SELF-DUAL SUBSTITUTIONS IN SMALL DIMENSIONS

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Abstract. There are several notions of the 'dual' of a word/tile substitution. We show that the most common ones are equivalent for substitutions in dimension one, where we restrict ourselves to the case of two letters/tiles. Furthermore, we obtain necessary and sufficient conditions for substitutions being selfdual in this case.

Contents

1. Introduction

Substitutions are simple but powerful tools to generate a large number of nonperiodic structures with a high degree of order. Examples include infinite words (e.g., the Thue Morse sequence), infinite tilings (e.g., Penrose tilings) and discrete point sets (e.g., models of atomic positions in quasicrystals). Here we consider several instances of the concept of substitutions:

(a) word substitutions
(b) endomorphisms of the free group $F_2 = \langle a, b | \varnothing \rangle$
(c) tile-substitutions
(d) dual maps of substitutions

Each of the concepts above gave rise to the concept of a dual substitution. Our first goal is to show the full equivalence of the distinct concepts of dual substitution in the contexts above for the case of two letters (tiles,...) in dimension one. Thus we will exclusively study substitutions on two letters (tiles, ...), and for the sake of clarity, we will define every term for this special case only.

Let us mention that there is a wealth of results for symbolic substitutions on two letters, thus for tile-substitutions in $\mathbb{R}^1$ with two tiles. For a start see [15] and references therein. The following theorem lists some interesting results which emerged in the work of many authors during the last decades.

Theorem 1.1. *Let σ be a primitive symbolic substitution on two letters. Let X_σ be the associated hull. Then the following are equivalent:*

(1) *Each biinfinite word $u \in X_\sigma$ is* Sturmian*, i.e., u contains exactly $n+1$ different words of length n for all $n \in \mathbb{N}$.*
(2) *The endomorphism $\sigma : F_2 \to F_2$ is invertible, i.e., $\sigma \in Aut(F_2)$.*
(3) *Each tiling generated by σ is a cut-and-project tiling which window is an interval.*
(4) *There is $k \geq 1$ such that the substitution σ^k is conjugate to $G^{a_1}\tilde{G}^{a_2}G^{a_3}\cdots\tilde{G}^{a_k}$, where $k, a_i \in \mathbb{N} \cup \{0\}$, $G:\ a \to a,\ b \to ab$, $\tilde{G}:\ a \to a,\ b \to ba$.*

The equivalence of (2) and (4) follows from [22]. The equivalence of (2) and (3) appears in [10] and relies on earlier results, see references in [10]. The equivalence of (1) and (3) is somehow folklore, see for instance [15], or Section 2.3 below.

2. Word substitutions

2.1. **Basic definitions.** Let $\mathcal{A} = \{a, b\}$ be an alphabet, let $\mathcal{A}^*$ be the set of all finite words over $\mathcal{A}$, and let $\mathcal{A}^{\mathbb{Z}} = \{(u_i)_{i\in\mathbb{Z}} \,|\, u_i \in \mathcal{A}\}$ be the set of all biinfinite words over $\mathcal{A}$. A substitution σ is a map from $\mathcal{A}$ to $\mathcal{A}^\star \setminus \{\varepsilon\}$, where ε denotes the empty word. By concatenation, a substitution extends to a map from $\mathcal{A}^\star$ to $\mathcal{A}^\star$, and to a map from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$. Thus a substitution can be iterated. For instance, consider the Fibonacci substitution

$$\sigma : \mathcal{A} \to \mathcal{A}^\star, \quad \sigma(a) = ab,\ \sigma(b) = a. \tag{1}$$

We abbreviate this long notation by $\sigma : a \to ab, b \to a$. Then, $\sigma(a) = ab$, $\sigma^2(a) = \sigma(\sigma(a)) = \sigma(ab) = \sigma(a)\sigma(b) = aba$, $\sigma^3(a) = abaab$, and so on. In order to rule out certain non-interesting cases, for instance $E\colon a \mapsto b, b \mapsto a$, that just exchanges letters a and b, the following definition is useful.

Definition 2.1. *Let σ be a substitution defined on $\mathcal{A}$. The* substitution matrix $\mathbf{M}_\sigma \in \mathbb{N}^{2\times 2}$ *associated with σ is defined as its Abelianisation , i.e.,*

$$(M_\sigma) = (|\sigma(j)|_i)_{1\le i,j\le 2},$$

where $|w|_i$ stands for the number of occurrences of the letter i of w. Furthermore, the length *of w is denoted by $|w|$.*
A substitution σ on $\mathcal{A}$ is called primitive, *if $M_\sigma^n > 0$ for some $n \in \mathbb{N}$.*
A substitution σ is called unimodular *if its substitution matrix M_σ has determinant* 1 *or* -1.

In the sequel, we will consider unimodular primitive substitutions only. The requirement of primitivity is a common one, essentially this rules out some pathological cases. The requirement of unimodularity is a restriction which we use because we will focus on the case where the substitution is invertible. Actually we will focus on biinfinite words that are preserved by some unimodular primitive two-letter substitution σ, that is, biinfinite words $u \in \mathcal{A}^{\mathbb{Z}}$ such that $\sigma(u) = u$. It is easy to see that for non-primitive substitutions, such a fixed word can only attain simple forms like $\dots aaa.aaa\dots$ or $\dots bbb.aaa\dots$. In contrast, primitive substitutions yield a wealth of biinfinite words.

Since we will consider fixed points of a substitution, it is beneficial to consider the set

$$\mathcal{X}_\sigma = \{u \,|\, \text{each subword of } u \text{ is a subword of } \sigma^n(a) \text{ or } \sigma^n(b) \text{ for some } n\}. \tag{2}$$

We will call $\mathcal{X}_\sigma$ the *hull* of the substitution σ.

Remark 2.2. For any substitution σ and for all $n \ge 1$ holds: $\mathcal{X}_{\sigma^n} = \mathcal{X}_\sigma$ [15].

In contrast, the hull $\mathcal{X}_u$ of a biinfinite word u is defined as the closure of the orbit $\{w \,|\, \exists k \in \mathbb{Z},\ \forall i \in \mathbb{Z}:\ w_i = u_{i+k}\}$ of u in a certain topology, see [15]. For primitive substitutions, it is easy to see that $\mathcal{X}_u = \mathcal{X}_\sigma$ for all $u \in \mathcal{X}_\sigma$, hence the two meanings of the term hull coincide in our context:

Proposition 2.3. *For a primitive substitution σ, $\mathcal{X}_\sigma$ is determined by each $u \in \mathcal{X}_\sigma$ uniquely.*

Let σ be a primitive substitution. By Perron Frobenius' Theorem, its substitution matrix M_σ admits a dominant eigenvalue $\lambda > 1$. We call it *substitution factor.* Let $\alpha \in (0,1)$ such that the vector $(1-\alpha, \alpha)$ is equal to the eigenvector of M_σ associated with the eigenvalueλ normalised so that the sum of its coordinates equals 1. The number α is called *frequency* of the substitution. Indeed, every biinfinite word in $\mathcal{X}_\sigma$ has well defined letter frequencies which are equal respectively to $1-\alpha$ and α [16].

2.2. **Substitutions as endomorphisms of F_2.** Naturally, any word substitution on $\mathcal{A}$ gives rise to an endomorphism σ of F_2, the free group on two letters. (Note that not every endomorphism of F_2 gives rise to a proper word substitution, consider for instance $\sigma: a \to ab^{-1}$, $b \to b^{-1}$). If this endomorphism σ happens to be an automorphism, then σ is said to be *invertible.*

It is well-known that the set of invertible word substitutions on a two-letter alphabet is a monoid, with one set of generators being

$$E\colon a \mapsto b,\ b \mapsto a,\ G\colon a \mapsto a,\ b \mapsto ab,\ \tilde{G}\colon a \mapsto a,\ b \mapsto ba.$$

For references, see [22] and Chap. 2 in [11].

We say that two given substitutions σ and σ', σ' are *conjugate*, if for some word $w \in \{a, b\}^*$, one has either $\sigma(a)w = w\sigma'(a),\ \sigma(b)w = w\sigma'(b)$ or $\sigma'(a)w = w\sigma(a),\ \sigma'(b)w = w\sigma(b)$. One checks easily that σ and σ' are conjugate if and only if $\sigma = \varphi\sigma'$ for some inner automorphism φ of F_2.

