,2,2,6,2,2)	A_5	(2,3)(4,5)	(1,2)(3,4)	(2,5)(3,4)	(1)	(1)	*P, R, S, T, QPRQPQ, QPTPQ* / (Figure 5.25)
	S_5	(2,3)(4,5)	(1,2)(3,4)	(2,5)(3,4)	(1)	(3,4)	*P, R, S, T, QPRQPQ, QPTQPQ QRPTPQ* / (Figure 5.26)
$\Gamma_{7,3}(5)$= Γ(4,2,2,0,3,3,2,5,2,2)	D_5	(2,3)(4,5)	(1)	(1)	(1)	(1,2)(3,4)	*P, Q, R, S, TPQPT, TPTPQPTPT, TPTPTPTPT* / (Figure 5.27)
	A_5	(1)	(2,3)(4,5)	(1,2)(4,5)	(1,4)(2,3)	(1)	*P, Q, T, RSRSR, RSQR, SRQS* / (Figure 5.15)
	S_5	(2,3)(4,5)	(3,4)	(4,5)	(4,5)	(1,2)	*P, Q, R, S, TPTPT, TPQRPQPT, TPQSPQPT* / (Figure 5.28)
$\Gamma_{8,4}(4)$= Γ(3,2,3,0,4,2,2,4,2,2)	A_5	(2,3)(4,5)	(1,2)(4,5)	(1)	(1,2)(4,5)	(1,4)(2,5)	*P, R, QS, QRQ, TPTQ, QTPT QPTPQ* / (Figure 5.29)
$\Gamma_{9,3}(5)$= Γ(5,2,2,0,3,3,2,5,2,2)	A_5	(2,3)(4,5)	(1,2)(3,4)	(2,5)(3,4)	(2,5)(3,4)	(1)	*P, R, S, T, QSRQ, QPRQPQ, QPTPQ* /(Figure 5.30)
	S_5	(2,3)(4,5)	(1,2)(3,4)	(2,5)(3,4)	(2,5)(3,4)	(3,4)	*P, R, S, T, QSRQ, QPRQPQ, QRSQ, QPTQPQ,QPSQPQ* / (Figure 5.31)

Table 5.1: The index 5 subgroup of some Coxeter groups $\Gamma = \langle P, Q, R, S, T\rangle$ with their generators.

In Table 5.1, we present the index 5 subgroups of some Coxeter groups $\Gamma = \langle P, Q, R, S, T\rangle$ derived in Section 3.3 together with their generating sets.

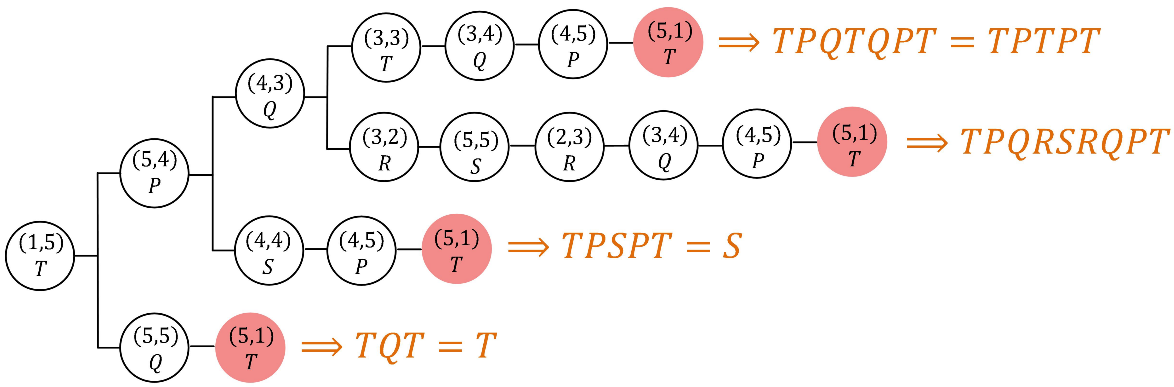


Figure 5.16: The *GenTree* for the index 5 subgroup of $\Gamma_{1,6}(6)$ arising from $\{(45),(34),(23),(1),(15)\}$.

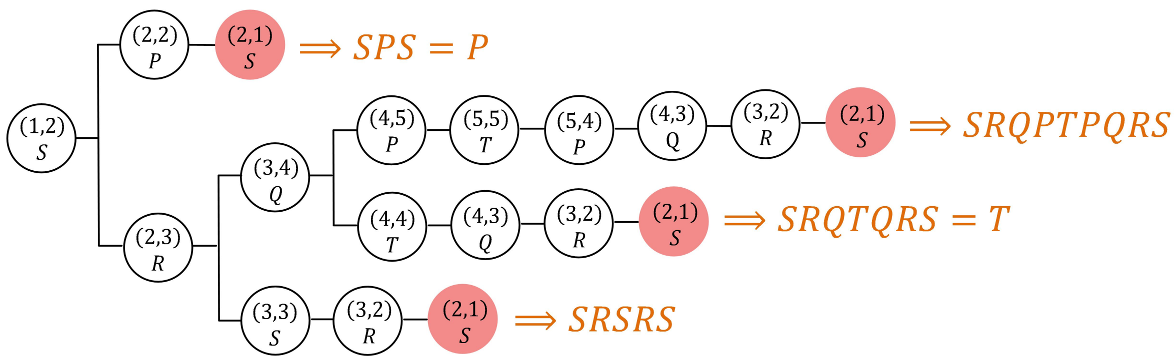


Figure 5.17: The *GenTree* for the index 5 subgroup of $\Gamma_{1,6}(6)$ arising from $\{(45),(34),(23),(12),(1)\}$.

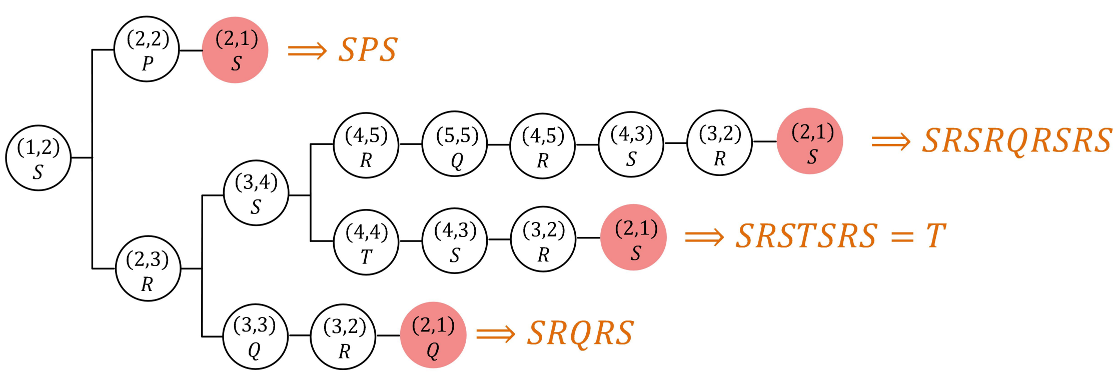


Figure 5.18: The *GenTree* for the index 5 subgroup of $\Gamma_{2,4}(5)$ arising from $\{(23)(45),(12)(45),(1),(24)(35),(1)\}$.

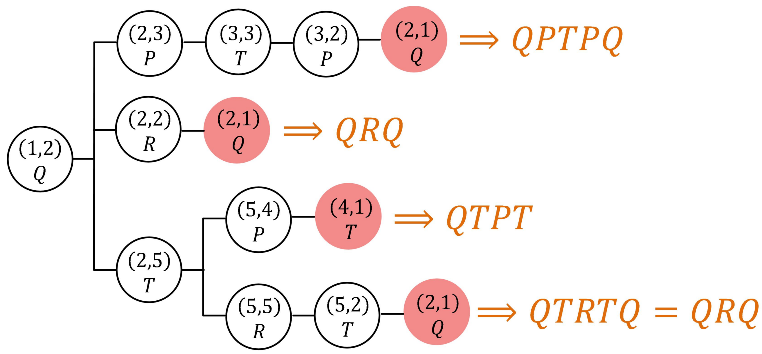


Figure 5.19: The *GenTree* for the index 5 subgroup of $\Gamma_{2,4}(5)$ arising from $\{(23)(45),(12)(45),(45),(24)(35),(1)\}$.

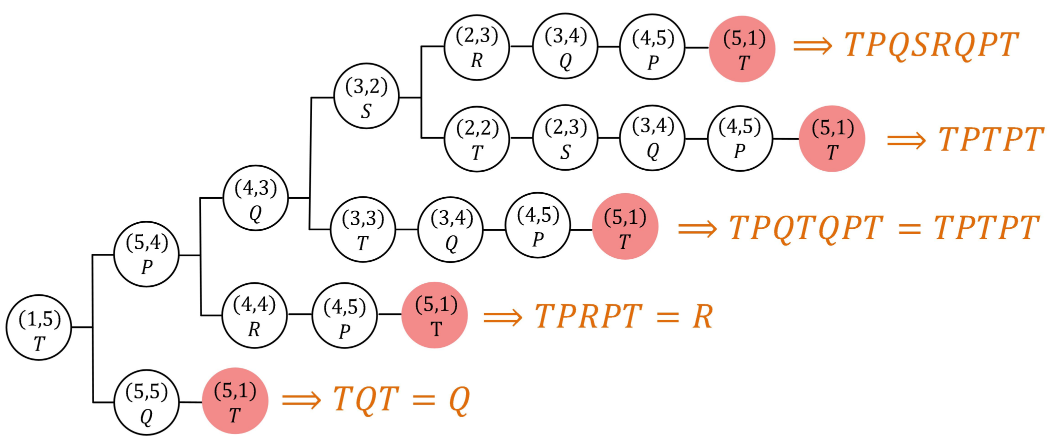


Figure 5.20: The *GenTree* for the index 5 subgroup of $\Gamma_{3,3}(5)$ arising from $\{(45),(34),(23),(23),(15)\}$.

