SELFDUAL SUBSTITUTIONS IN DIMENSION ONE

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Abstract. There are several notions of the 'dual' of a word/tile substitution. We show that the most common ones are equivalent for substitutions in dimension one, where we restrict ourselves to the case of two letters/tiles. Furthermore, we obtain necessary and sufficient arithmetic conditions for substitutions being selfdual in this case. Since many connections between the different notions of word/tile substitution are discussed, this paper may also serve as a survey paper on this topic.

1. Introduction

Substitutions are simple but powerful tools to generate a large number of nonperiodic structures with a high degree of order. Examples include infinite words (e.g., the Thue Morse sequence, see [36, 35]), infinite tilings (e.g., Penrose tilings, see [38, 43]) and discrete point sets (e.g., models of atomic positions in quasicrystals, see [3, 31]). Here we consider several instances of the concept of substitutions:

(a) word substitutions
(b) nonnegative endomorphisms of the free group $F_2 = \langle a, b \rangle$
(c) tile-substitutions
(d) dual maps of substitutions

Each of the concepts above gives rise to the concept of a dual substitution. Our first goal is to show the full equivalence of the distinct notions of dual substitution with respect to the concepts above for the case of two letters, or of two tiles in $\mathbb{R}^1$. Thus we will exclusively study substitutions on two letters, and for the sake of clarity, we will define every term for this special case only. From here on, we will consider only this case, whether or not this is stated explicitly.

Let us mention that there is a wealth of results for word substitutions on two letters, thus for tile-substitutions in $\mathbb{R}^1$ with two tiles. For a start see [36, 29, 35] and references therein. The following theorem lists some interesting results which emerged in the work of many authors during the last decades. We recall that a word substitution is a morphism of the free monoid. A word substitution is said to be primitive if there exists a power n such that the image of any letter by σ^n contains all the letters of the alphabet.

Theorem 1.1. *Let σ be a primitive word substitution on two letters. Then the following are equivalent:*

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(1) *Each bi-infinite word u generated by σ is* Sturmian*, i.e., u contains exactly $n+1$ different words of length n for all $n \in \mathbb{N}$.*
(2) *The endomorphism $\sigma : F_2 \to F_2$ is invertible, i.e., $\sigma \in Aut(F_2)$.*
(3) *Each tiling generated by σ is a cut and project tiling whose window is an interval.*
(4) *There is $k \geq 1$ such that the substitution σ^k is conjugate to $L^{a_1} E L^{a_2} E L^{a_3} \cdots E L^{a_m}$, where $a_i \in \mathbb{N} \cup \{0\}$, $L : a \to a$, $b \to ab$, $E : a \to b$, $b \to a$.*

The equivalence of (2) and (4) follows from [45]. The equivalence of (2) and (3) appears in [26], and relies on earlier results, see references in [26, 11]. The equivalence of (1) and (3) is due to [32, 14]. For the equivalence of (1) and (4), see [40]. For more details, see [29, 35].

This theorem links automorphisms of the free group F_2 with bi-infinite words over two letters, and with tilings of the line by intervals of two distinct lengths. Our motivation was to work out this correspondence.

The paper is organised as follows: Section 2 provides the necessary definitions and facts about word substitutions, and about (inverses of) substitutions as automorphisms of F_2. Section 3 contains the relevant definitions and facts about (duals of) substitution tilings and cut and project tilings. The very close relation of the natural decomposition method of [42] to duals of cut and project tilings via so-called Rauzy fractals is given in Section 3.3. Section 4 introduces generalised substitutions and the essential definitions and facts for strand spaces and dual maps of substitutions. Any reader familiar with any of this may skip the corresponding section(s). Section 5 explains how the distinct concepts of dual substitutions are related. The first main result, Theorem 5.10, shows that they all are equivalent, in the sense, that they generate equivalent hulls. (For the definition of the hull of a word substitution see Equation (2) below. For a precise definition of equivalence of hulls, see Section 5.) In Section 6 we give several necessary and sufficient conditions for a substitution to be selfdual, namely, Theorems 6.4 and 6.7. These results yield a complete classification of selfdual word substitutions on two letters.