Let us note that we can extend the notion of substitution matrix to endomorphisms of the free group by taking here again the Abelianisation. Let σ be an endomorphism of the free group on two letters F_2 over the alphabet $\{a, b\}$. The (i, j)-entry of the *substitution matrix* $M_\sigma \in \mathbb{Z}^{2\times 2}$ associated with σ is defined as the number of occurrences of the letter i in $\sigma(j)$ minus the number of occurrences of the letter i^{-1} in $\sigma(j)$, where $i, j \in \{a, b\}$.

Recall the following theorem of Nielsen [14]: given two automorphisms σ and σ' of the free group F_2, they have the same substitution matrix if and only if $\sigma = \varphi\sigma'$ for some inner automorphism of F_2. We thus conclude in terms of substitutions (see [18] for a combinatorial proof):

Theorem 2.4. *Given two invertible substitutions σ and σ', they are conjugate if and only if they have the same substitution matrix.*

2.3. **Sturmian words.** *Sturmian words* are infinite words over a binary alphabet that have exactly $n + 1$ factors of length n for every positive integer n. Sturmian words can also be defined in a constructive way as follows. Let $0 < \alpha < 1$. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x + \alpha$. For a given real number α, we introduce the following two partitions of $\mathbb{T}^1$:

$$\underline{I}_a = [0, 1 - \alpha),\ \ \underline{I}_b = [1 - \alpha, 1);\ \ \overline{I}_a = (0, 1 - \alpha],\ \ \overline{I}_b = (1 - \alpha, 1].$$

Tracing the two-sided orbit of $R_\alpha^n(\varrho)$, we define two biinfinite words for $\varrho \in \mathbb{T}^1$:

$$\underline{s}_{\alpha,\varrho}(n) = \begin{cases} a & \text{if } R_\alpha^n(\varrho) \in \underline{I}_1, \\ b & \text{if } R_\alpha^n(\varrho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\varrho}(n) = \begin{cases} a & \text{if } R_\alpha^n(\varrho) \in \overline{I}_1, \\ b & \text{if } R_\alpha^n(\varrho) \in \overline{I}_2. \end{cases}$$

It is well known ([6, 13]) that a biinfinite word is a Sturmian word if and only if it is equal either to $\overline{s}_{\alpha,\varrho}$ or to $\underline{s}_{\alpha,\varrho}$ for some irrational number α. The word $\underline{s}_{\alpha,\varrho}$ is called *lower Sturmian word* whereas the word $\overline{s}_{\alpha,\varrho}$ is called *upper Sturmian word*. The notation c_α stands in all that follows for $\overline{s}_{\alpha,\alpha} = \underline{s}_{\alpha,\alpha}$. This particular Sturmian word is called *characteristic word*. A detailed description of Sturmian words can be found in Chapter 2 of [11], see also [15].

Note that frequently Sturmian words are defined as one-sided infinite words. For our purposes it is rather natural to consider biinfinite Sturmian words.

Example 2.5. The Fibonacci sequences arising from the Fibonacci substitution (1) are Sturmian words with parameter $\alpha = \frac{\sqrt{5}+1}{2}$.

3. Tile-substitutions

3.1. **Substitution tilings.** Rather than substituting symbolic objects, like word substitutions, tile-substitutions replace geometric objects (tiles) by larger geometric objects (collections of tiles). A tiling of $\mathbb{R}^d$ is a collection $(t_1, t_2, \dots)$ of compact sets t_i, such that the union

$\bigcup t_i$ is $\mathbb{R}^d$ and the interiors of the tiles a pairwise disjoint. Whereas in $\mathbb{R}^1$ there is a natural correspondence between biinfinite words and tilings (just assign to each letter an interval of specified length), in higher dimensions tilings show a richer structure. One motivation to study tilings is that they can serve as models for quasicrystals (for instance Penrose tilings, see [2]). One aim of this paper is to connect the theories of nonperiodic tilings to the theory of combinatorics of words.

As in the case of biinfinite words, substitutions are a powerful method to generate interesting tilings. In general, a tile-substitution in $\mathbb{R}^d$ is given by a set of *prototiles* $T_1, \dots, T_m \subset \mathbb{R}^d$, a *substitution factor* $\lambda > 1$, and a rule how to dissect each expanded prototile λT_j into isometric copies of some prototiles T_i. Here, we restrict ourselves to two prototiles. Moreover, we want to consider tilings of the line only. (For the discussion of analogues of some results of the present paper for the case of plane tilings, see [8]). The precise definition of a tile-substitution in $\mathbb{R}^d$ goes as follows.

Definition 3.1. *A (self-similar) tile-substitution in $\mathbb{R}^d$ is defined via a set of prototiles and a map σ. Let $T_1, T_2, \dots T_m$ be nonempty compact sets — the* prototiles *— in $\mathbb{R}^d$, such that the closure of the interior of each T_i is T_i itself. Let*

$$\lambda T_j = \bigcup_{i=1}^{m} T_i + \mathcal{D}_{ij} \quad (1 \le j \le m), \tag{3}$$

where the union is not overlapping (i.e., the interiors of the tiles in the union are pairwise disjoint); and each $\mathcal{D}_{ij}$ is a finite (possibly empty) subset of $\mathbb{R}^d$, called digit set*. Then*

$$\sigma(T_j) := \{T_i + \mathcal{D}_{ij} \mid i = 1 \dots m\}$$

is called a tile-substitution*.*

By $\sigma(T_j + x) := \sigma(T_j) + \lambda x$ and $\sigma(\{T, T'\}) := \{\sigma(T), \sigma(T')\}$, σ extends in a natural way to all sets $\{T_{i(k)} + x_k\}_{k \in I}$.

Example 3.2. Consider the square of the Fibonacci substitution from Equation (1): σ^2 : $a \to aba$, $b \to ab$. We will realize it as a tiling by assigning the unit interval to the letter b and the longer interval $[0, \tau]$ to the letter a, where $\tau = \frac{\sqrt{5}+1}{2}$, see Figure 1. So, let $T_1 = [0, \tau]$, $T_2 = [0, 1]$. Then $\tau^2 T_1 = [0, 2\tau + 1]$, $\tau^2 T_2 = [0, \tau + 1]$, thus

$$\tau^2 T_1 = T_1 \cup T_2 + \tau \cup T_1 + \tau + 1; \quad \tau^2 T_2 = T_1 \cup T_2 + \tau,$$

where $A \cup B$ denotes the union of A and B where the interiors of A and B are disjoint. Therefore the last equation yields a tile-substitution ϱ:

$$\varrho(T_1) = \{T_1, T_2 + \tau, T_1 + \tau + 1\}, \quad \sigma(T_2) = \{T_1, T_2 + \tau\}.$$

For an illustration of this substitution, see Figure 1. It corresponds to the squared Fibonacci substitution σ^2, compare (1). This substitution ϱ can be encoded in the digit sets $\mathcal{D}_{1,1} = \{0, \tau + 1\}, \mathcal{D}_{2,1} = \{\tau\}, \mathcal{D}_{1,2} = \{0\}, \mathcal{D}_{2,2} = \{\tau\}$. This can be written conveniently as a digit set matrix:

$$\mathcal{D} = \begin{pmatrix} \{0, \tau + 1\} & \{0\} \\ \{\tau\} & \{\tau\} \end{pmatrix}. \tag{4}$$

By comparison with 2.1 we note that we can derive the substitution matrix from the digit set matrix simply as follows: $M_\sigma = (|\mathcal{D}_{ij}|)_{1 \le i,j \le 2}$.

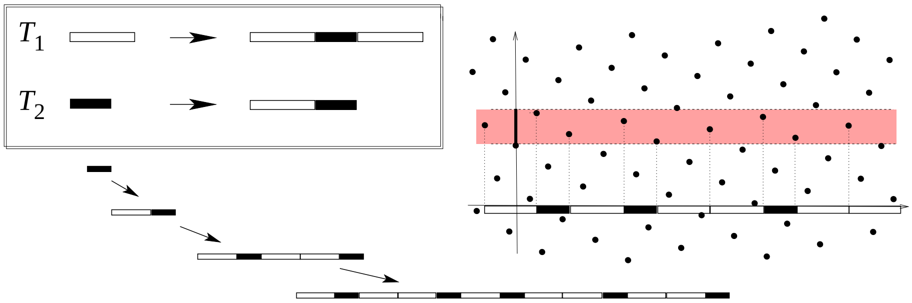

FIGURE 1. The Fibonacci tile-substitution σ (box top left), some iterates of σ on T_2, and the generation of a Fibonacci tiling as a cut-and-project tiling (right). The interval W defines a horizontal strip, all lattice points within this strip are projected down to the line.

Remark 3.3. Any self-similar tile-substitution is uniquely defined by its digit set matrix $\mathcal{D}$.