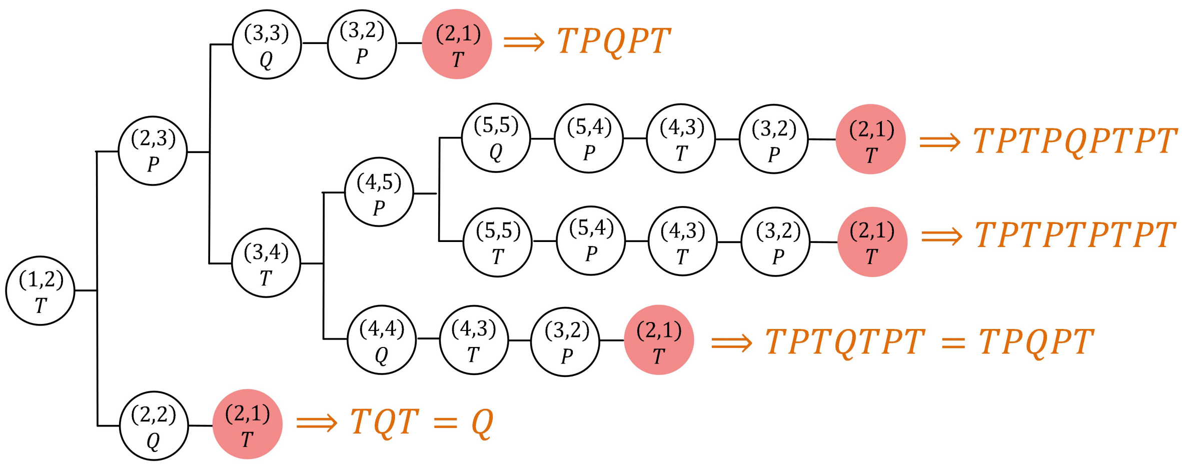


Figure 5.21: The *GenTree* for the index 5 subgroup of $\Gamma_{4,6}(6)$ arising from $\{(23)(45),(1),(1),(1),(12)(34)\}$.

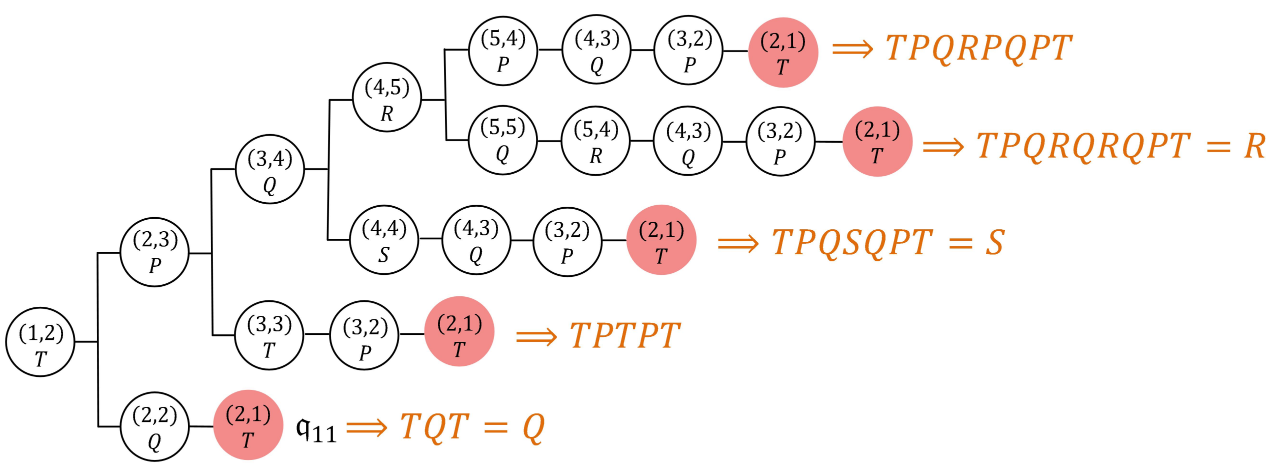


Figure 5.22: The *GenTree* for the index 5 subgroup of $\Gamma_{4,6}(6)$ arising from $\{(23)(45),(34),(45),(1),(12)\}$.

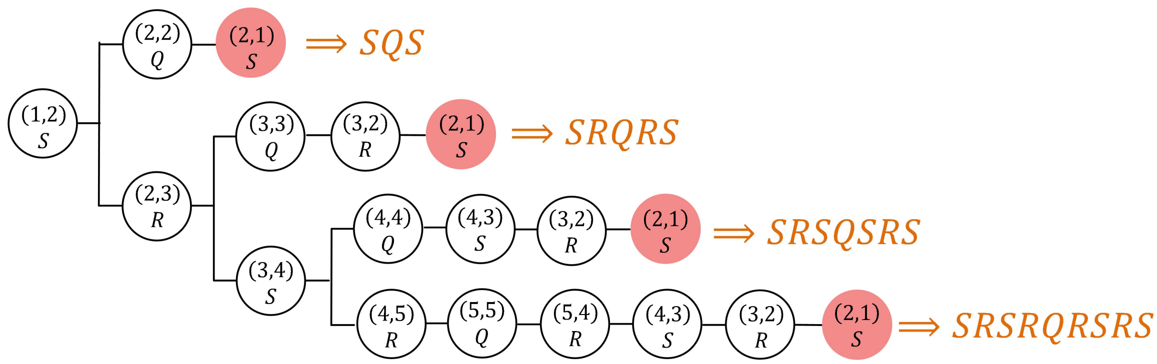


Figure 5.23: The *GenTree* for the index 5 subgroup of $\Gamma_{5,2}(5)$ arising from $\{(1), (1), (23)(45), (12)(34), (1)\}$.

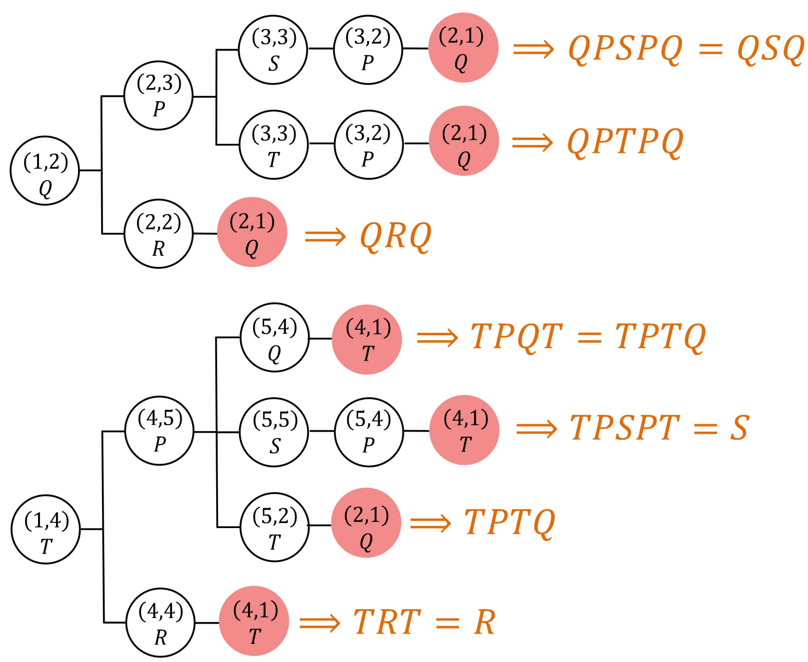


Figure 5.24: The *GenTree* for the index 5 subgroup of $\Gamma_{5,2}(5)$ arising from $\{(23)(45), (12)(45), (1), (1), (14)(25)\}$.

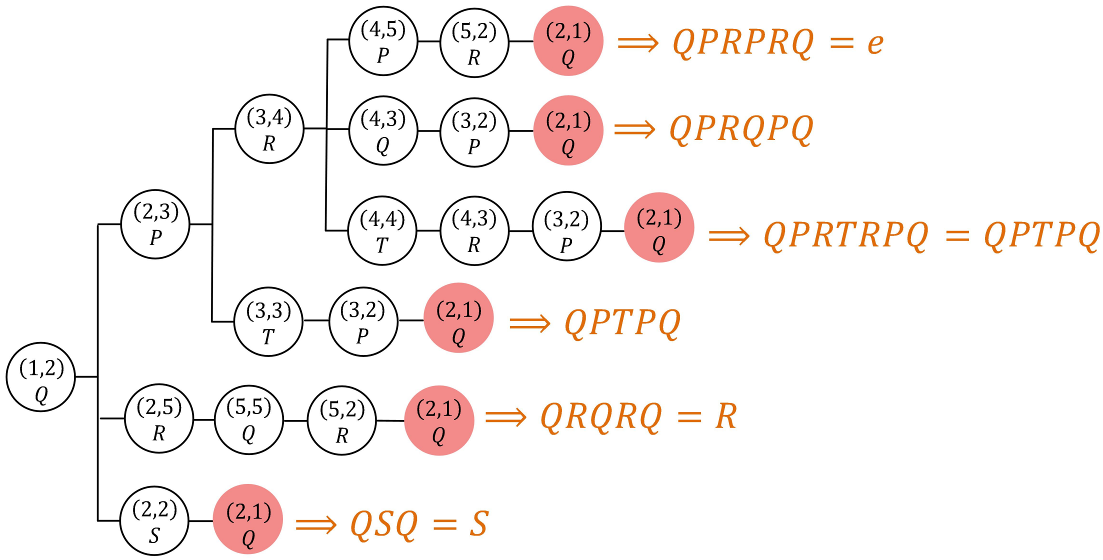


Figure 5.25: The *GenTree* for the index 5 subgroup of $\Gamma_{6,6}(6)$ arising from $\{(23)(45), (12)(34), (25)(34), (1), (1)\}$.

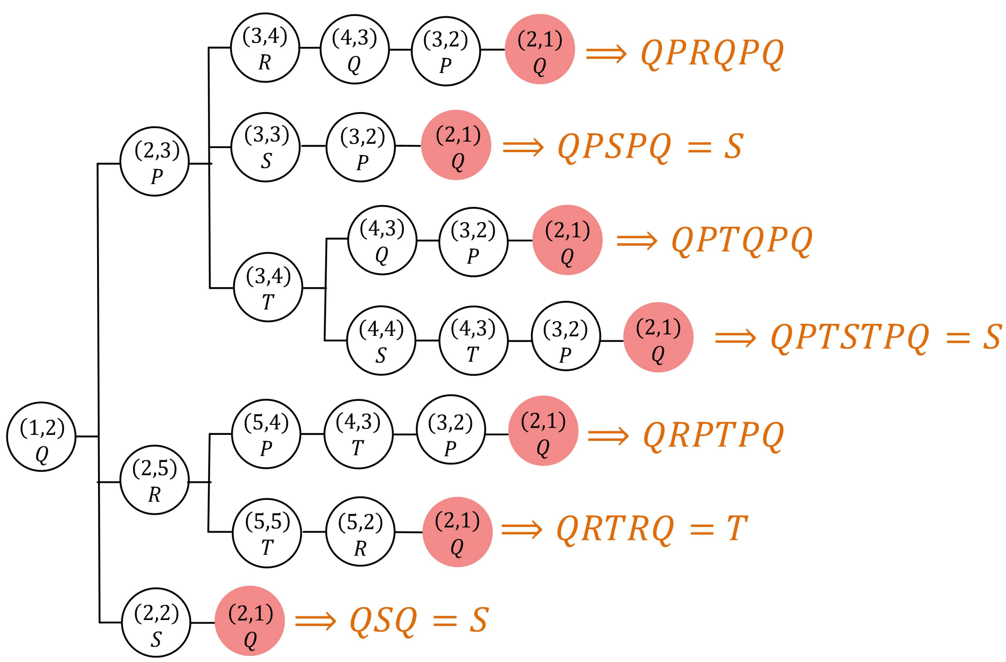


Figure 5.26: The *GenTree* for the index 5 subgroup of $\Gamma_{6,6}(6)$ arising from $\{(23)(45), (12)(34), (25)(34), (1), (34)\}$.