2. Word substitutions

2.1. **Basic definitions and notation.** Let $\mathcal{A} = \{a, b\}$ be a finite alphabet, let $\mathcal{A}^*$ be the set of all finite words over $\mathcal{A}$, and let $\mathcal{A}^{\mathbb{Z}} = \{(u_i)_{i \in \mathbb{Z}} \mid u_i \in \mathcal{A}\}$ be the set of all bi-infinite words over $\mathcal{A}$. A substitution σ is a map from $\mathcal{A}$ to $\mathcal{A}^* \setminus \{\varepsilon\}$; it is called *non-erasing* if it maps $\mathcal{A}$ to $\mathcal{A}^* \setminus \{\varepsilon\}$, where ε denotes the empty word. We will exclusively consider here non-erasing substitutions. By concatenation, a substitution extends to a map from $\mathcal{A}^*$ to $\mathcal{A}^*$, and to a map from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$. Thus a substitution can be iterated. For instance, consider the Fibonacci substitution

$$\sigma : \mathcal{A} \to \mathcal{A}^*, \quad \sigma(a) = ab, \ \sigma(b) = a. \tag{1}$$

We abbreviate this long notation by $\sigma : a \to ab, b \to a$. Then, $\sigma(a) = ab$, $\sigma^2(a) = \sigma(\sigma(a)) = \sigma(ab) = \sigma(a)\sigma(b) = aba$, $\sigma^3(a) = abaab$, and so on. In order to rule out certain non-interesting cases, for instance $E \colon a \mapsto b, b \mapsto a$, the following definition is useful.

Definition 2.1. *Let σ be a substitution defined on $\mathcal{A}$. The* substitution matrix M_σ *associated with σ is defined as its Abelianisation, i.e.,*

$$M_\sigma = (|\sigma(j)|_i)_{1 \leq i,j \leq 2},$$

where $|w|_i$ *stands for the number of occurrences of the letter* i *in* w.
A substitution σ *on* $\mathcal{A}$ *is called* primitive*, if its substitution matrix is primitive, that is, if* $(M_\sigma)^n > 0$ *for some* $n \in \mathbb{N}$.
A substitution σ *is called* unimodular*, if its substitution matrix* M_σ *has determinant* 1 *or* -1.

In the sequel, we will consider unimodular primitive substitutions only. The requirement of primitivity is a common one, essentially this rules out some pathological cases. The requirement of unimodularity is a restriction which we use because we will focus on the case where the substitution is invertible (see Section 2.2 for a definition).

We will consider the set of *all* bi-infinite words which are 'legal' with respect to a substitution σ. One reason for this choice of working with bi-infinite words (instead of infinite words) comes from the connections with tile-substitutions that we stress in Section 3. Thus we define the *hull* $\mathcal{X}_\sigma$ of σ as follows.

$$\mathcal{X}_\sigma = \{u \in \mathcal{A}^{\mathbb{Z}} \mid \text{each subword of } u \text{ is a subword of } \sigma^n(a) \text{ or } \sigma^n(b) \text{ for some } n\}. \tag{2}$$

Subwords of a word are commonly also called 'factors' of that word.

Proposition 2.2. [36, 35] *For any primitive substitution* σ *and for all* $n \geq 1$ *holds:* $\mathcal{X}_{\sigma^n} = \mathcal{X}_\sigma$.

There is a further notion of a hull, namely, the hull of a bi-infinite word. In our framework (substitutions are assumed to be primitive) the two notions coincide. The hull $\mathcal{X}_u$ of a bi-infinite word u is defined as the closure of the orbit $\{S^k u \mid k \in \mathbb{Z}\}$ in the obvious topology, see [36, 35]. Here S denotes the shift operator $Su = S(u_i)_{i\in\mathbb{Z}} = (u_{i+1})_{i\in\mathbb{Z}}$, which yields $S^{-1}u = (u_{i-1})_{i\in\mathbb{Z}}$.