This holds because one can derive the substitution factor λ and the prototiles T_i from the digit set matrix $\mathcal{D}$. This is not only true for two tiles in one dimension, but for any self-similar tile-substitution in $\mathbb{R}^d$. For details, see [8]. Here we just mention two facts: the substitution factor λ is the larger eigenvalue of the substitution matrix $M_\sigma = (|\mathcal{D}_{ij}|)_{ij}$. And the prototiles are the unique compact solution of the multi component IFS (iterated function system) which is obtained by dividing (3) by λ.

In analogy to word substitutions, we want to deal with the space $\mathbb{X}_\sigma$ of all substitution tilings arising from a given tile-substitution.

Definition 3.4. *Let σ be a primitive tile-substitution with prototiles T_1, T_2. The* tiling space *$\mathbb{X}_\sigma$ is the set of all tilings T, such that each finite patch of T is contained in some translate of $\sigma^n(T_1)$ or $\sigma^n(T_2)$.*

3.2. **Cut and project tilings.** Certain substitution tilings can be obtained by a cut and project method. There is a large number of results about such cut and project sets, or model sets, see [12] and references therein. In our setting, it is pretty simple to explain, compare Figure 1. Let $G = H = \mathbb{R}$, let $\pi_1 : \mathbb{R}^2 \to G$, $\pi_2 : \mathbb{R}^2 \to H$ be the canonical projections, and let Λ be a lattice in $\mathbb{R}^2$, such that $\pi_1 : \Lambda \to G$ is one-to-one, and $\pi_2(\Lambda)$ is dense in H. Then, choose some compact set $W \subset H$, and let

$$V = \{\pi_1(x) \,|\, x \in \Lambda,\ \pi_2(x) \in W\}.$$

Then V is a *cut and project set* (or model set). Since V is a discrete point set in $\mathbb{R} = G$, it induces a partition of $\mathbb{R}$ into intervals. Regarding these (closed) intervals yields a tiling of $\mathbb{R}$. Such a tiling is called *cut and project tiling.*

Given a substitution σ which is known to yield cut and project tilings, one can construct Λ and W out of σ in a standard way. In general, Λ and W are not unique. The following construction has the advantage that everything can be expressed in some algebraic number field, which allows the use of algebraic tools. For clarity, we explain the construction for the case of $G = H = \mathbb{R}$ only.

Start with a symbolic substitution σ. Consider the substitution matrix M_σ with eigenvalues $\lambda > \lambda' \notin \mathbb{Z}$. Since M_σ is an integer matrix, λ is an quadratic irrational. (The case where λ

is an integer requires that the internal space H is non Euclidean, see [3], citesing2). Since our substitutions will always be unimodular in the sequel, this case cannot occur here.) Let $\binom{1}{\ell}$ be the left eigenvector to the larger eigenvalue λ. It is known (and easy to see) that this eigenvector yields the 'natural' lengths of the prototiles. Thus let $T_1 = [0,1]$, $T_2 = [0,\ell]$. Since λ is a quadratic irrational, ℓ is of the form $\alpha + \beta\sqrt{k}$, where k is a square-free integer, $\alpha, \beta \in \mathbb{Z}$ (respectively $\alpha, \beta \in \frac{1}{\mathbb{Z}}, \alpha + \beta \in \mathbb{Z}$, if $k \cong 1 \mod 4$). Now, let $\Lambda = \langle \binom{1}{1}, \binom{\ell}{\ell'} \rangle_{\mathbb{Z}}$, where ℓ' denotes the algebraic conjugate $\alpha - \beta\sqrt{k}$ of λ.

Now, consider the set V of endpoints of the intervals in a tiling in $\mathbb{X}_\sigma$. Wlog, let one endpoint be 0. Then all other endpoints are of the form $a + b\ell \in \mathbb{Z}[\ell]$. Now, let

$$\Lambda = \langle v, w \rangle, \quad \text{where } v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, w = \begin{pmatrix} \ell \\ \ell' \end{pmatrix}.$$

Any point $a + b\ell \in V$ has a unique preimage in Λ, namely, $av + bw$. Thus, each point $a + b\ell$ in V has a unique counterpart in the internal space H, namely $\pi_2 \circ \pi_1^{-1}(a + b\ell) = a + b\ell'$. The window W is now obtained as the closure of $\pi_2 \circ \pi_1^{-1}(V)$. The fact that V *is* a cut and project tiling guarantees that W is indeed compact. The map $\pi_2 \circ \pi_1^{-1}$ can be conveniently abbreviated by $\star$. For instance, we may write shortly $W = \mathrm{cl}(V^\star)$. In our case, the star map is just mapping an element of $\mathbb{Z}[\ell]$ to its Galois dual.

Now we can define the star-dual of a tile-substitution. In our framework, this is exactly the same as Thurston's Galois dual substitution, see [20], [21]. In general, star-duality is a generalization of Galois duality [8]. Recall that any self-similar tile-substitution is uniquely defined by its digit set matrix $\mathcal{D}$ (Remark 3.3). The star-dual substitution of a self-similar cut and project tiling is obtained by applying the star map to $\mathcal{D}^T$.

Definition 3.5. *Let σ be a self-similar substitution yielding cut and project tilings, with digit set matrix $(\mathcal{D}_{ij})_{ij}$. Then the* star-dual *substitution of σ is the unique substitution defined by $(\mathcal{D}_{ij}^\star)_{ij}^T$.*

Here $A^\star$ means the application of the star map to each element of $A \in \mathbb{Z}[\lambda]$. This definition together with Definition 3.4 defines the *star-dual* tiling space $\mathbb{X}_{\sigma^\star}$.

Example 3.6. The star-dual of the squared Fibonacci substitution in Example 5.5 is easily obtained by applying the star map to the transpose of the digit set matrix in (4). We obtain

$$(\mathcal{D}^T)^\star = \begin{pmatrix} \{0, 1 - \tau^{-1}\} & \{-\tau^{-1}\} \\ \{0\} & \{-\tau^{-1}\} \end{pmatrix}.$$

4. Dual maps of substitutions

4.1. **Generalized substitutions.** We follow here the formalism introduced in [1, 17] defined originally on a d-letter alphabet. We restrict ourselves here to the case $d = 2$. Let $\mathcal{A}$ be the finite alphabet $\{a, b\}$.

We denote by L the *Abelianisation map* from $\mathcal{A}^*$ to $\mathbb{Z}^2$: if w is a word in $\mathcal{A}^*$, then $L(w)$ is the vector that counts the number of occurrence of each letter in w, i.e., $L\colon \mathcal{A}^* \to \mathbb{Z}^2,\ w \mapsto (|w|_a, |w|_b)$. There is an obvious commutative diagram, where M_σ stands for the substitution matrix of σ:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ L\downarrow & & \downarrow L \\ \mathbb{Z}^2 & \xrightarrow{M_\sigma} & \mathbb{Z}^2. \end{array}$$

Finite strand. Let (e_a, e_b) stand for the canonical basis of $\mathbb{R}^2$. It is natural to associate with each finite word $w = w_1 w_2 \dots w_n$ on the two-letter alphabet $\mathcal{A}$ a path in the two-dimensional space, starting from 0 and ending in $L(w)$, with vertices in $L(w_1 \dots w_i)$ for $i = 1 \dots n$: we start from 0, advance by $\vec{e}_i$ if the first letter is i, and so on.

More generally, we define the notion of strand by following the formalism of [4]. A *finite strand* is a subset of $\mathbb{R}^2$ defined as the image by a piecewise isometric map $\gamma\colon [i,j] \to \mathbb{R}^2$, where $i, j \in \mathbb{Z}$, which satisfies the following: for any integer $k \in [a, b)$, there is a letter $x \in \{a, b\}$ such that $\gamma(k+1) - \gamma(k) = e_x$. If we replace $[i,j]$ by $\mathbb{Z}$, we get the notion of *biinfinite strand.* A strand is thus a connected union of segments with integer vertices which projects orthogonally in a one-to-one way on the line $x = y$. The path associated with a finite word w such as defined in the previous paragraph is a finite strand.