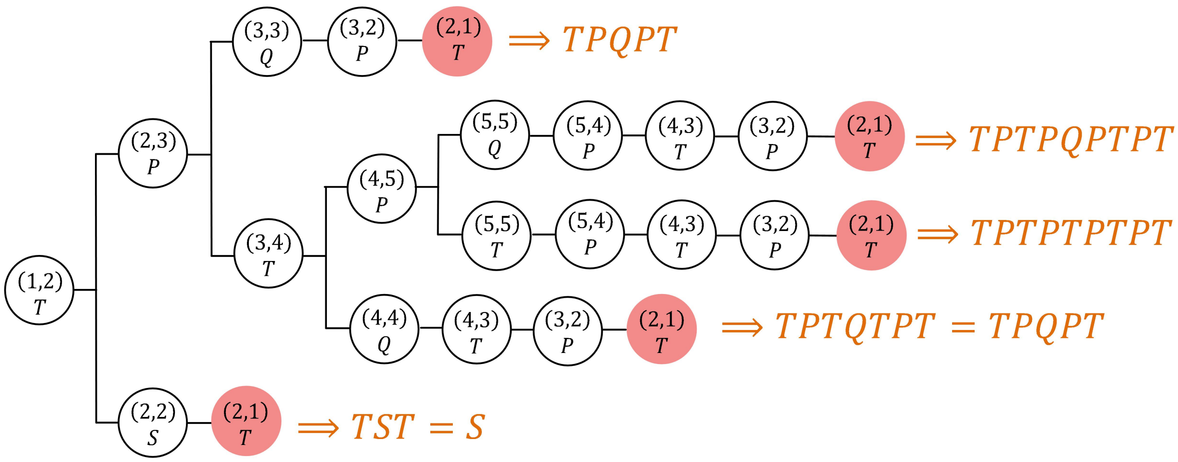


Figure 5.27: The *GenTree* for the index 5 subgroup of $\Gamma_{7,3}(5)$ arising from $\{(23)(45),(1),(1),(1),(12)(34)\}$.

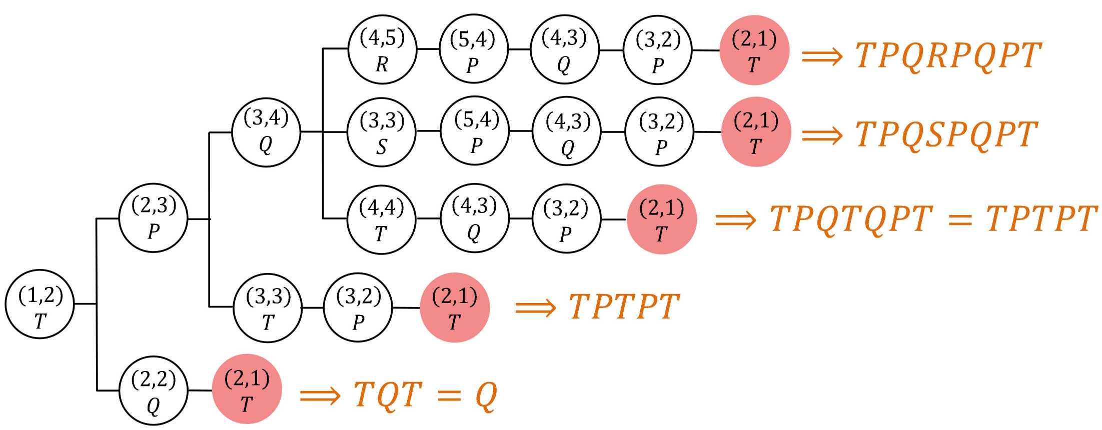


Figure 5.28: The *GenTree* for the index 5 subgroup of $\Gamma_{7,3}(5)$ arising from $\{(23)(45),(34),(45),(45),(12)\}$.

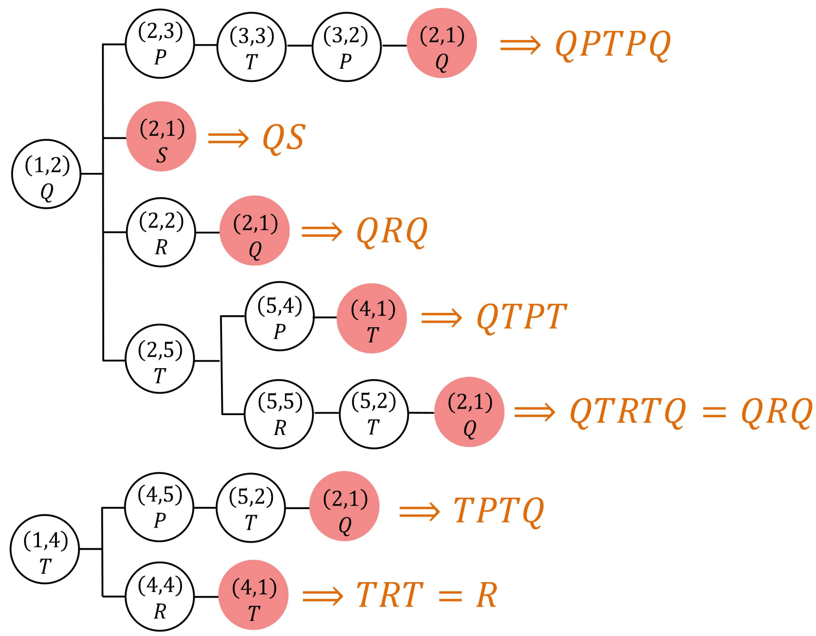


Figure 5.29: The *GenTree* for the index 5 subgroup of $\Gamma_{8,4}(4)$ arising from $\{(23)(45), (12)(45), (1), (12)(45), (14)(25)\}$.

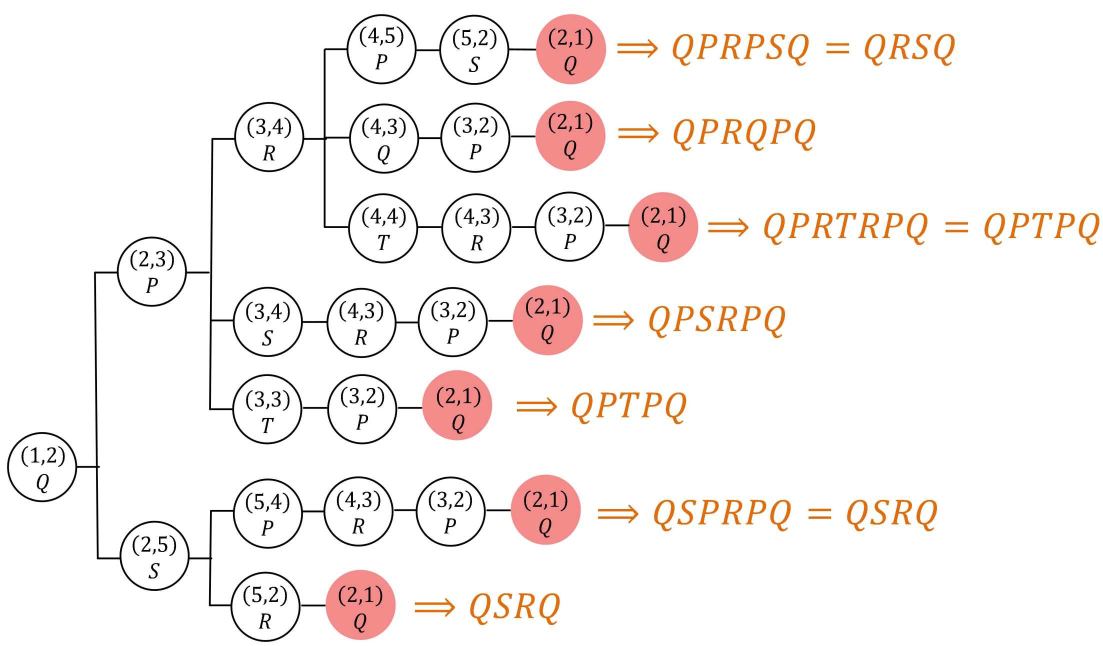


Figure 5.30: The *GenTree* for the index 5 subgroup of $\Gamma_{9,3}(5)$ arising from $\{(23)(45), (12)(34), (25)(34), (25)(34), (1)\}$.

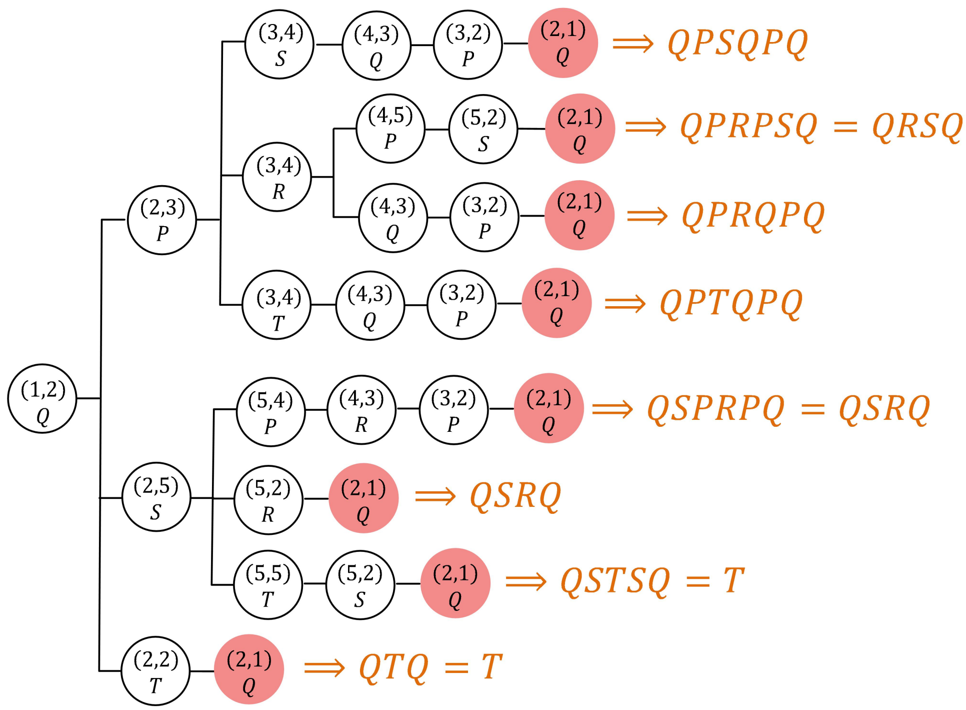


Figure 5.31: The $GenTree$ for the index 5 subgroup of $\Gamma_{9,3}(5)$ arising from $\{(23)(45), (12)(34), (25)(34), (25)(34), (34)\}$.