Proposition 2.3. *Let* σ *be a primitive substitution. Then, for each* $u \in \mathcal{X}_\sigma$, $\mathcal{X}_\sigma = \mathcal{X}_u$. *In particular,* $\mathcal{X}_\sigma$ *is determined by each* $u \in \mathcal{X}_\sigma$ *uniquely.*

Let σ be a primitive substitution. By the Perron-Frobenius theorem, its substitution matrix M_σ has a dominant real eigenvalue $\lambda > 1$, which has a positive eigenvector (i.e., an eigenvector having all its components > 0). We call λ the *inflation factor* of σ (see Section 3.1 for a justification of this term). If $(1-\alpha, \alpha)$ is the normed eigenvector of λ ($0 < \alpha < 1$), that is, the vector whose sum of coordinates equals 1, then α is called the *frequency* of the substitution. Indeed, it is not hard to see that every bi-infinite word in $\mathcal{X}_\sigma$ has well defined letter frequencies $1-\alpha$ and α, see for instance [36]. Left eigenvectors associated with the inflation factor λ will also play an important role here. We will work with the positive left eigenvector $v_\lambda = (1, l_\lambda)$ with first coordinate 1.

In all that follows, $\sigma(i)_k$ stands for the k-th letter of $\sigma(i)$, $|\sigma(i)|$ is the length of $\sigma(i)$ and $\sigma(i)[k-1]$ is the prefix of length $k-1$ of $\sigma(i)$ (with $\sigma(i)_0$ being the empty word), i.e.,

$$\sigma(i) = \sigma(i)_1 \cdots \sigma(i)_{|\sigma(i)|} = \sigma(i)[k-1]\sigma(i)_k\sigma(i)_{k+1} \cdots \sigma(i)_{|\sigma(i)|}.$$

We denote by A the *Abelianisation map* from $\mathcal{A}^*$ to $\mathbb{Z}^2$: if w is a word in $\mathcal{A}^*$, then $A(w)$ is the vector that counts the number of occurrences of each letter in w, i.e., $A\colon \mathcal{A}^* \to \mathbb{Z}^2$, $w \mapsto (|w|_a, |w|_b)$. There is an obvious commutative diagram, where M_σ stands for the substitution matrix of σ:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ A\downarrow & & \downarrow A \\ \mathbb{Z}^2 & \xrightarrow{M_\sigma} & \mathbb{Z}^2. \end{array}$$

2.2. **Substitutions as endomorphisms of F_2.** Naturally, any word substitution on $\mathcal{A}$ gives rise to an endomorphism σ of F_2, the free group on two letters. (Note that not every endomorphism of F_2 gives rise to a proper word substitution, consider for instance $\sigma : a \to ab^{-1}$, $b \to b^{-1}$.) If this endomorphism σ happens to be an automorphism, that is, $\sigma \in \mathrm{Aut}(F_2)$, then σ is said to be *invertible*. It is well-known that the set of invertible word substitutions on a two-letter alphabet is a finitely generated monoid, with one set of generators being

$$E\colon a \mapsto b,\ b \mapsto a,\ L\colon a \mapsto a,\ b \mapsto ab,\ \tilde{L}\colon a \mapsto a,\ b \mapsto ba. \tag{3}$$

For references, see [45] and Chap. 2 in [29].

In the sequel, we want to consider whether two substitutions generate the same sequences, that is, the same hull. Recall that an automorphism γ is called *inner* automorphism, if there exists $w \in F_2$ such that $\gamma(x) = wxw^{-1}$ for every $x \in F_2$. We let γ_w denote this inner automorphism.

Definition 2.4. *We say that two given substitutions σ and ϱ are conjugate, if $\sigma = \gamma_w \circ \varrho$ for some $w \in F_2$. In this case, we will write $\sigma \sim \varrho$.*

Remark 2.5. We use here the term 'conjugate' but a more precise terminology would be the following: σ and ϱ belong to the same outer class, or else, ϱ is obtained from σ by action of an inner automorphism (also called conjugation). For convenience, we take the freedom to use the short version. Note that if σ and ϱ are conjugate, then w is a suffix or a prefix of $\varrho(x)$ for $x = a, b$, and $\sigma(x)$ has the same length as the word $\varrho(x)$ for every $x \in \mathcal{A}$.