Any biinfinite strand defines a biinfinite word $w = (w_k)_{k\in\mathbb{Z}} \in \{a,b\}^{\mathbb{Z}}$ that satisfies $\gamma(k+1) - \gamma(k) = e_{w_k}$. The corresponding map which sends biinfinite strands on biinfinite words is called *strand coding.*

This allows us to define a map on strands, coming from the substitution, by taking the strand for w to the strand for $\sigma(w)$. In fact, this map can be made in a linear map, in the following way. Let σ be a substitution on $\mathcal{A}$. We will take the notation for $i \in \{a, b\}$

$$\sigma(i) = w^{(i)} = w_1^{(i)} \dots w_{l_i}^{(i)} = p_n^{(i)} w_n^{(i)} s_n^{(i)},$$

for $1 \le n \le l_i$, where l_i is the length of $\sigma(i)$, $p_n^{(i)}$ is the prefix of length $n-1$ of $\sigma(i)$ (the empty word for $n = 1$), and $s_n^{(i)}$ is the suffix of length $l_i - n$ of $\sigma(i)$ (the empty word for $n = l_i$).

Definition 4.1. *We denote by $(W, i) \in \mathbb{Z}^2 \times \mathcal{A}$ an* elementary strand *(that is, a segment from W to $W + e_i$); we denote by $\mathcal{G}$ the real vector space of formal finite weighted sums of elementary strands. Let $E_1(\sigma)$ be the linear map defined on $\mathcal{G}$ by:*

$$E_1(\sigma)(W, i) = \sum_{n=1}^{l_i} ((M_\sigma . W + L(p_n^{(i)}), w_n^{(i)}).$$

We call $E_1(\sigma)$ one-dimensional extension *of σ,*

It is easily checked that this formula is such that σ takes the finite strand corresponding to a word w to the finite strand corresponding to $\sigma(w)$.

Definition 4.2. *Let σ be a primitive substitution. The* strand space $\mathbf{X}_\sigma$ *is the set of biinfinite strands η such that each finite substrand ξ of η is a substrand of some $E_1(\sigma)^n(W, x)$, for $W \in \mathbb{Z}^2$, $n \in \mathbb{N}$ and $x \in \{a, b\}$.*

4.2. **Dual maps.** From now on, we suppose that σ is a unimodular substitution. Upon considering the square of σ, we will often take as assumption that σ has determinant $+1$.

We want to study the dual map $E_1^*(\sigma)$ of $E_1(\sigma)$, as a linear map on $\mathcal{G}$. We thus denote by $\mathcal{G}^*$ the space of dual maps with finite support (that is, dual maps that give value 0 to all but a finite number of the vectors of the canonical basis).

The space $\mathcal{G}^*$ has a natural basis (W, i^*), for $i = a, b$, defined as the map that gives value 1 to (W, i) and 0 to all other elements of $\mathcal{G}$. It is possible to give a geometric meaning to this dual space: for $i = a, b$, we represent the element (W, i^*) by the finite strand defined as the lower face perpendicular to the direction $\vec{e}_i$ of the unit square with lowest vertex W.

The map $E_1(\sigma)$ has a dual map, which is easily computed:

Theorem 4.3. [1] *Let σ be a unimodular substitution. The dual map $E_1^*(\sigma)$ is defined on $\mathcal{G}^*$ by*

$$E_1^*(\sigma)(W, i^*) = \sum_{n,j : w_n^{(j)} = i} \left(M_\sigma^{-1}(W + L(s_n^{(j)})), j^*\right).$$

Furthermore, if τ is also a unimodular substitution, then

$$E_1^*(\sigma \circ \tau) = E_1^*(\tau) \circ E_1^*(\sigma).$$

Example 4.4. Let $G\colon a \mapsto a, b \mapsto ab$. One has $M_G \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M_G^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. We deduce that $\begin{cases} E_1(G)^*(0, a)^* = (0, a)^* + (-e_a + e_b, b)^* \\ E_1(G)^*(0, b)^* = (0, b)^*. \end{cases}$

Let $\tilde{G}\colon a \mapsto a, b \mapsto ba$. One has $M_G = M_{\tilde{G}}$. Hence $\begin{cases} E_1(\tilde{G})^*(0, a)^* = (0, a)^* + (0, b)^* \\ E_1(\tilde{G})^*(0, b)^* = (e_a, b)^*. \end{cases}$

Let $E\colon a \mapsto b, b \mapsto a$. One has $M_E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_E^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We thus get

$$\begin{cases} E_1(E)^*(0, a)^* = (0, b^*) \\ E_1(E)^*(0, b)^* = (0, a)^*. \end{cases}$$

To be more precise, the map $E_1^*(\sigma)$ is defined in [1] with $-L(p_n^{(j)})$ instead of $L(s_n^{(j)})$, whereas, for $i = a, b$, the element (W, i^*) is represented by the finite strand defined as the *upper* face perpendicular to the direction $\vec{e}_i$ of the unit square with lowest vertex W. Nevertheless, an easy computation shows that both formula coincide.

Dual strands. We can also define a notion of strand associated with this dual formalism. A *finite dual strand* is a subset of $\mathbb{R}^2$ defined as the image by a piecewise isometric map $\gamma\colon [i, j] \to \mathbb{R}^2$, where $i, j \in \mathbb{Z}$, which satisfies the following: for any integer $k \in [a, b)$, there is a letter $x \in \{a, b\}$ such that

$$\gamma(k+1) - \gamma(k) = e_x \text{ if } x = a,\ \gamma(k+1) - \gamma(k) = -e_x, \text{ otherwise.}$$

If we replace $[i, j]$ by $\mathbb{Z}$, we get the notion of *biinfinite dual strand.* Any biinfinite dual strand defines a biinfinite word $(w_k)_{k \in \mathbb{Z}} \in \{a, b\}^{\mathbb{Z}}$ that satisfies $\gamma(k+1) - \gamma(k) = e_{w_k}$. Similarly, any finite dual strand s defines a finite word w as follows: let γ be a piecewise isometric map $\gamma\colon [i, j] \to \mathbb{R}^2$, where $i, j \in \mathbb{Z}$, whose image equals s which satisfies for any integer $k \in [a, b)$, there is a letter $x \in \{a, b\}$ such that $\gamma(k+1) - \gamma(k) = e_x$ if $x = a$, $\gamma(k+1) - \gamma(k) = -e_x$, otherwise. We set $w_1 = \gamma(i+1) - \gamma(i), \cdots, w_{j-i} = \gamma(j) - \gamma(j-1)$. This definition does not depend on the choice of γ. The map ψ^* that sends finite dual strands on words in $\mathcal{A}^*$ is called *dual coding*.

Theorem 4.5. [1] *Let τ be a unimodular substitution. The map $E_1^*(\tau)$ maps any finite substrand of $\mathcal{S}_\alpha$ on a finite substrand of $\mathcal{S}_\alpha$. Furthermore, if (V, i^*) and (W, j^*) are two distinct segments included in some $\mathcal{S}_\alpha$, for $\alpha \in (0,1)$, then their images by $E_1^*(\tau)$ have disjoint interiors.*

Finite dual substrands of $\mathcal{S}_\alpha$ are preserved under the action of an invertible two-letter substitution (a different proof of this result can be found in [7]):

Proposition 4.6. *Let σ be a two-letter substitution. The map $E_1^*(\sigma)$ maps every finite strand onto a finite strand if and only if σ is invertible.*

Proof. We first check that the the generators $E, G, \tilde{G}$ of the monoid of invertible two-letter substitutions map every finite dual strand made of two adjacent segments onto a finite dual strand (see also Figure ???):

$$\begin{cases} E_1^*(G)((x,a^*)+(x,b^*)) = (0,a^*)+(0,b^*)+(-e_a+e_b,b^*) \\ E_1^*(G)((x,a^*)+(x+e_b,a^*)) = (0,a^*)+(-e_a+e_b,a^*)+(-e_a+e_b,b^*)+(-2e_a+2e_b,b^*) \\ E_1^*(G)((x,b^*)+(x+e_a,b^*)) = (0,b^*)+(e_a,b^*) \\ E_1^*(G)((x,b^*)+(x+e_a-e_b,a^*)) = (0,b^*)+(e_a,b^*)+(2e_a-e_b,a^*). \end{cases}$$

$$\begin{cases} E_1^*(G)((x,a^*)+(x,b^*)) = (0,a^*)+(0,b^*)+(-e_a+e_b,b^* \\ E_1^*(G)((x,a^*)+(x+e_b,a^*)) = ((0,a^*)+(0,b^*)+(-e_a+e_b,a^*)+(-e_a+e_b,b^*) \\ E_1^*(G)((x,b^*)+(x+e_a,b^*)) = (e_a,b^*)+(2e_a,b^*) \\ E_1^*(G)((x,b^*)+(x+e_a-e_b,a^*)) = (e_a,b^*)+(2e_a-e_b,a^*)+(2e_a-e_b,b^*). \end{cases}$$

$$\begin{cases} E_1^*(E)((x,a^*)+(x,b^*)) = (0,a^*)+(0,b^*) \\ E_1^*(E)((x,a^*)+(x+e_b,a^*)) = ((0,b^*)+(e_a,b^*) \\ E_1^*(E)((x,b^*)+(x+e_a,b^*)) = (0,a^*)+(e_b,a^*) \\ E_1^*(E)((x,b^*)+(x+e_a-e_b,a^*)) = (0,a^*)+(-e_a+e_b,b^*). \end{cases}$$

Hence the generators map finite dual strands onto connected unions of unit segments with integer vertices. It remains to check that these unions are indeed dual strands. This is a consequence of Theorem 4.5 since they all are substrands of $\mathcal{S}_\alpha$. We deduce that the generators map finite dual strands onto finite dual strands.