For our last example, we are going to derive a set of generators for each of the index 6 subgroup Ω of the orientation-preserving group $\Gamma^0(3,2,\infty)$ by constructing its corresponding $GenTree(\Omega)$. In the following discussion, we are going to look closely at the construction of $GenTree(\Omega''')$ where Ω''' is the index 6 subgroup of $\Gamma^0(3,2,\infty)$, where $\pi(\Gamma^0(3,2,\infty)) \cong F_{18}(6)$, arising from the permutation assignment $\tau''' = \{(123456), (12)(34)(56)\}$.

From the permutation assignment $\tau''' = \{(123456), (12)(34)(56)\}$, we have $\pi(Q'P') = (123456)$, $\pi(P'Q') = (165432)$ and $\pi(R'P') = (12)(34)(56)$. The set of vertices $\mathcal{V}'''$ of $GenTree(\Omega''')$ is $\{(Q'P', (1,2)), (P'Q', (2,1)), (Q'P',\ (2,3)), (P'Q',\ (3,2)), (Q'P',\ (3,4)), (P'Q',\ (4,3)), (Q'P',\ (4,5)), (P'Q',\ (5,4)), (Q'P', (5,6)), (P'Q',\ (6,5)), (Q'P',\ (6,1)), (P'Q',\ (1,6)), (R'P',\ (1,2)), (R'P',\ (2,1)), (R'P', (3,4)), (R'P',\ (4,3)), (R'P', (5,6)), (R'P', (6,5))\}$. Now, the set of roots for $GenTree(\Omega''')$ is $\mathcal{W}''' = \{(P'Q', (1,6)), (Q'P', (1,2)), (R'P', (1,2))\}$. Since the roots $(Q'P', (1,2))$ and $(R'P', (1,2))$ have the same ordered pair, by Rule 3, we use $(R'P',\ (1,2))$ together with $(P'Q', (1,6))$ in the construction.

We connect the root $(P'Q', (1,6))$ to the vertices $(P'Q', (6,5))$ and $(R'P', (6,5))$. Since the vertices $(P'Q', (6,5))$ and $(R'P', (6,5))$ have the same ordered pair, by Rule 3, we only branch out from one of them, say $(R'P', (6,5))$, and so we connect the vertex $(R'P', (6,5))$ to $(P'Q', (5,4))$ and $(Q'P', (5,6))$. Appying Rule 5, since $(R'P', (6,5))$ and $(Q'P', (5,6))$ are two consecutive vertices with reverse ordered pairs, branch out from $(Q'P', (5,6))$ to $(Q'P', (6,1))$ arising in the path $\mathfrak{p}_{41}$. We will show later that we can consider $(P'Q', (5,4))$ a dead node.

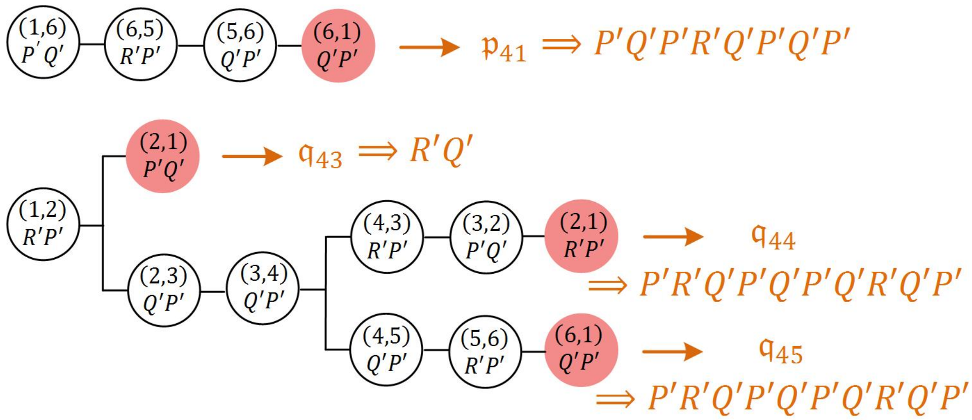


Figure 5.32: The $GenTree$ for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong F_{18}(6)$.

On the other hand, we connect the root $(R'P', (1,2))$ to $(Q'P', (2,3))$ and $(P'Q', (2,1))$ which yields the path $\mathfrak{q}_{43}$. We connect $(Q'P', (2,3))$ to $(Q'P', (3,4))$ and $(R'P', (3,4))$ but, by applying Rule 3, we only branch out from one of them, say $(Q'P', (3,4))$. Now, we connect $(Q'P', (3,4))$ to $(R'P', (4,3))$ and $(Q'P', (4,5))$. Since $(Q'P', (3,4))$ and $(R'P', (4,3))$ are two consecutive vertices with reverse ordered pairs, by Rule 4, we connect $(R'P', (4,3))$ to the subpath $(P'Q', (3,2))$, $(R'P', (2,1))$ giving rise to the path $\mathfrak{q}_{44}$. The vertex $(Q'P', (4,5))$ has the same ordered pair as the vertex $(P'Q', (4,5))$, and so, by Rule 2, we can consider $(P'Q', (4,5))$ a dead node and connect $(Q'P', (4,5))$ to the subpath $(R'P', (5,6))$, $(Q'P', (6,1))$ which yields the path $\mathfrak{q}_{45}$.

In Figure 5.32, we give $GenTree(\Omega''')$ with the corresponding elements in

Ω''' of each path. Hence, we obtain the set

$$\{R'Q', P'Q'P'R'Q'P'Q'P', P'R'Q'P'Q'P'Q'R'Q'P'\}$$

as set of generators for Ω'''.

In Table 5.2, we present the index 6 subgroups of $\Gamma^0(\infty, 2, 3)$ together with their generators.

$\pi(\Gamma^0(\infty,2,3))$	$\pi\ (QP)$	$\pi\ (RP)$	$\pi\ (QR)$	Generators	GenTree
C_6	(123456)	(14)(25)(36)	(153)(264)	$Q'P'Q'R'Q'P', Q'R'Q'P'Q'P'$	Figure 5.33
$S_3(6)$	(12)(34)(56)	(13)(25)(46)	(154)(236)	$Q'P'Q'P', R'Q'P'Q'R'P'$	Figure 5.34
$A_4(6)$	(123)(456)	(25)(36)	(153)(264)	$R'P', Q'P'Q'P'Q'P',$ $Q'P'R'Q'P'Q'R'P'Q'P'$	Figure 5.35
$F_{18}(6)$	(123456)	(12)(34)(56)	(246)	$R'Q', P'Q'P'R'Q'P'Q'P',$ $P'R'Q'P'Q'P'Q'R'Q'P'$	Figure 5.32
$2A_4(6)$	(123456)	(36)	(126)(345)	$Q'R'P'Q', P'Q'P'Q'R'P'Q'P'Q'P',$ $P'Q'P'R'Q'P'Q'P'Q'P'Q'P', R'P'$	Figure 5.36
$S_4(6c)$	(2356)	(12)(36)(45)	(126)(345)	$Q'P', R'Q'P'Q'P'Q'P'Q'R'P'$	Figure 5.37
$S_4(6d)$	(1245)(36)	(23)(56)	(135)(246)	$R'P', Q'P'Q'P'Q'P'Q'P',$ $Q'P'Q'R'Q'P'Q'P'$	Figure 5.38
$A_5(6)$	(23564)	(12)(34)	(124)(356)	$Q'P', R'Q'P'Q'R'P'Q'P'Q'R',$ $P'R'Q'P'Q'R'P'Q'P'Q'R'P'$	Figure 5.39

Table 5.2: The index 6 subgroups of $\Gamma^0(\infty, 2, 3)$ with their generators.

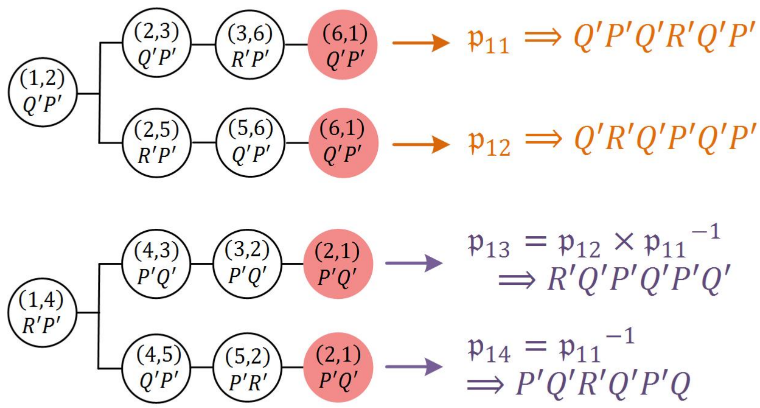


Figure 5.33: The *GenTree* for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong C_6$.

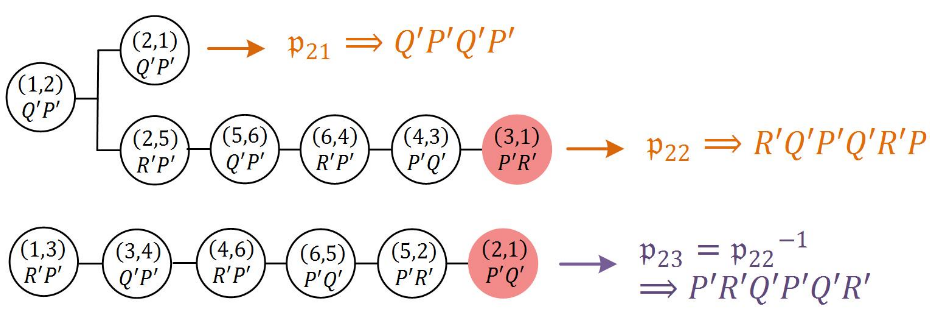


Figure 5.34: The *GenTree* for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong S_3$.