Recall the following theorem of Nielsen [34]: given two automorphisms σ and ϱ of the free group F_2, they have the same substitution matrix if and only if they are conjugate. (Here, the substitution matrix relies on the Abelianisation map from F_2 to $\mathbb{Z}^2$ that counts the number of occurrences of each 'positive' letter minus the number of occurrences of each 'negative' letter.) We thus conclude in terms of substitutions (see [40] for a combinatorial proof):

Theorem 2.6. *Two invertible substitutions σ and ϱ are conjugate if and only if they have the same substitution matrix.*

2.3. **A rigidity result.** The following theorem is a classical rigidity result for two-letter primitive substitutions by P. Séébold [40], see also [25]. Here we provide two alternative proofs. Let us note that the apparent simplicity of the first proof relies on the use of Theorem 2.6 and the defect theorem (Theorem 1.2.5 in [28]), whereas the original proof of [40] was self-contained.

Theorem 2.7. *Let σ, ϱ be primitive substitutions on the two-letter alphabet $\mathcal{A}$. If $\sigma^k \sim \varrho^m$ for some k, m, then $\mathcal{X}_\sigma = \mathcal{X}_\varrho$.*
Furthermore, if σ and ϱ are invertible, then $\mathcal{X}_\sigma = \mathcal{X}_\varrho$ if and only if $\sigma^k \sim \varrho^m$ for some k, m.

In plain words, this theorem states that two invertible substitutions are conjugate (up to powers) if and only if their hulls are equal. In even different words (compare Proposition 2.3): if u is a bi-infinite word obtained by some invertible primitive substitution on two letters, where the substitution is unknown, then u determines the substitution uniquely, up to conjugation and up to powers of the inflation factor. An immediate consequence is the following result.

Corollary 2.8. *Let σ, ϱ be primitive invertible substitutions on the alphabet $\mathcal{A} = \{a, b\}$ with the same inflation factor. Then, $\sigma \sim \varrho$ if and only if $\mathcal{X}_\sigma = \mathcal{X}_\varrho$, which is also equivalent to σ and ϱ having the same frequency α.* □

Proof. Let us prove Theorem 2.7. Because of Proposition 2.2 we can restrict ourselves to the case $k = m = 1$. First we prove that if $\sigma \sim \varrho$, then $\mathcal{X}_\sigma = \mathcal{X}_\varrho$. Thus let $\sigma \sim \varrho$. This means there is $w \in F_2$ such that $\sigma(x) = w\varrho(x)w^{-1}$ for $x = a, b$. Consequently, for $x \in \mathcal{A}$ holds: $w\varrho(x) = \sigma(x)w$, and in general for $u \in \mathcal{A}^*$, $w\varrho(u) = \sigma(u)w$, which yields for $x \in \mathcal{A}$,

$$w\varrho(w)\cdots\varrho^{k-1}(w)\varrho^k(x) = \sigma^k(x)w\varrho(w)\cdots\varrho^{k-1}(w). \tag{4}$$

Setting $w^{(k)} := w\varrho(w)\cdots\varrho^{k-1}(w)$, we can write shortly

$$w^{(k)}\varrho^k(x) = \sigma^k(x)w^{(k)}.$$

Let $x = a$. By the defect theorem (Theorem 1.2.5 in [28]) this commutation relation implies

$$\varrho^k(a) = u_k v_k, \quad \sigma^k(a) = v_k u_k \tag{5}$$

for some words u_k and v_k, with u_k being nonempty. Now, let $u \in \mathcal{X}_\sigma$, and let v be some subword of u. Then, by primitivity, v is a subword of $\sigma^k(a)$ for some k. Since $\max(|u_k|, |v_k|) \to \infty$ for $k \to \infty$ (σ is primitive), v is also contained in u_k or v_k for k large enough. Thus it is also contained in $\varrho^k(a)$ for some k, thus v is a word in $\mathcal{X}_\varrho$. The same argument holds vice versa, thus $\mathcal{X}_\sigma = \mathcal{X}_\varrho$.

For the other direction, we first note that if λ is an irrational eigenvalue of a 2×2 integer matrix M then it is an algebraic integer, and the algebraic conjugate λ' of λ is the second eigenvalue of M. If $v = (1, v_\lambda)$ is an eigenvector of M corresponding to λ, then $v' = (1, v'_\lambda)$ (again, v'_λ denotes the algebraic conjugate of v_λ) is an eigenvector corresponding to λ'. If the matrix is furthermore primitive, both vectors are distinct by the Perron-Frobenius theorem, and thus linearly independent.