Let τ be a two-letter substitution that maps every finite dual strand onto a finite dual one. Now, if $\sigma = \tau \circ G$, then we deduce from $E_1^*(\tau) = E_1^*(G) \circ E_1^*(\tau)$, that the map $E_1^*(\sigma)$ also maps every finite dual strand onto a connected union of unit segments, and hence by Theorem 4.5, one a finite dual strand. The same holds true for the other generators. We thus conclude by induction on the length of a decomposition on the generators $E, G, \tilde{G}$. □

We now can introduce the notion of dual strand space:

Definition 4.7. *Let σ be a primitive invertible substitution over a two-letter alphabet. The* dual strand space $\mathbf{X}_\sigma^*$ *is the set of bi-infinite strands η such that each finite substrand ξ of η is a substrand of some $E_1^*(\sigma)^n(W, x^*)$, for $W \in \mathbb{Z}^2$, $n \in \mathbb{N}$ and $x \in \{a,b\}$.*

Definition 4.8. *Let σ be a primitive invertible substitution over a two-letter alphabet. The* dual substitution σ^* *is defined on the alphabet $\{a, b\}$ as*

$$\sigma^*(x) = \psi^*(E_1^*(\sigma)(0, x^*)) \text{ for } x = a, b.$$

The dual frequency of α^* of σ is defined as the frequency of σ.

Let us observe that the substitution matrix of σ^* is the transpose of the substitution matrix of σ. Hence the dual substitution factor λ^* of σ is equal to the substitution factor of σ.

Example 4.9. One has $(G^*)\colon a \mapsto ba$, $b \mapsto b$, $(\tilde{G}^*)\colon a \mapsto ab$, $b \mapsto b$ and $E^*\colon a \mapsto b$, $b \mapsto a$.

Theorem 4.10. *Let σ be a primitive invertible substitution over a two-letter alphabet. The dual of σ is invertible. Furthermore, σ is conjugate to the substitution τ if and only if σ^* is conjugate to τ^*.*

Proof. Consider $G \circ \tau$. Recall that $\begin{cases} E_1(G)^*(0,a)^* = (0,a)^* + (-e_a + e_b, b)^* \\ E_1(G)^*(0,b)^* = (0,b)^*. \end{cases}$

One has

$$E_1^*(\sigma) \circ E_1^*(G)(0, a^*) = E_1^*(\sigma)(0,a)^* + E_1^*(\sigma)(-e_a + e_b, b^*).$$

The dual strands $E_1^*(\sigma)(0,a)^*$ and $E_1^*(\sigma)(-e_a + e_b, b^*)$ have disjoint interiors by Theorem 4.5. Furthermore, $E_1^*(\sigma)(-e_a + e_b, b^*)$ is located of the left of $E_1^*(\sigma)(0,a)^*$. Indeed we recall that λ is the maximal eigenvalue of M_σ. Its algebraic conjugate λ' is also an eigenvalue of M_σ. By Perron-Frobenius' theorem, we have $\lambda > 1$. Now $\lambda\lambda' = \det M_\sigma = 1$ implies $0 < \lambda'$.

We thus deduce that for $x = a, b$

$$\psi^*(E_1^*(\sigma) \circ E_1^*(G)(0, x^*)) = \psi^*(E_1^*(G \circ \sigma)(0, x^*)).$$

The same holds for $\tilde{G}$ and E.

One checks that G^* , $\tilde{G}^*$, and E^* (see Example 4.9) are invertible substitutions. We conclude here again by induction.

Now σ is conjugate to τ if and only if σ and τ have the same substitution matrix, which is also equivalent to σ^* and τ^* having the same substitution matrix (by transposition), and thus to σ^* and τ^* being conjugate by Theorem 2.4. □

4.3. **Arithmetic duality.** Let us now express the notion of duality in terms of continued fraction expansion.

Theorem 4.11. *Let σ be a primitive invertible substitution over $\{a, b\}$. The continued fraction expansion of the dual frequency α^* of α satisfies*

1. *If $\alpha < 1/2$, then $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_{k+1} + n_1}]$, with $n_{k+1} \geq 0$ and $n_1 \geq 1$*
   - $\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}]$ *if $n_{k+1} \geq 1$*
   - $\alpha^* = [0; 1, \overline{n_k, \cdots, n_2, n_1}]$ *otherwise.*
2. *If $\alpha > 1/2$, then $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k + n_2}]$, with $n_k \geq 0$ and $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}]$.*

Proof. We follow here the proof of Theorem 3.7 of [5] (see also the proof of Theorem 2.3.25 of [11]). We define

$$G\colon 0 \mapsto 0, 1 \mapsto 01,\ D\colon 0 \mapsto 10, 1 \mapsto 1,\ E\colon 0 \mapsto 1,\ 1 \mapsto 0.$$

Let $\alpha^* = [0; m_1, m_2, \cdots]$. One has $\sigma(c_\alpha) = c_\alpha$ and $\sigma^*(c_{\alpha^*}) = c_{\alpha^*}$.

One has $G^* = D$ and $E^* = E$. The substitution σ can be decomposed as

$$\sigma = G^{n_1} E G^{n_2} \cdots E G^{n_{k+1}},$$

with $k \geq 1$, $n_1, n_{k+1} \geq 0$, $n_2, \cdots, n_k \geq 1$. One has

$$\sigma^* = D^{n_{k+1}} E \cdots E D^{n_1}.$$

Furthermore $G \circ E = E \circ D$, and $D \circ E = E \circ G$. Let $\theta_m := G^{m-1} E G$, for $m \geq 1$. One has $\theta_m(c_\alpha) = c_{1/(m+\alpha)}$. Furthermore, $G(c_\alpha) = c_{\frac{1}{1+1/\alpha}}$.

- We first assume $n_1 > 0$. From $D \circ E = E \circ G$, we deduce that

  $$\sigma^* = E G^{n_{k+1}} \cdots E G^{n_1} E,$$

  and thus

  $$E \circ \sigma^* \circ E = G^{n_{k+1}} E \cdots E G^{n_1} = \theta_{1+n_{k+1}} \theta_{n_k} \cdots \theta_{n_2} G^{n_1 - 1}.$$

  Let $1 - \alpha^* = [0; m'_1, m'_2, \cdots]$. From $E \circ \sigma^* \circ E(c_{1-\alpha^*}) = c_{1-\alpha^*}$, we deduce that

  $$1 - \alpha^* = [0; 1 + n_{k+1}, n_k, \cdots, n_1 - 1 + m'_1, m'_2, \cdots],$$

  and thus $m'_1 = 1 + n_{k+1}$, $m'_2 = n_k, \cdots, m'_{k+1} = n_1 + n_{k+1}$, and $m'_{k+j} = m'_j$, for $j \geq 2$. We deduce that

  $$1 - \alpha^* = [0; 1 + n_{k+1}, \overline{n_k, \cdots, n_1 + n_{k+1}}].$$

  We thus deduce that

  $$\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}] \text{ if } n_{k+1} > 0$$

  and that

  $$\alpha^* = [0; 1, \overline{n_k, \cdots, n_2, n_1}] \text{ if } n_{k+1} = 0.$$

- We now assume $n_1 = 0$. One has

  $$\sigma^* = D^{n_{k+1}} E \cdots D^{n_2} E = E G^{n_{k+1}} \cdots E G^{n_2}.$$

  - We assume $n_{k+1} > 0$. Hence

    $$\sigma^* = \theta_1 \theta_{n_{k+1}} \cdots \theta_{n_3} G^{n_2 - 1}.$$

    Consequently $[0; m_1, m_2, \cdots] = [0; 1, n_{k+1}, \cdots, n_3, n_2 - 1 + m_1, m_2, \cdots]$. We deduce that $m_1 = 1$, $m_2 = n_{k+1}, \cdots, m_k = n_3$, $m_{k+1} = n_2$, and $m_{k+j} = m_{j+1}$, for $j \geq 1$. We thus get

    $$\alpha^* = [0; 1, \overline{n_{k+1}, \cdots n_3, n_2}].$$

  - We now assume $n_{k+1} = 0$. One has

    $$\sigma^* = E D^{n_k} E \cdots D^{n_2} E = G^{n_k} E \cdots E G^{n_2} = \theta_{1+n_k} \cdots \theta_{n_3} G^{n_2 - 1}.$$

    We deduce that $[0; m_1, m_2, \cdots] = [0; 1 + n_k, \cdots, n_3, n_2 - 1 + m_1, m_2, \cdots]$ and thus

    $$\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}].$$

In conclusion, we have proved that

(1) $\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}]$ if $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_{k+1} + n_1}]$, with $n_1 > 0$ and $n_{k+1} > 0$.
(2) $\alpha^* = [0; 1, \overline{n_k, \cdots, n_2, n_1}]$ if $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_1}]$.
(3) $\alpha^* = [0; 1, \overline{n_{k+1}, \cdots n_3, n_2}]$ if $\alpha = [0; 1, \overline{n_2, \cdots, n_k, n_{k+1}}]$.