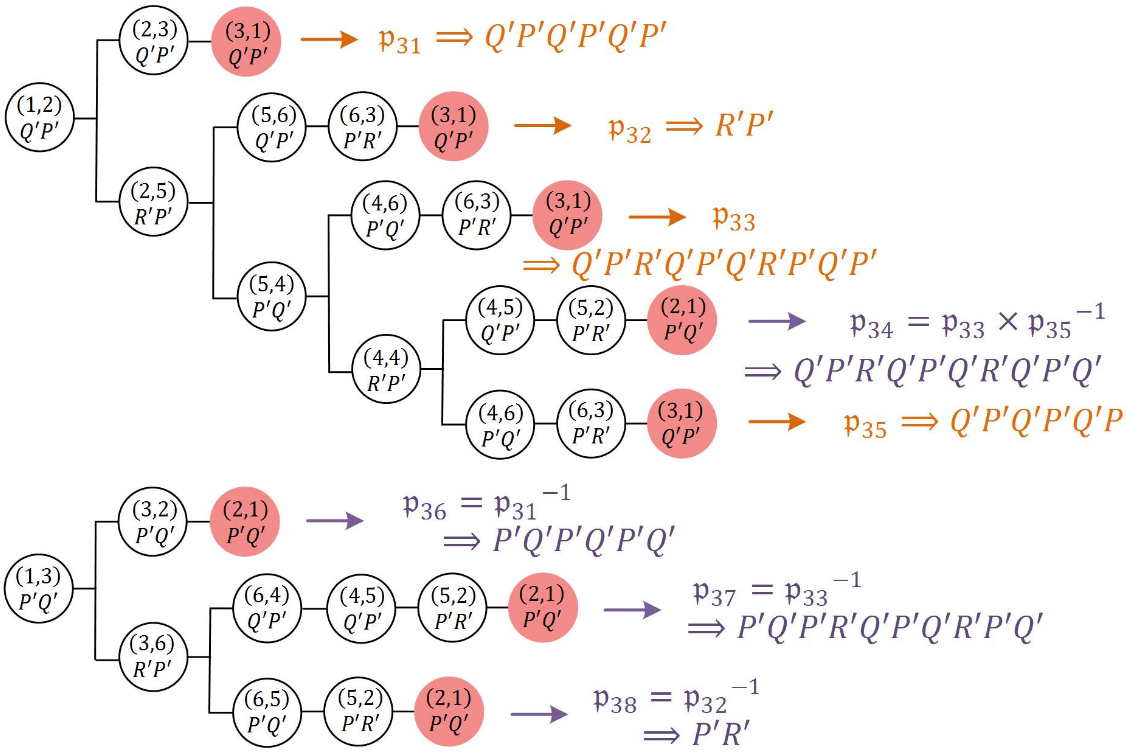


Figure 5.35: The $GenTree$ for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong A_4(6)$.

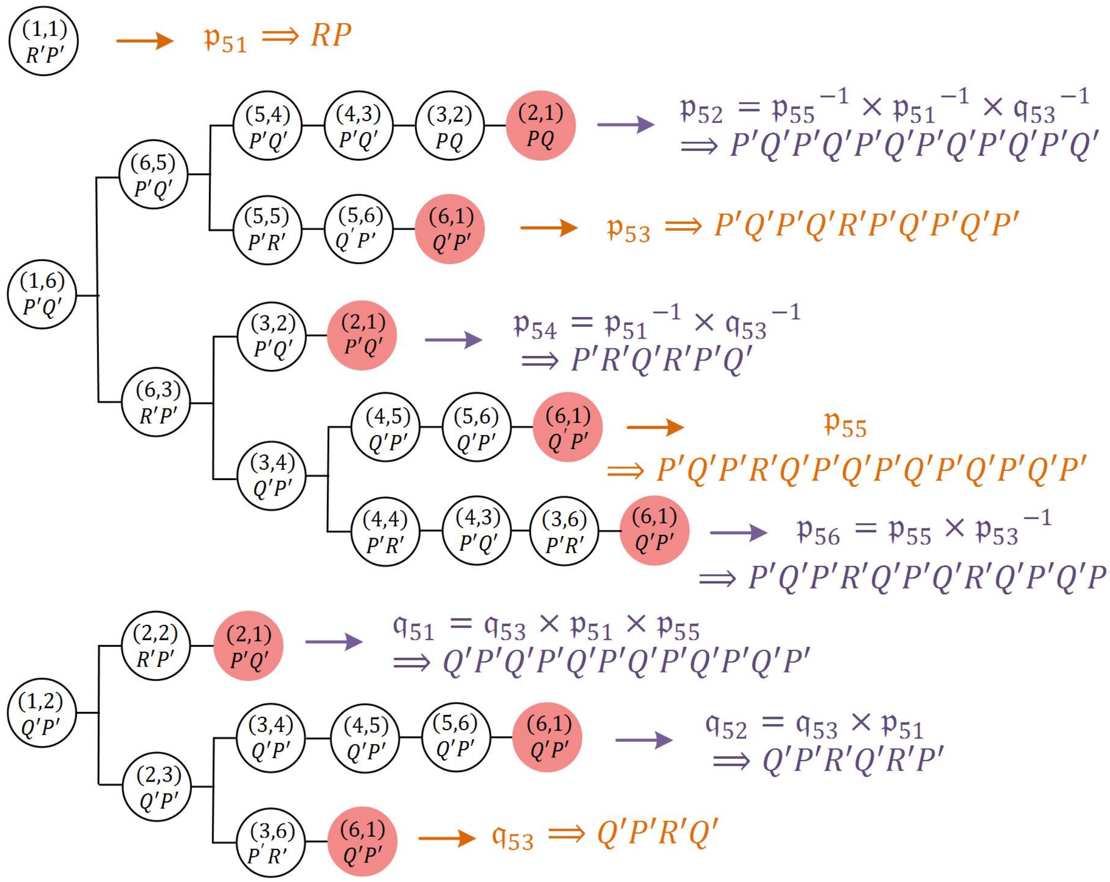


Figure 5.36: The $GenTree$ for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong 2A_4(6)$.

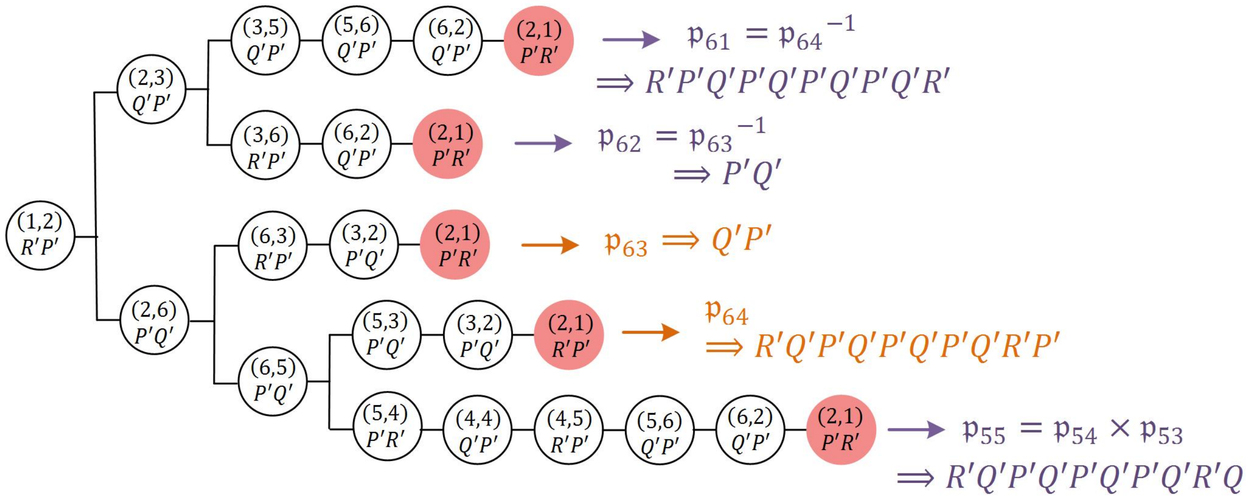


Figure 5.37: The *GenTree* for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong S4(6c)$.

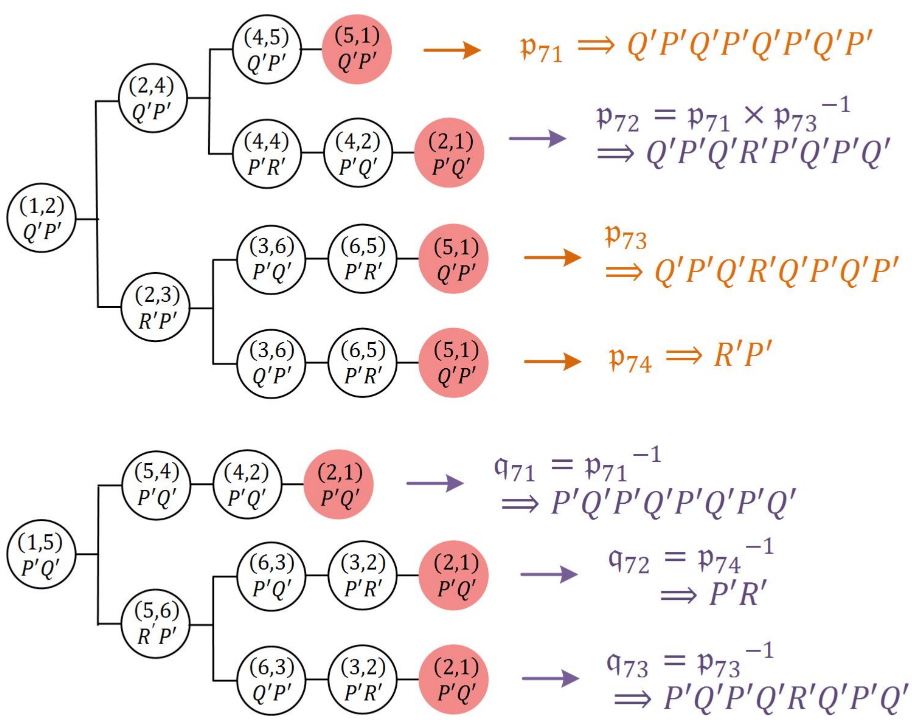


Figure 5.38: The *GenTree* for the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$ where $\pi(\Gamma^0(\infty, 2, 3)) \cong S4(6d)$.