Let σ, ϱ be two primitive and invertible substitutions such that $\mathcal{X}_\sigma = \mathcal{X}_\varrho$. Let $u \in \mathcal{X}_\sigma$. Let us recall that the the vector of frequencies $(1 - \alpha, \alpha)$ of letters in u is an eigenvector of the (up to here unknown) substitution matrices M_σ and M_ϱ. Hence the vectors $(1, \ell) = (1, \frac{\alpha}{1-\alpha})$ and $(1, \ell')$ are eigenvectors for both matrices. We now consider the eigenvalues associated with the previous eigenvectors. Since σ is invertible, M_σ is unimodular. Its (up to here unknown) inflation factor λ is therefore a unit in the underlying ring of integers of the form $\mathbb{Z}[\sqrt{k}]$ for some $k \geq 1$. It is well known that the unit group of $\mathbb{Z}[\sqrt{k}]$ is generated by a *fundamental unit* z. (This is a consequence of the fact that there is a fundamental unit for the solution of the corresponding Pell's equation, or a consequence of Dirichlet's unit theorem, see for instance [27] or [33]). Thus λ is a power of the generating element z. Let $\lambda = z^n$, where n is arbitrary but fixed. The same holds for the inflation factor of M_ϱ which also belongs to $\mathbb{Z}[\sqrt{k}]$, and which is thus of the form z^m. By algebraic conjugation we obtain the second eigenvalue, and the second eigenvector of M_ϱ.

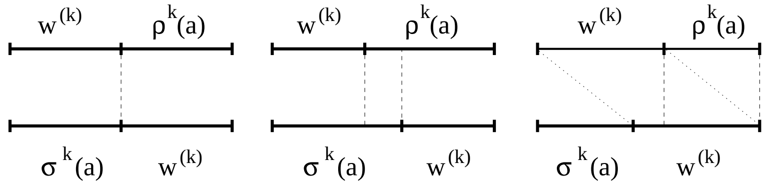

FIGURE 1. Three cases in the proof of Theorem 2.7

Since the eigenvectors $(1, \ell)$ and $(1, \ell')$ are linearly independent eigenvectors, on which the substitution matrices of σ^n and ϱ^m act in the same way, σ^n and ϱ^m have thus the same substitution matrix. By the fact that all invertible substitutions with the same substitution matrix are conjugate (see Theorem 2.6), the claim of the theorem follows. □

Remark 2.9. One can prove also directly from (4) that $\varrho^k(x)$ and $\sigma^k(x)$ share a common subword, without using the defect theorem. This can be done as follows: We prove Equation (5) by the following argument:

We have to distinguish three cases (compare Figure 1).

Case 1: If $|w^{(k)}| = |\varrho^k(a)| = |\sigma^k(a)|$, then $w^{(k)} = \varrho^k(a) = \sigma^k(a)$ (Figure 1, left).

Case 2: If $|w^{(k)}| < |\varrho^k(a)| = |\sigma^k(a)|$ (Figure 1, centre), then $w^{(k)}$ is a prefix of $\varrho^k(a)$ and a suffix of $\sigma^k(a)$. Moreover, the remaining suffix of $\varrho^k(a)$ overlaps the remaining prefix of $\sigma^k(a)$. Thus we obtain again Equation (5):

$$\varrho^k(a) = u_k v_k, \quad \sigma^k(a) = v_k u_k \tag{6}$$

for some nonempty words $u_k (= w^{(k)})$ and v_k.

Case 3:: If $|w^{(k)}| > |\varrho^k(a)| = |\sigma^k(a)|$ (Figure 1, right), then $\varrho^k(a)$ is a suffix of $w^{(k)}$. Either this suffix of $w^{(k)}$ overlaps already with $\sigma^k(a)$ (as in the figure), then (5) holds for some nonempty words u_k, v_k. Or (if $|w^{(k)}(a)| \geq 2|\varrho^k(a)|$) $\varrho^k(a)\varrho^k(a)$ is a suffix of $w^{(k)}$, and we continue with the shorter words $w^{(k)}$ vs $\sigma^k(a)w^{(k)}$ without the suffix $\varrho^k(a)$. After finitely many steps, we are in one of the first two cases.