(4) $\alpha^* = [0; 1+n_k, \overline{n_{k-1}, \cdots, n_3, n_2+n_k}]$ if $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k+n_2}]$.

□

Theorem 4.12. *Let σ be a primitive two-letter invertible substitution. Let α be its frequency and α' its algebraic conjugate.*

- *Let $\alpha = [0; 1+n_1, \overline{n_2, \cdots, n_k, n_{k+1}+n_1}]$, with $n_{k+1} \geq 0$ and $n_1 \geq 1$. Then*
$$\alpha^* = \frac{\alpha'}{2\alpha'-1}.$$
- *Let $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k+n_2}]$, with $n_k \geq 1$. Then*
$$\alpha^* = \frac{1-\alpha'}{2\alpha'-1}.$$

Proof. We first note that if $\gamma = [\overline{a_1, \cdots, a_n}]$, then $-1/\gamma' = [\overline{a_n, \cdots, a_1}]$:

- If $\alpha < 1/2$, then $\alpha = [0; 1+n_1, \overline{n_2, \cdots, n_k, n_{k+1}+n_1}]$, with $n_{k+1} \geq 0$ and $n_1 \geq 1$. Let
$$\gamma = [\overline{n_2, \cdots, n_k, n_{k+1}+n_1}].$$
We have $1/\alpha = 1+n_1+1/\gamma$ and $1/\alpha' = 1+n_1+1/\gamma'$. One has
$$-1/\gamma' = [\overline{n_{k+1}+n_1, n_k, \cdots, n_2}].$$
We deduce that $-(1/\gamma'+n_1+n_{k+1}) = [0; \overline{n_k, \cdots, n_2, n_1+n_{k+1}}]$.
  - If $n_{k+1} \geq 1$, then $\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1+n_{k+1}}]$. We deduce that
$$\frac{\alpha^*}{1-\alpha^*} = n_{k+1} - (1/\gamma'+n_1+n_{k+1}) = 1 - 1/\alpha',$$
and thus
$$\alpha^* = \frac{\alpha'}{2\alpha'-1}.$$
  - If $n_{k+1} = 0$, then $\alpha^* = [0; 1, \overline{n_k, \cdots, n_2, n_1}]$. One has $-(1/\gamma'+n_1) = [0; \overline{n_k, \cdots, n_2, n_1}]$. We deduce that
$$1/\alpha^* = 1 - 1/\gamma' - n_1 = 2 - 1/\alpha'$$
and thus
$$\alpha^* = \frac{\alpha'}{2\alpha'-1}.$$
- If $\alpha > 1/2$, then $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k+n_2}]$, with $n_k \geq 1$. Let
$$\gamma = [\overline{n_3, \cdots, n_{k-1}, n_k+n_2}].$$
We have $\frac{\alpha}{1-\alpha} = n_2 + 1/\gamma$ and $\frac{\alpha'}{1-\alpha'} = n_2 + 1/\gamma'$. One has
$$-1/\gamma' = [\overline{n_k+n_2, n_{k-1}, \cdots, n_3}].$$
We deduce that $-(1/\gamma'+n_k+n_2) = [0; \overline{n_{k-1}, \cdots, n_3, n_2+n_k}]$.
One has $\alpha^* = [0; 1+n_k, \overline{n_{k-1}, \cdots, n_3, n_2+n_k}]$. Hence
$$1/\alpha' = 1+n_k - (1/\gamma'+n_k+n_2) = 1 - \frac{\alpha'}{2\alpha'-1},$$
and thus
$$\alpha^* = \frac{1-\alpha'}{2\alpha'-1}.$$

□

5. Relations between distinct concepts of 'dual substitution'

Let us recall that two substitutions are said to be *conjugate* up to an *inner automorphism* of F_2: $\sigma \sim \sigma'$, if there exists $w \in \mathcal{A}^*$ such that for

$$x \in \{a, b\},\ w^{-1}\sigma(x)w = \sigma'(x).$$

5.1. Sturmian substitutions and rigidity.

We would like to compare the various notions of duality we have introduced so far. For that purpose we need to introduce several analogous equivalence relations among hulls, tiling spaces etc. We first recall the classical rigidity result for two-letter primitive substitutions (Theorem 5.1below).

The following result is due to P. Séébold [18], see also [9]. Here we provide an alternative and simple proof.

Theorem 5.1. *Let σ, σ' be primitive invertible substitutions on the alphabet $\mathcal{A} = \{a, b\}$. Then, $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$ if and only if there are k, n such that $\sigma^k \sim (\sigma')^m$.*

In plain words, this theorem states that two substitutions are conjugate (up to powers) if and only if their hulls are equal. In even different words (compare Proposition 2.3): if u is a biinfinite word obtained by a primitive substitution on two letters, where the substitution is unknown, then u determines the substitution uniquely, up to conjugation and up to powers of the substitution factor.

Corollary 5.2. *Let σ, σ' be primitive invertible substitutions on the alphabet $\mathcal{A} = \{a, b\}$ with the same substitution factor. Then, $\sigma \sim \sigma'$ if and only if $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$. Furthermore, $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$ if and only if σ and σ' have the same density α.*

Proof. First we prove that if $\sigma \sim \sigma'$, then $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$.

Let $\sigma \sim \sigma'$. This means there is w such that $\sigma'(x) = w^{-1}\sigma(a)w$ for $x = a, b$. Consequently, for each finite word $u = (u_1, u_2, \ldots, u_n)$ holds $\sigma'(u) = w^{-1}\sigma(u_1)ww^{-1}\sigma(u_2)ww^{-1}\cdots ww^{-1}\sigma(u_n)w = w^{-1}\sigma(u)w$, since each w but the leftmost one is cancelled by a w^{-1}. Therefore, $(\sigma')^n(u) = w^{-n}\sigma^n(u)w^n$ for all $u \in \mathcal{A}^*, n \in \mathbb{N}$. Thus, any biinfinite word $\bigcup_{k\in\mathbb{N}} \sigma^{nk}(u)$ which is a fixed point of some σ^n is also a fixed point of $(\sigma')^n$. By the definition of equality for hulls, it follows $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$.

For the other direction, we first note that if $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ is the eigenvalue of a 2×2 integer matrix M (so $\lambda = a + b\sqrt{k} \in \mathbb{Z}[\sqrt{k}]$, k square-free), then the algebraic conjugate of λ, $\lambda' = a - b\sqrt{k}$, is the second eigenvalue of M. If $v = (1, v_2)$ is an eigenvalue of M corresponding to λ, then $v' = (1, v_2')$, where v_2' denotes the algebraic conjugate of v_2, is an eigenvector corresponding to λ'. We then need the following lemma.

Lemma 5.3. *Let M, M' be two primitive 2×2 matrices with the same pair of real eigenvalues $\lambda > \lambda'$ and the same pair of eigenvectors $v = (1, v_2)$ and $v' = (1, v_2')$. Then $M = M'$.*

Proof. By assumption, there is $A \in GL(2,\mathbb{R})$ such that $A^{-1}MA = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda'\end{smallmatrix}\right)$. By elementary linear algebra, the columns of A are eigenvectors of M. Therefore, A can be written as

$A = r\tilde{A}$ with $\tilde{A} = \left(\begin{smallmatrix} 1 & s \\ v_2 & sv_2' \end{smallmatrix}\right)$, $r \neq 0 \neq s$. We obtain $M = A\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}\right)A^{-1} = \tilde{A}\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}\right)\tilde{A}^{-1}$. A simple computation yields $M = \frac{1}{v_2 - v_2'}\left(\begin{smallmatrix} v_2\lambda' - v_2'\lambda & \lambda - \lambda' \\ v_2 v_2'(\lambda' - \lambda) & v_2\lambda - v_2'\lambda' \end{smallmatrix}\right)$, which is independent of r and s.