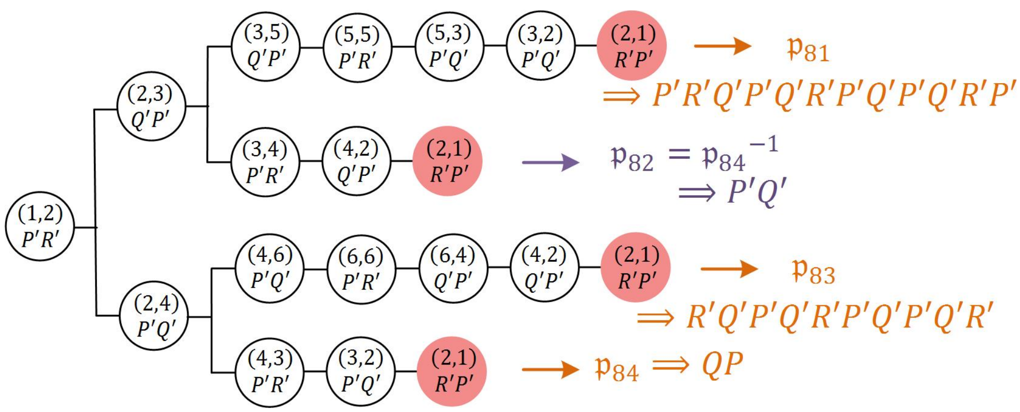


Figure 5.39: The $GenTree$ for the index 6 subgroup of $\Gamma^0(\infty,2,3)$ where $\pi(\Gamma^0(\infty,2,3)) \cong A_5$.

Chapter 6

TORSION-FREE SUBGROUPS OF COXETER GROUPS

In this chapter, we present a method for identifying subgroups of Coxeter groups which are torsion-free; that is, subgroups which contain no nontrivial elements of finite order. In identifying torsion-free subgroups the starting point would be to work inside Γ^0, the group of orientation-preserving isometries in Γ to avoid the presence of reflections, which are order two elements.

6.1 Torsion-free Subgroups of Γ^0

Let us consider the set $\mathbb{P}$ of permutation assignments corresponding to the Γ^0-transitive n-colorings of the Γ^0-orbit of D that give rise to index n subgroups of Γ^0. We want to address the question: among these index n subgroups of Γ^0, which are torsion-free?

Let us first look at the following result which contributes to the solution of the problem.

Theorem 6.1 *Let τ be a permutation assignment in $\mathbb{P}$ which gives rise to an index n subgroup Ω of Γ^0. Suppose τ corresponds to an n-coloring of the Γ^0-*

orbit of D where Γ^0 acts transitively on the set of colors $c_1, c_2, \ldots, c_n$ (denoted by $1, 2, \ldots, n$). Assume that for such a coloring, a homomorphism $\pi : \Gamma^0 \to S_n$ is defined. Consider $\tilde{x} \in \Gamma^0$.

(i) If $\pi(\tilde{x})(f) = f$ for $f \in \{2, \ldots, n\}$, then $\sigma\tilde{x}\sigma^{-1} \in \Omega$ where $\sigma \in \Gamma^0$.

(ii) If $\pi(\tilde{x})$ has order ϑ then $\tilde{x}^\vartheta \in \Omega$.

Proof

(i) Since Γ^0 acts transitively on C, there exists $\sigma \in \Gamma^0$ such that $\sigma c_1 = c_f$, that is, $\pi(\sigma)(1) = f$. This implies that $\pi(\sigma^{-1})(f) = 1$. Thus, $[\pi(\sigma\tilde{x}\sigma^{-1})](1) = 1$ or $\sigma\tilde{x}\sigma^{-1} \in \Omega$.

(ii) Note that $\pi(\tilde{x}^\vartheta)(1) = [\pi(\tilde{x})]^\vartheta(1) = 1$. Thus $\tilde{x}^\vartheta \in \Omega$. ∎

Aside from the previous theorem, the following well-known result from [54, 30, 13] will also be used in determining torsion-free subgroups of Γ^0.

Lemma 6.2 *An element $\zeta \in \Gamma^0$ is of finite order if and only if it is a conjugate to a power of a relator of Γ^0.*

Let $\tilde{y}$ be a relator of Γ^0 of order d. By Theorem 6.1 (i), if $\pi(\tilde{y})$ fixes a number from the set $\{2, 3, \ldots, n\}$, then $\sigma'\tilde{y}\sigma'^{-1} \in \Omega$ for some $\sigma' \in \Gamma^0$. From Lemma 6.2, $\sigma'\tilde{y}\sigma'^{-1}$ has finite order and Ω is not torsion-free.

Moreover, suppose $\pi(\tilde{y})$ is of order $d' < d$. By Theorem 6.1 (ii) $\tilde{y}^{d'} \in \Omega$. Now, since d' divides d, we can write $d = d'z'$ where z' is an integer not equal to 1 so that $\tilde{y}^{d'} = \tilde{y}^{\frac{d}{z'}}$. Note that $(\tilde{y}^{d'})^{z'} = (\tilde{y}^{\frac{d}{z'}})^{z'} = \tilde{y}^d = e$. This implies $\tilde{y}^{d'}$ is of order z' and thus is of finite order. Hence, Ω is not torsion-free.

Based on these arguments, we can conclude that Ω avoids any torsion when for each relator $\tilde{y}$ of Γ^0, $\pi(\tilde{y})$ does not fix any number $f \in \{1, 2, \ldots, n\}$, and the order of $\pi(\tilde{y})$ is equal to the order of $\tilde{y}$. To satisfy these two conditions, $\pi(\tilde{y})$ should be a permutation having $\frac{n}{d}$ disjoint d-cycles.

Consider $\tau \in \mathbb{P}$. Suppose that, with respect to this permutation assignment, each relator $\tilde{y}$ of Γ^0 of order d is such that $\pi(\tilde{y})$ is a permutation having $\frac{n}{d}$ disjoint d-cycles. Then we say that τ is a *semiregular permutation assignment*.

We now present the main theorem of this chapter.

Theorem 6.3 *A subgroup Ω of Γ^0 is torsion-free if and only if the permutation assignment $\tau \in \mathbb{P}$ that gives rise to Ω is a semiregular permutation assignment.*

Proof

Suppose $\tau \in \mathbb{P}$ is a semiregular permutation assignment that gives rise to a subgroup Ω of Γ^0. Consider $\tilde{z}$ an element of order g in Γ^0. Then, by Lemma 6.2, $\tilde{z}$ is a conjugate to a power of a relator, say $\tilde{v}$; that is, $\tilde{z} = \tilde{w}\tilde{v}\tilde{w}^{-1}$ for some $\tilde{w} \in \Gamma^0$. Since $\pi(\tilde{z}) = \pi(\tilde{w}\tilde{v}\tilde{w}^{-1}) = \pi(\tilde{w})\pi(\tilde{v})\pi(\tilde{w})^{-1}$ and $\pi(\tilde{v})$ is a permutation having $\frac{n}{g}$ disjoint g-cycles, $\pi(\tilde{z})$ is a permutation having $\frac{n}{g}$ disjoint g-cycles. This implies that $\tilde{z}$ does not fix 1. Thus, every element of finite order in Γ^0 is not an element of Ω and Ω is torsion-free.

Conversely, suppose Ω is a torsion-free subgroup of Γ^0. We want to show that its corresponding permutation assignment $\tau \in \mathbb{P}$ is semiregular.

Let $\tilde{u}$ be an element of Γ^0 of order g'. Since Ω is torsion-free, $\tilde{u}^{s'} \in \Omega$ if and only if g' divides s', that is $y^{s'} = (y^{g'})^{r'} = e$, $r' \in \mathbb{Z}$. If s' is not divisible by g', then $\pi(\tilde{u}^{s'})$ does not fix 1. Consequently, $\pi(\tilde{u}^{s'})$ does not fix $j \in \{1, 2, \ldots, n\}$ since $\pi(\tilde{b}\tilde{u}^{s'}\tilde{b}^{-1})$ does not fix 1 for any $\tilde{b} \in \Gamma^0$ by Theorem 6.1. Therefore, $\pi(\tilde{u})$ is a permutation of order g', and no nontrivial power of $\pi(\tilde{u})$ fixes any number in $\{1, 2, \ldots, n\}$. This implies that $\pi(\tilde{u})$ is a permutation having $\frac{n}{g'}$ disjoint g'-cycles. Particularly, for every relator $\tilde{a}$ of order t', $\pi(\tilde{a})$ is necessarily represented as a product of $\frac{n}{t'}$ disjoint t'-cycles. Hence, the permutation assignment τ corresponding to the torsion-free subgroup Ω is semiregular. ∎

At this point, it is naturally of interest to know what the smallest possible index of a torsion-free subgroup of Γ^0 is. A simple corollary to the previous theorem gives us a useful lower bound.

Corollary 6.4 *The index of any finite-index torsion-free subgroup of a group* Γ^0 *is divisible by the lowest common multiple of the orders of the relators.*

Proof Suppose Ω is an index n subgroup which arises from the permutation assignment τ. By Theorem 6.3, τ is a semiregular permutation. Hence, for each relator $\tilde{c}$ of order m', $\pi(\tilde{c})$ is a permutation having $\frac{n}{m'}$ disjoint m'-cycles. Thus, n is divisible by the lowest common multiple of the orders of the relators. ∎

To illustrate the process for determining torsion-free subgroups of a Coxeter group, let us consider the Euclidean triangle group $*442$ given by

$$\Gamma(2,4,4) = \langle P', Q', R' | P'^{\,2} = Q'^{\,2} = R'^{\,2} = (Q'P')^2 = (R'P')^4 = (Q'R')^4 = e\rangle.$$

The orientation-preserving subgroup of $\Gamma(2,4,4)$ is

$$\Gamma^0(2,4,4) = \langle Q'P', R'P' | (Q'P')^2 = (R'P')^4 = (Q'R')^4 = e\rangle.$$

The group $*442$ using the International Union of Crystallographers (IUC) notation is the group $p4m$.

Now, the relators of $\Gamma(2,4,4)$ are $Q'P'$, $R'P'$, and $Q'R$ which are of orders 2,4 and 4, respectively. Hence, by Corollary 6.4, the torsion-free subgroups of $\Gamma^0(2,4,4)$ is at least of index 4.

We consider the permutation assignments that give rise to index 4 subgroups of $\Gamma^0(2,4,4)$ up to conjugacy and determine which yield a torsion-free subgroup. Using the method discussed in Chapter 4, we obtain the permutation assignments presented in Table 6.1.

Consider the permutation assignment $\tau = \{(12)(34), (1324)\}$ from the list. Consequently, $\pi(Q'P') = (12)(34)$, $\pi(R'P') = (1324)$ and $\pi(Q'R') = (1324)$. Since $Q'P'$ is of order 2, $R'P'$ is of order 4 and $Q'R'$ is of order 4 and $\pi(Q'P')$, $\pi(R'P')$ and $\pi(Q'R')$ are permutations having two 2-cycles, a 4-cycle and a 4-cycle, respectively, then the permutation assignment τ is semiregular. Hence, this permutation assignment gives rise to an index 4 torsion-free subgroup Ω' of $\Gamma^0(2,4,4)$.