In either case, (5) holds for some nonempty word u_k. (In Case 1, just let v_k be the empty word.)

Remark 2.10. Let us note that the assumption that σ is invertible is crucial in Theorem 2.7 as shown by the following example (see [25]). Consider on the alphabet $\{a, b\}$ the following two primitive substitutions:

$$\sigma : \ a \mapsto ab, \ b \mapsto baabbaabbaabba, \ \varrho : \ a \mapsto abbaab, \ b \mapsto baabbaabba.$$

One has $\sigma(ab) = \varrho(ab)$ and $\sigma(ba) = \varrho(ba)$. We deduce that σ and ϱ have the same fixed point beginning by a, and thus $\mathcal{X}_\sigma = \mathcal{X}_\varrho$. Nevertheless, σ and ϱ are neither conjugate, nor conjugate up to a power of a common substitution (their substitution matrices are neither conjugate in $GL(2, \mathbb{Z})$, nor conjugate up to a power to a common matrix).

3. TILE-SUBSTITUTIONS

3.1. **Substitution tilings.** In contrast to word substitutions, which act on symbolic objects like words, tile-substitutions act on geometric objects, like tiles or tilings. A *tiling* of $\mathbb{R}^d$ is a collection of compact sets which cover topologically $\mathbb{R}^d$ in a non-overlapping way, that is, the interiors of the tiles are pairwise disjoint. In $\mathbb{R}^1$ there is a natural correspondence between bi-infinite words and tilings when the tiles are intervals: just assign to each letter an interval of specified length.

In general, a tile-substitution in $\mathbb{R}^d$ is given by a set of *prototiles* $T_1, \dots, T_m \subset \mathbb{R}^d$, an expanding map and a rule how to dissect each expanded prototile into isometric copies of some prototiles T_i. Here, we restrict ourselves to two prototiles in dimension one, and, moreover, our prototiles are always intervals, with the expanding map being given by an *inflation factor* $\lambda > 1$. (For the discussion of analogues of some results of the present paper for the case of more general tilings, see [38, 43, 18]). The precise definition of a tile-substitution in $\mathbb{R}$, where the prototiles are intervals, goes as follows.

Definition 3.1. *A (self-similar) tile-substitution in* $\mathbb{R}$ *is defined via a set of intervals* $T_1, \dots T_m$ *— the* prototiles *— and a map* s*. Let*

$$\lambda T_j = \bigcup_{i=1}^{m} T_i + \mathcal{D}_{ij} \quad (1 \le j \le m), \tag{7}$$

where the union is not overlapping (i.e., the interiors of the tiles in the union are pairwise disjoint), and each $\mathcal{D}_{ij}$ *is a finite (possibly empty) subset of* $\mathbb{R}^d$*, called* digit set*. Then*

$$s(T_j) := \{T_i + \mathcal{D}_{ij} \mid i = 1 \dots m\}$$

is called a tile-substitution*. It is called* primitive *if the substitution matrix* $M_s := (|\mathcal{D}_{ij}|)_{1 \le i,j \le 2}$ *is primitive, where* $|\mathcal{D}_{ij}|$ *stands for the cardinality of the set* $\mathcal{D}_{ij}$*.*

By $s(T_j + x) := s(T_j) + \lambda x$ and $s(\{T, T'\}) := \{s(T), s(T')\}$, s extends in a natural way to all finite or infinite sets of copies of the prototiles.

In analogy to word substitutions we want to deal with the space $\mathbb{X}_s$ of all substitution tilings arising from a given tile-substitution. Note the correspondence with the definition of the hull of a word substitution, see Equation (2). The main difference between both types of associated dynamical systems is that $\mathcal{X}_s$ is endowed with a $\mathbb{Z}$-action by the shift, whereas $\mathbb{X}_s$ is endowed with an $\mathbb{R}$-action defined by the action of translations.