Let M' be another matrix with the same eigenvalues and the same eigenvectors. Then, the above holds also for M'. It follows $M = M'$. □

Back to the proof of Theorem 5.1. Let σ, σ' such that $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$. Let $u \in \mathcal{X}_\sigma$. Let us recall that the the vector of frequencies (f_a, f_b) of letters in u is an eigenvector of the (up to here unknown) substitution matrix M_σ. In the normed eigenvector $f_a^{-1}(f_a, f_b) = (1, \ell)$ we have $\ell = a + b\sqrt{k}$, $a, b \in \mathbb{Z}$ (resp. $a, b \in \frac{1}{2}\mathbb{Z}$, $a - b \in \mathbb{Z}$, if $k \equiv 1 \mod 4$). Since σ is invertible, M_σ is unimodular. The (up to here unknown) substitution factor λ is therefore a unit in $\mathbb{Z}[\sqrt{k}]$. It is well known that the unit group of $\mathbb{Z}[\sqrt{k}]$ is generated by a *fundamental unit* z. (This is a consequence of the fact that there is a fundamental unit for the solution of the corresponding Pell's equation, or a consequence of Dirichlet's unit theorem, see for instance []). Thus λ is a power of the generating element z. Let $\lambda = z^n$, where n is arbitrary but fixed. By conjugacy, we obtain the second eigenvalue, and the second eigenvector of M_σ. By the Lemma 5.3, M_σ and $M_{\sigma'}$ have thus the same substitution matrix. By the fact that all invertible substitutions with the same substitution matrix are conjugate (see Theorem 2.4), the claim of the theorem follows. □

Let us note that the assumption that σ is invertible is crucial in Theorem 5.1 as shown by the following example (see also [9]). Consider on the alphabet $\{a, b\}$ the following two substitutions:

$$\sigma : \ a \mapsto ab, \ b \mapsto baabbaabbaabba, \ \sigma' : \ a \mapsto abbaab, \ b \mapsto baabbaabba.$$

One has $\sigma(ab) = \sigma'(ab)$ and $\sigma(ba) = \sigma'(ba)$. We deduce that σ and σ' have the same fixed point beginning by a. Nevertheless, σ and σ' are neither conjugate, nor conjugate to a power of a common substitution, since their substitution matrices are not conjugate in $GL(2, \mathbb{Z})$.

5.2. **Conjugation and equivalence.** In order to compare strand or tiling spaces with their duals, we introduce a new type of conjugation of substitutions in $\mathrm{Aut}(F_2)$ w.r.t. certain outer automorphisms.

Definition 5.4. *Let $\langle E, I\rangle$ be the group of (outer) automorphisms of F_2 generated by*

$$E : a \mapsto b, b \mapsto a, \ I : a \mapsto a^{-1}, b \mapsto b.$$

We say that two automorphisms σ and σ' of F_2 are equivalent*: $\sigma \cong \sigma'$, if*

$$\tau^{-1}\sigma\tau \sim \sigma' \text{ for some } \tau \in \langle E, I\rangle.$$

In other words, σ and σ' are equivalent if they are conjugate as two-letter alphabet substitutions up to a renaming of the letters.

If $\tau = E$, then the frequency α of σ satisfies: $\alpha = 1 - \alpha'$, where α' is the frequency of σ'.

Let us note that if σ is an invertible substitution of the two-letter alphabet $\mathcal{A}$, then its inverse yields another substitution, but now on the two-letter alphabet $\{a^{-1}, b\}$. Indeed this can be easily checked by an induction on the length of a decomposition of σ on the set of generators $\{E, G, \tilde{G}\}$. Hence, there exists $\tau \in \langle E, I\rangle$ such that $\tau^{-1}\sigma^{-1}\tau$ is a substitution over $\mathcal{A}$. We call this new substitution *inverse substitution* of σ. We thus can define the hull $\mathcal{X}_\sigma^{-1}$ of the inverse of an invertible substitution over $\mathcal{A}$.

Example 5.5. With σ being the Fibonacci substitution, the map σ^2 is the substitution $\varrho := \sigma^2 : a \to aba, b \to ab$. Then, $\varrho^{-1} = a \to b^{-1}a, b^{-1} \to b^{-1}b^{-1}a$ is a substitution on $\{a, b^{-1}\}$.

Theorem 2.4 extends in a natural with respect to the equivalence relation. We first introduce the following equivalence relation on matrices: two 2×2 matrices M and M' with entries in $\mathbb{Z}$ are said to be *equivalent*: $M \cong M'$, if there exists a matrix Q in the set $\{\left(\begin{smallmatrix} \pm1 & 0 \\ 0 & \pm1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \pm1 \\ \pm1 & 0 \end{smallmatrix}\right)\}$ such that $M' = Q^{-1}MQ$.

Theorem 2.4 becomes: if σ is an invertible substitution over $\mathcal{A}$ and τ an endomorphism of F_2, one has $\tau \cong \sigma$ if, and only if, their substitution matrices are equivalent.

We now extend the notion of conjugation to tilings, hulls and strand spaces by introducing suitable equivalence classes on these spaces.

Two hulls $\mathcal{X}_\sigma, \mathcal{X}_{\sigma'}$ over two two-letter alphabets $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent*, if there is a letter-to-letter morphism τ with $\tau(\mathcal{X}_\sigma) = \mathcal{X}_{\sigma'}$: $\mathcal{X}_\sigma \cong \mathcal{X}_\sigma$. In particular, if σ' is an automorphism of F_2 that is equivalent to an invertible primitive substitution σ, then we can define $\mathcal{X}_{\sigma'}$. Furthermore, we have seen in the proof of Theorem 5.1 that if σ and σ' are conjugate primitive substitutions over $\{a, b\}$, then $\mathcal{X}_\sigma = \mathcal{X}_{\sigma'}$.

Theorem 5.1 becomes for σ, σ' two-letter invertible primitive substitutions: $\mathcal{X}_\sigma \cong \mathcal{X}_{\sigma'}$ if and only if there are k, n such that $\sigma^k \cong (\sigma')^m$. In a more formal way, it reads:

$$\mathcal{X}_\sigma \cong \mathcal{X}_{\sigma'} \quad \text{iff} \quad \exists k, n \in \mathbb{N}, \tau \in \{\text{id},\ E\} : \ \tau^{-1}\sigma^k\tau \sim (\sigma')^n.$$

Two tilings $\mathcal{T}, \mathcal{T}'$ are said *equivalent*, if they are *similar*, i.e., there are $c > 0, t \in \mathbb{R}$ such that $c\mathcal{T} + t = \mathcal{T}'$.

Assume that $\mathcal{T}$ and $\mathcal{T}'$ are two equivalent tilings of two tiling spaces associated with two primitive tile-substitutions σ and σ'. Then, by primitivity, all the tilings of both tilings spaces are equivalent. The two tiling spaces $\mathbb{X}_\sigma$ and $\mathbb{X}_{\sigma'}$ are said to be *equivalent*: $\mathbb{X}_\sigma \cong \mathbb{X}_{\sigma'}$.

By abuse of notation, we extend the notion of equivalence to the set of hulls, strand spaces, dual strand spaces and tiling spaces by using the coding maps (tiling coding, strand coding, dual strand coding). For instance, we say that a dual strand space is conjugate to a hull if $\psi^*(\mathbf{X}_\sigma)$ is equivalent to $\mathcal{X}_\sigma$. In particular, one has $\mathbb{X}_\sigma \cong \mathbf{X}_\sigma \cong \mathcal{X}_\sigma$. One has also $\mathbf{X}_{\sigma^*} \cong \mathbf{X}_\sigma^*$.

Example 5.6. Our definitions of conjugate and equivalent take care that sequences are related up to renaming of letters. For instance,

$$\ldots abaababaabaabab \ldots \cong \ldots ba^{-1}bba^{-1}ba^{-1}bba^{-1}bba^{-1}ba^{-1} \ldots,$$

since one sequence is obtained from the other by replacing a with b, and b with a^{-1}. Moreover, we can compare tiling spaces with hulls of words and strand spaces. For instance, for Example 5.5 we obtain $\mathbb{X}_{Fib^2} \cong \mathcal{X}_{Fib^2} \cong \mathbf{X}_{Fib^2}$.