$\pi(\Gamma^0(2,4,4))$	$\pi(Q'P')$	$\pi(R'P')$	$\pi(Q'R')$
D_4	(34)	(1324)	$(13)(24)$
D_4	(34)	$(13)(24)$	(1324)
D_4	$(13)(24)$	(12)	(1324)
Z_4	$(12)(34)$	(1324)	(1324)
K_4	$(12)(34)$	$(14)(23)$	$(13)(24)$

Table 6.1: Permutation assignments that give rise to the index 4 subgroups of $\Gamma^0(2,4,4)$.

By constructing GenTree(Ω') (given in Figure 6.1), we obtain a set of generators $\{Q'R'P'R', R'P'R'Q', R'Q'R'P\}$ for this subgroup.

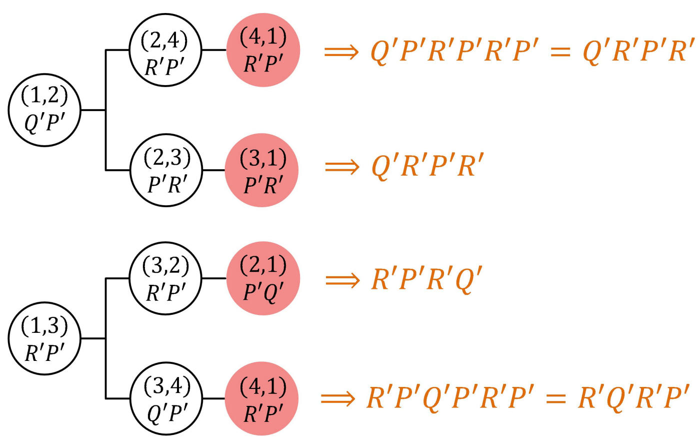


Figure 6.1: The GenTree(Ω') for $\tau = \{(12)(34), (1324)\}$ of $\Gamma^0(2,4,4)$.

Observe that the other permutation assignments appearing in Table 6.1 are not semiregular permutation assignments. Thus, $\Gamma^0(2,4,4)$ has only one index 4 subgroup which is torsion-free. This torsion-free subgroup is more commonly known in IUC notation as $p1$ generated by two translations.

For our next example, let us consider the extended Hecke group $*p'2\infty$

given by

$$\Gamma(\infty, 2, p') = \langle P', Q', R' | P'^{\,2} = Q'^{\,2} = R'^{\,2} = (R'P')^2 = (Q'R')^{p'} = e \rangle.$$

Its orientation preserving subgroup known as the Hecke group $p'2\infty$ is

$$\Gamma^0(\infty, 2, p') = \langle Q'P', R'P' | (R'P')^2 = (Q'R')^{p'} = e \rangle.$$

Let us focus on three examples of $p'2\infty$, namely: the modular group 32∞ and the groups 42∞ and 62∞. By Corollary 6.4, the torsion-free subgroups of these groups are at least of index 6, 4 and 6, respectively.

Based on the discussion in Section 4.4, we obtain the permutation assignments in Table 6.2, Tables 6.3 and 6.4, Table 6.5 that will give rise to index 6, index 6 and index 4 subgroups of $\Gamma^0(\infty, 2, 3)$, $\Gamma^0(\infty, 2, 6)$, $\Gamma^0(\infty, 2, 4)$, respectively. It is easy to verify that the permutation assignments

$$\tau_1' = \{(123456), (14)(25)(36)\},$$

$$\tau_2' = \{(12)(34)(56), (13)(25)(46)\},$$

$$\tau_3' = \{(2356), (12)(36)(45)\}$$

given in Table 6.2 are semiregular.

Hence, the index 6 subgroups

$$\Omega_1' = \langle Q'P'Q'R'Q'P', Q'R'Q'P'Q'P' \rangle,$$

$$\Omega_2' = \langle Q'P'Q'P', R'Q'P'Q'R'P' \rangle,$$

$$\Omega'_3 = \langle Q'P', R'Q'P'Q'P'Q'P'Q'R'P' \rangle$$

which arise from τ'_1, τ'_2 and τ'_3, respectively, are torsion-free subgroups of $\Gamma^0(\infty, 2, 3)$.

$\Gamma^0(\infty,2,3)$	$\pi(Q'P')$	$\pi(R'P')$	$\pi(Q'R')$
C_6	(123456)	$(14)(25)(36)$	$(153)(264)$
$S_3(6)$	$(12)(34)(56)$	$(13)(25)(46)$	$(154)(236)$
$A_4(6)$	$(123)(456)$	$(25)(36)$	$(153)(264)$
$F_{18}(6)$	(123456)	$(12)(34)(56)$	(246)
$2A_4(6)$	$(12)(3546)$	$(13)(24)$	$(146)(235)$
$S_4(6c)$	(2356)	$(12)(36)(45)$	$(126)(345)$
$S_4(6d)$	(3546)	$(13)(24)(56)$	$(136)(245)$
$A_5(6)$	(23564)	$(12)(34)$	$(124)(356)$

Table 6.2: Permutation assignments that give rise to the index 6 subgroup of $\Gamma^0(\infty, 2, 3)$.

Similarly, the permutation assignments $\tau''_1 = \{ (135)(246), (14)(25)(36), (165432) \}$, $\tau''_2 = \{ (34)(56), (15)(23)(46), (154236) \}$, $\tau''_3 = \{ (246), (12)(34)(56), (123456) \}$, $\tau''_4 = \{(12)(3546), (13)(24)(56), (145236)\}$ and $\tau''_5 = \{(23645), (12)(34)(56), (124635)\}$ in Tables 6.3 and 6.4, $\tau'''_1 = \{(34), (13)(24), (1324)\}$ and $\tau'''_2 = \{(1234), (13)(24), (1432)\}$ in Table 6.5 are semiregular. Thus, $\Gamma^0(\infty, 2, 6)$ has 5 index 6 torsion-free subgroups which arise from τ''_1, τ''_2, τ''_3,τ''_4, τ''_5, and $\Gamma^0(\infty, 2, 4)$ has 2 index 4 torsion-free subgroups which arise from τ'''_1,τ'''_2.

For our final example, we derive an index 96 torsion-free subgroup Ω^* of the triangle group $*642$ given by

$$\Gamma(2,4,6) = \langle P', Q', R' | P'^2 = Q'^2 = R'^2 = (Q'P')^2 = (R'P')^4 = (Q'R')^6 = e \rangle.$$

This subgroup arises from a permutation assignment τ^* where

$$\pi(Q'P') = (1\ 18)(2\ 36)(3\ 10)(4\ 15)(5\ 33)(6\ 7)(8\ 42)(9\ 22)(11\ 39)(12\ 19)(13\ 24)$$

$$(14\ 28)(16\ 21)(17\ 25)(20\ 46)(23\ 43)(27\ 34)(29\ 48)(30\ 31)(32\ 37)(35\ 40)(38\ 47)(41\ 44);$$

$$\pi(R'P') = (1\ 7\ 19\ 13)(2\ 18\ 25\ 31)(3\ 36\ 40\ 11)(4\ 10\ 22\ 16)(5\ 15\ 28\ 34)(6\ 33\ 37\ 8)$$

$$(9\ 42\ 44\ 23)(12\ 39\ 47\ 20)(14\ 24\ 43\ 29)(17\ 21\ 46\ 26)(27\ 45\ 41\ 35)(30\ 48\ 38\ 32);$$

$$\pi(Q'R') = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)(13\ 14\ 15\ 16\ 17\ 18)(19\ 20\ 21\ 22\ 23\ 24)$$

$$(25\ 26\ 27\ 28\ 29\ 30)(31\ 32\ 33\ 34\ 35\ 36)(37\ 38\ 39\ 40\ 41\ 42)(43\ 44\ 45\ 46\ 47\ 48).$$

Note that τ^* is a semiregular permutation assignment since $\pi(Q'P')$, $\pi(R'P')$, and $\pi(Q'R')$ are permutations having respectively 24 disjoint 2-cycles, 12 disjoint 4-cycles, and 8 disjoint 6-cycles.

The $\Gamma^0(2,4,6)$-transitive 48-coloring of the $\Gamma^0(2,4,6)$-orbit of $\triangle$ arising from τ^* is given in Figure 6.2. The colors are labeled $1, 2, \ldots, 48$.