5.3. **Inversion and duality.** We now can state the main theorem of this section:

Theorem 5.7. *Let σ be a primitive invertible substitution on two letters. Then*

$$\mathbf{X}_\sigma^\star \cong \mathcal{X}_{\sigma^{-1}} \cong \mathbb{X}_{\sigma^\star}.$$

Furthermore, $\sigma^* \cong \sigma^{-1}$.

Proof. In order to show the equivalence of $\sigma^\star$ and $E_1^*(\sigma)$, note that it does not matter on which lattice the paths in $E_1(\sigma)$ is defined: All lattices of full rank in $\mathbb{R}^2$ are isomorphic. So, let the underlying lattice be $\Lambda = \langle \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}\ell\\\ell^\star\end{smallmatrix}\right)\rangle_{\mathbb{Z}}$, rather than $\mathbb{Z}^2$. The substitution matrix M_σ acts as an automorphism on Λ: The image of $\left(\begin{smallmatrix}1 & \ell\\1 & \ell^\star\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right) \in \Lambda$ under M_σ is $\left(\begin{smallmatrix}1 & \ell\\1 & \ell^\star\end{smallmatrix}\right) M_\sigma \left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right)$. [Similar for $\mathbb{Z}[\ell]$]

In the next paragraph, the important idea is that M acts also as automorphism in $\mathbb{Z}[\ell]$, and that a stepped path considered here is in one-to-one correspondence with a tiling of the line.

We proceed by translating the formal sum

$$E_1^*(\sigma)(W, i^*) = \sum_{n,j: w_n^{(j)} = i} \left(M_\sigma^{-1}(W + L(s_n^{(j)})), j^*\right)$$

into the language of digit sets and tile-substitutions, and into the internal space H, resp. into the module $\mathbb{Z}[\lambda] \subset H$. Then, multiplication by M_σ^{-1} in the integers $\mathbb{Z}[\lambda]$, embedded in G, is just multiplication by λ^{-1} in G. And, by construction of the lattice Λ, multiplication by M_σ^{-1} in the integers $\mathbb{Z}[\lambda]$, embedded in H, is multiplication by $(\lambda^{-1})^{-1} = \lambda$ in H. Furthermore, since

$$f : \mathcal{A}^\star \to \mathbb{Z}^2, \quad f(u) = (\#a \text{ in } u, \#b \text{ in } u),$$

the term $M_\sigma^{-1} f(u)$, projected to H, reads in $\mathbb{Z}[\ell]$ as $f(u) = (\#a \text{ in } u) + (\#b \text{ in } u)\ell^\star$. The formal sum above translates into

$$x + T_i^\star \mapsto \{\lambda x + T_j^\star - t_{ijn} \,|\, n, j, \text{ such that the } n\text{-th letter in } \sigma(j) \text{ is } i\},$$

where t_{ijn} denotes the 'prefix' of the n-th letter, this means here: the digit d_{ijn} stared, that is, $d_{jin}^\star$. In other words, this means

$$x + T_i^\star \mapsto \lambda x + \{T_j^\star - d_{ji}^\star \,|\, i = 1, \ldots, m\} = \lambda x + \{T_j^\star - \mathcal{D}_{ji}^\star \,|\, i = 1, \ldots, m\}.$$

This shows that $\sigma^\star$ and $E_1^*(\sigma)$ are equivalent.

The equivalence $\mathcal{X}_{\sigma^{-1}} \cong \mathcal{X}_{\sigma^*}$ comes from the fact that σ^{-1} and σ^* have equivalent Abelianisation matrices. Indeed, assume σ has determinant 1. The substitution matrix of σ^{-1} is equal to $\left(\begin{smallmatrix}d & -b\\-c & a\end{smallmatrix}\right)$ whereas σ^* has as substitution matrix the transpose of M_σ. We conclude by noticing that

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

We can even say more: $\sigma^* = \mu^{-1} \circ \sigma^{-1}\mu$ for μ: $a \mapsto b^{-1}$, $b \mapsto a$. Indeed we check it on the generators $E, G, \tilde{G}$. We then prove it by induction by sing the fact that $(\sigma \circ \tau)^{-1} = \tau^{-1} \circ \sigma^{-1}$ and $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$. □

6. Selfduality

It is natural to ask which tiling spaces are selfdual, i.e., for which σ holds $\mathbb{X}_\sigma \cong \mathbb{X}_{\sigma^*}$.

Definition 6.1. *A primitive invertible two-letter substitution is said* selfdual, *if* $\sigma = \sigma^*$. *The hull of a primitive invertible two-letter substitution is said* selfdual, *if* $\mathcal{X}_\sigma \cong \mathcal{X}_{\sigma^*}$.

According to Theorem 5.7, on a two-letter alphabet , selfduality for the hull $\mathcal{X}_\sigma$ means that

$$\mathcal{X}_\sigma \cong \mathcal{X}_{\sigma^{-1}} \cong \mathcal{X}_{\sigma^*},$$

or equivalently $\sigma \cong \sigma^*$, since σ and σ^* have the same substitution factor.

By the equivalence of (a) and (b), it follows from [8]:

Proposition 6.2. *If σ is selfdual, then $P^{-1}M_\sigma P = M_\sigma^T$ for some permutation matrix P.*

The following theorem gives necessary and sufficient conditions for a primitive invertible substitution on two letters to be self-dual.

Theorem 6.3. *Let $P_1 = \mathrm{id}$, $P_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $Q_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $Q_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. If σ is a two-letter primitive invertible substitution with $\det M_\sigma = 1$, then TFAE*

(1) *σ is selfdual;*
(2) *M_σ is of the form*

$$M_\sigma = \begin{pmatrix} m & k \\ \frac{m^2-1}{k} & m \end{pmatrix} \quad or \quad M_\sigma = \begin{pmatrix} m & k \\ k & \frac{k^2+1}{m} \end{pmatrix},$$

where $k \geq 1$ divides $m^2 - 1$, respectively $m \geq 1$ divides $k^2 + 1$.
(3) *$Q_i^T M_\sigma Q_i = (M_\sigma)^{-1}$ for $i = 1$ or $i = 2$;*
(4) *$P_i^T M_\sigma P_i = (M_\sigma)^T$ for $i = 1$ or $i = 2$.*

Proof. Let us prove that (1) $\Rightarrow$ (2). Let σ be selfdual. Consequently,

$$\exists \tau \in S,\ \exists w \in \mathcal{A}^*,\ \forall x \in \{a, b\} : w^{-1}\tau^{-1}\Big(\sigma\big(\tau(x)\big)\Big)w = \sigma^{-1}(x).$$

It follows for the Abelianisation:

$$\forall x \in \{a, b\} : -L(w) + (M_\tau)^{-1} M_\sigma M_\tau(e_x) + L(w) = M_{\sigma^{-1}}(e_x),$$

thus $(M_\tau)^{-1} M_\sigma M_\tau = M_{\sigma^{-1}}$ with M_τ in the set $\{\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}\right)\}$, that is, $M_\sigma \cong M_\sigma^{-1}$.

Since the matrices of M_σ^{-1} and M_σ^T are also equivalent, we deduce that $M_\sigma \cong M_\sigma^T$. One checks that the only possible matrices for the congruence are the permutation matrices P_1 or P_2. From $\det M = 1$, we then deduce that M_σ is either of the form $M = \left(\begin{smallmatrix} m & k \\ \frac{m^2-1}{k} & m \end{smallmatrix}\right)$ or $M' = \left(\begin{smallmatrix} m & k \\ k & \frac{k^2+1}{m} \end{smallmatrix}\right)$.

Le us prove that (2) $\Rightarrow$ (1). Let σ be invertible with substitution matrix of the form in Theorem 6.3 (3). One checks that $M_{\sigma^\star} = M_\sigma^T \cong M_\sigma$. The claim follows from the extension of Theorem 2.4 quoted in Section 5.2.

It is easily seen that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) by simple computation. Indeed, compute the inverse matrices: $M^{-1} = \left(\begin{smallmatrix} m & -k \\ -\frac{m^2-1}{k} & m \end{smallmatrix}\right)$, resp. $M'^{-1} = \left(\begin{smallmatrix} \frac{k^2+1}{m} & -k \\ -k & m \end{smallmatrix}\right)$. These are obtained as $Q_1^{-1} M Q_1$, resp. $Q_2^{-1} M' Q_2$, with Q_i as in the theorem. Obviously, $Q_i^T = Q_i^{-1}$. $\square$

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