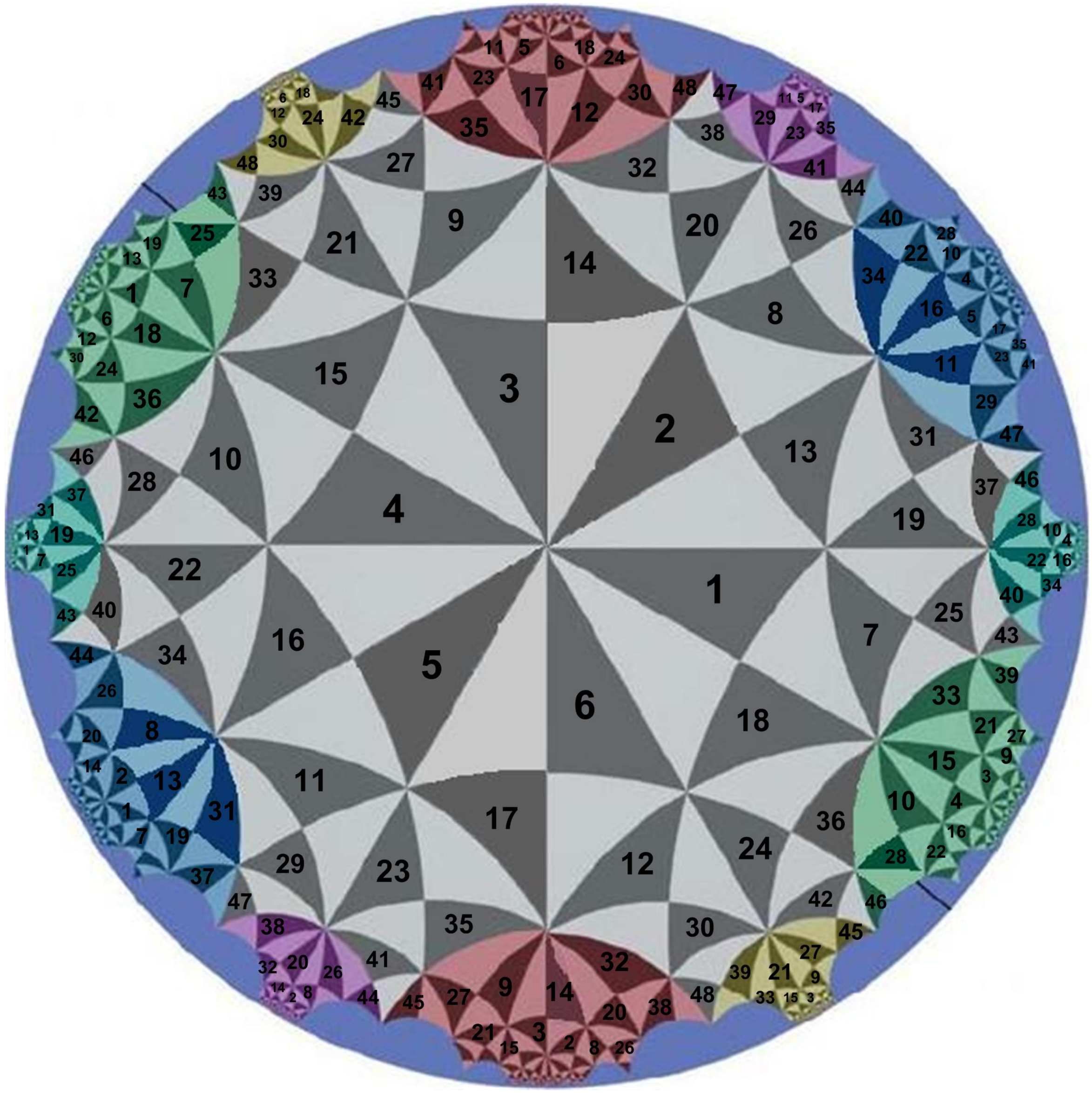


Figure 6.2: A coloring that gives rise to the index 96 torsion-free subgroup of $\Gamma(2,4,6)$.

$\Gamma^0(\infty,2,6)$	$\pi(Q'P')$	$\pi(R'P')$	$\pi(Q'R')$
C_6	$(1,2,3,4,5,6)$	$()$	$(1,2,3,4,5,6)$
C_6	$(1,2,3,4,5,6)$	$(1,4)(2,5)(3,6)$	$(1,5,3)(2,6,4)$
C_6	$(1,3,5)(2,4,6)$	$(1,4)(2,5)(3,6)$	$(1,6,5,4,3,2)$
$S_3(6)$	$(1,2)(3,4)(5,6)$	$(1,3)(2,5)(4,6)$	$(1,5,4)(2,3,6)$
$S_3(6)$	$(1,4,5)(2,6,3)$	$(1,2)(3,4)(5,6)$	$(1,3)(2,5)(4,6)$
$A_4(6)$	$(1,2,3)(4,5,6)$	$(2,5)(3,6)$	$(1,5,3)(2,6,4)$
D_6	$(3,4)(5,6)$	$(1,5)(2,3)(4,6)$	$(1,5,4,2,3,6)$
D_6	$(1,2)(3,6)(4,5)$	$(1,3)(2,5)$	$(1,5,4,2,3,6)$
D_6	$(1,5,4,2,3,6)$	$(3,4)(5,6)$	$(1,6)(2,4)(3,5)$
D_6	$(1,5,4,2,3,6)$	$(1,2)(3,6)(4,5)$	$(1,4)(2,6)$
$F_{18}(6)$	$(2,4,6)$	$(1,2)(3,4)(5,6)$	$(1,2,3,4,5,6)$
$F_{18}(6)$	$(1,2,3,4,5,6)$	$(1,2)(3,4)(5,6)$	$(2,4,6)$
$2A_4(6)$	$(1,2,3)(4,5,6)$	$(3,6)$	$(1,2,6,4,5,3)$
$2A_4(6)$	$(1,2,3,4,5,6)$	$(3,6)$	$(1,2,6)(3,4,5)$
$2A_4(6)$	$(1,2,3,4,5,6)$	$(2,5)(3,6)$	$(1,5,3,4,2,6)$
$S_4(6c)$	$(2,3,5,6)$	$(1,2)(3,6)(4,5)$	$(1,2,6)(3,4,5)$
$S_4(6d)$	$(3,5,4,6)$	$(1,3)(2,4)(5,6)$	$(1,3,6)(2,4,5)$
$2F_{18}(6)$	$(1,2,3,4,5,6)$	$(3,5)(4,6)$	$(1,2,5,4,3,6)$
$2S_4(6)$	$(3,5,4,6)$	$(1,3)(2,4)$	$(1,3,5,2,4,6)$
$2S_4(6)$	$(1,2)(3,5,4,6)$	$(1,3)(2,4)(5,6)$	$(1,4,5,2,3,6)$

Table 6.3: Permutation assignments that give rise to the index 6 subgroups of $\Gamma^0(\infty,2,6)$.

$\Gamma^0(\infty,2,6)$	$\pi(Q'P')$	$\pi(R'P')$	$\pi(Q'R')$
$A_5(6)$	$(2,3,5,6,4)$	$(1,2)(3,4)$	$(1,2,4)(3,5,6)$
$2F_{36}(6)$	$(1,4)(2,5,3,6)$	$(4,5)$	$(1,5,3,6,2,4)$
$2F_{36}(6)$	$(1,4)(2,5,3,6)$	$(1,5)(2,4)(3,6)$	$(1,2)(4,5,6)$
$S_5(6)$	$(3,4,5,6)$	$(1,3)(2,6)$	$(1,3,4,5,2,6)$
$S_5(6)$	$(2,3,6,4,5)$	$(1,2)(3,4)(5,6)$	$(1,2,4,6,3,5)$
$S_5(6)$	$(1,2,3,5,6,4)$	$(2,5)(3,4)$	$(1,5,6,3,2,4)$
S_6	$(2,3)(4,5,6)$	$(1,2)(3,4)$	$(1,2,4,5,6,3)$
S_6	$(2,3)(4,5,6)$	$(1,4)(3,5)$	$(1,4,3,2,5,6)$
S_6	$(2,3)(4,5,6)$	$(1,4)(3,6)$	$(1,4,5,3,2,6)$
S_6	$(2,3,4,5,6)$	$(1,2)$	$(1,2,3,4,5,6)$
S_6	$(2,3,4,5,6)$	$(1,2)(3,6)(4,5)$	$(1,2,6)(3,5)$
S_6	$(1,2,3,4,5,6)$	$(3,6)(4,5)$	$(1,2,6)(3,5)$
S_6	$(1,2,3,4,5,6)$	$(2,3)(4,6)$	$(1,3,6)(4,5)$
S_6	$(1,2,3,4,5,6)$	$(2,4)(5,6)$	$(1,4,6)(2,3)$

Table 6.4: Permutation assignments that give rise to the index 6 subgroups of $\Gamma^0(\infty,2,6)$.

$\Gamma^0(\infty, 2, 4)$	$\pi(Q'P')$	$\pi(R'P')$	$\pi(Q'R')$
D_4	(34)	(13)(24)	(1324)
D_4	(13)(24)	(34)	(1423)
D_4	(1324)	(34)	(14)(23)
D_4	(1324)	(13)(24)	(34)
K_4	(12)(34)	(13)(24)	(14)(23)
S_4	(234)	(12)	(1234)
C_4	(1234)	()	(1234)
C_4	(1234)	(13)(24)	(1432)

Table 6.5: Permutation assignments that give rise to the index 4 subgroups of $\Gamma^0(\infty, 2, 4)$.

Chapter 7

CONCLUSION AND OUTLOOK

In this study, we have presented a method for characterizing the subgroups of the Coxeter group Γ and its subgroups, where Γ is generated by reflections in the hyperplanes containing the sides of a Coxeter polytope D.

The approach that we employ is geometric in nature, and uses tools in color symmetry theory. In determining the index n subgroups of a subgroup Λ of Γ, we constructed Λ-transitive n-colorings of the Λ-orbit of D. Most of the information pertaining to the subgroups rely on the algebraic structure of $\pi(\Lambda)$, where π is the homomorphism that results from the action of Λ on the set of colors in the Λ-orbit of D. The computations are aided by the *GAP4* routines *TranSub*, *Coloring*, *Sieve* and *ListSG*.

The process that we provide to come up with a particular subgroup Ω of Λ includes arriving at a generating set for Ω. We also develop a method which involves the construction of trees whose paths represent generators of Ω. The explorations on the color permutation assignments to the generators of Λ that give rise to subgroups of Λ led us to formulate the rules for the construction of trees and allowed us to gain more insights on the nature of each subgroup. In fact, the other main result of this study which centers on determining whether a subgroup Ω of Λ is torsion-free was largely motivated by such insights. We were able to characterize color permutation assignments which give rise to torsion free subgroups of Λ.

This study addresses the important problem of enumerating and characterizing the subgroups of Coxeter groups and their subgroups. With the results provided in this work, we are able to explore the subgroup structure of hyperbolic groups in higher dimension and derive higher index subgroups of Coxeter groups and their subgroups. For example, we can continue the derivation of the subgroups of Coxeter groups whose fundamental polytopes are simplices in $\mathbb{H}^{\mathsf{d}}$, $\mathsf{d} = 5, 6, 7, 8, 9$ and determine subgroups of hyperbolic groups in higher dimensional space such as the quaternionic modular groups, and the higher dimensional analogues of the modular and Picard groups. Moreover, it is now possible to construct the subgroup lattice of Coxeter groups, such as that of the triangle groups, which is essential in the study of crystal nets. It is suggested that the method and algorithm given here be implemented on a high performance computing system to accommodate these calculations.

The method and results given here pertaining to the derivation of subgroups of Coxeter groups and their subgroups will also provide a springboard in studying the subgroup structure of other classes of symmetry groups in hyperbolic space.

In recent years, the geometrical interest in the study of torsion-free subgroups of Coxeter groups and their subgroups is in manifold theory. Having established means to identify torsion-free subgroups of Coxeter groups, it would be interesting to explore its applications in the construction of manifolds. For example, the torsion-free subgroups of finite index in the Picard group are the fundamental groups of hyperbolic 3-manifolds.

The types of torsion-free crystallographic groups in Euclidean space (known as Bieberbach groups) up to dimension 6 have already been classified [9]. As a continuation of the study on the theory of Bieberbach groups, it will be interesting to know the isomorphism types of such groups in dimension greater than 6. We also hope to generalize the results we present here on torsion-free subgroups to allow us to apply it to a larger class of crystallographic groups.

Appendix A

GAP4 **ROUTINE:** *TranSub*

Appendix B

GAP4 **ROUTINE:** *Coloring*

Appendix C

GAP4 **ROUTINE:** *Sieve*

Appendix D

GAP4 **ROUTINE:** *ListSG*

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