Definition 3.2. *Let* s *be a primitive tile-substitution with prototiles* T_1, T_2*. The* tiling space $\mathbb{X}_s$ *is the set of all tilings* T*, such that each finite set of tiles of* T *is contained in some translate of* $s^n(T_1)$ *or* $s^n(T_2)$*. Any element of* $\mathbb{X}_s$ *is called a* substitution tiling *generated by* s*.*

Remark 3.3. Any primitive self-similar tile-substitution is uniquely defined by its digit set matrix $\mathcal{D}$. This holds because one can derive the inflation factor λ and the prototiles T_i from the digit set matrix $\mathcal{D} := \big(\mathcal{D}_{ij}\big)_{ij}$. This is not only true for two tiles in one dimension, but for any self-similar tile-substitution in $\mathbb{R}^d$. For details, see [18]. Here we just mention two facts: the inflation factor λ is the largest eigenvalue of the primitive substitution matrix M_s. And the prototiles are the unique compact solution of the multi component IFS (iterated function system) in the sense of [30] (also called graph-directed IFS), which is obtained by dividing (7) by λ. In particular the vector of lengths of the tiles is a *left* eigenvector of the substitution matrix M_σ.

Remark 3.4. A one-dimensional tile-substitution (where the tiles are intervals) yields a unique word substitution: just replace the tiles by symbols. Conversely, one can realize any primitive word substitution as a tile-substitution by taking as lengths l_a, l_b for the prototiles the coordinates of a positive left eigenvector associated with its inflation factor λ: the intervals T_i are chosen so that they line up with the action of the word substitution. We chose here to normalise the eigenvector $v_\lambda = (1, \ell_\lambda) = (l_a, l_b)$ by taking its first coordinate equal to 1. For $j = a, b$, let $T_j = [0, l_j]$. For $j = a, b$, if $\sigma(j) = \sigma(j)_1 \cdots \sigma(j)_{|\sigma(j)|}$ then $\lambda l_j = \sum_{k=1}^{|\sigma(j)|} l_{\sigma(j)_k}$, i.e., the tile T_j is inflated by the factor λ, so it can be subdivided into translates of the prototiles according to the substitution rule:

$$\lambda T_j = [0, \lambda l_j] \mapsto \{T_{\sigma(j)_1}, T_{\sigma(j)_2} + l_{\sigma(j)_1}, \ldots, T_{\sigma(j)_{|\sigma(j)|}} + l_{\sigma(j)_1} + \cdots + l_{\sigma(j)_{|\sigma(j)|-1}}\}.$$

This can be written as

$$\lambda T_j = \bigcup_{i,j\,:\,(j,k)\in F_i} T_j + \delta([\sigma(j)]_{k-1}),$$

where

$$F = \{(j,k) \mid j \in \mathcal{A},\ 1 \le k \le |\sigma(j)|\}$$

and

$$F_i = \{(j,k) \in F \mid \sigma(j)_k = i\},$$

with the valuation map $\delta\colon \mathcal{A}^* \to \mathbb{R}^+$ being defined for any $w = w_1 \cdots w_m \in \mathcal{A}^*$ as

$$\delta(w_1 \ldots w_m) = l_{w_1} + \cdots + l_{w_m} = |w|_a + |w|_b \ell_\lambda = \langle A(w), v_\lambda \rangle,$$

by recalling that A stands for the Abelianisation map. We then set for all i, j

$$\mathcal{D}_{ij} = \{\delta([\sigma(j)]_{k-1}) \mid (j,k) \in F_i\}.$$

We illustrate this by the following example.

Example 3.5. Consider the square of the Fibonacci substitution from Equation (1), namely, $\varrho = \sigma^2 : a \to aba,\ b \to ab$. We work here with the square of the Fibonacci substitution since the determinant of its substitution matrix equals 1. We will realize it as a tile-substitution as follows (see Figure 2): Let $T_1 = [0,1]$, $T_2 = [0, 1/\tau]$, where $\tau = \frac{\sqrt{5}+1}{2}$. Note that $\lambda = \tau^2$ is the dominant eigenvalue for M_{σ^2}, with $(1, 1/\tau)$ being an associated left eigenvector. Then

$$\tau^2 T_1 = [0, 2+1/\tau] = T_1 \cup (T_2+1) \cup (T_1 + 1 + 1/\tau); \quad \tau^2 T_2 = [0, 1+1/\tau] = T_1 \cup (T_2+1),$$

where the unions are disjoint in measure. (Note that $\tau = 1 + 1/\tau$.) Hence the last equation yields a tile-substitution s:

$$s(T_1) = \{T_1, T_2+1, T_1+\tau\}, \quad s(T_2) = \{T_1, T_2+1\}.$$

For an illustration of this substitution, see Figure 2. This substitution s can be encoded in the digit sets $\mathcal{D}_{1,1} = \{0,\tau\}, \mathcal{D}_{2,1} = \{1\}, \mathcal{D}_{1,2} = \{0\}, \mathcal{D}_{2,2} = \{1\}$. This can be written conveniently as a digit set matrix:

$$\mathcal{D} = \begin{pmatrix} \{0,\tau\} & \{0\} \\ \{1\} & \{1\} \end{pmatrix}. \tag{8}$$

By comparison with Definition 2.1 we note that we can derive the substitution matrix from the digit set matrix simply as follows: $M_{\sigma^2} = \mathcal{D} = (|\mathcal{D}_{ij}|)_{1 \le i,j \le 2}$. In this case we get the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Its dominant eigenvalue is the inflation factor $\lambda = \tau^2$.

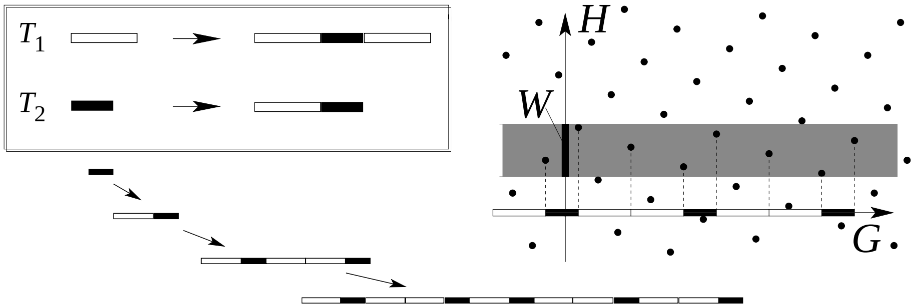

FIGURE 2. The Fibonacci tile-substitution s (box top left), some iterates of s on T_2, and the generation of a Fibonacci tiling as a cut and project tiling (right). The interval W defines a horizontal strip, all lattice points within this strip are projected down to the line.

3.2. **Cut and project tilings.** In order to define a notion of duality for tile-substitutions, we now work in the framework of cut and project sets.

Certain substitution tilings can be obtained by a cut and project method. There is a large number of results about such cut and project sets, or model sets, see [31] and references therein. In our setting, this is pretty simple to explain, compare Figure 2. Let $G = H = \mathbb{R}$, let $\pi_1 : G \times H \to G$, $\pi_2 : G \times H \to H$ be the canonical projections, and let Λ be a lattice in $G \times H = \mathbb{R}^2$, such that $\pi_1 : \Lambda \to G$ is one-to-one, and $\pi_2(\Lambda)$ is dense in H. Then, choose some compact set $W \subset H$ with W being the closure of its interior, and let

$$V = \{\pi_1(x) \mid x \in \Lambda,\ \pi_2(x) \in W\}.$$

Then V is a *cut and project set* (or model set). Since V is a discrete point set in $\mathbb{R} = G$ (W is bounded), it induces a partition of $\mathbb{R}$ into intervals. Regarding these (closed) intervals as tiles yields a tiling of $\mathbb{R}$. Such a tiling is called *cut and project tiling.* Furthermore, a tile-substitution s is said to yield the cut and project set V if V is the set of left endpoints of an element of $\mathbb{X}_s$. According to Theorem 1.1, a primitive word substitution σ on two letters yields cut and project tilings whose window W is an interval if and only if it is invertible [26].

Given a primitive tile-substitution s which is known to yield a cut and project set V, one can construct Λ and W out of s in a standard way. In general, Λ and W are not unique. The following construction provides a choice of Λ and W when s comes from a primitive two-letter substitution, according to Remark 3.4. It has the advantage that everything can be expressed in some algebraic number field, which allows the use of algebraic tools. See also [20] for more details on cut and project schemes in the case of $G = H = \mathbb{R}$.

We thus start with a primitive and unimodular word substitution σ on a two-letter alphabet such that its associated tile-substitution s yields the cut and project set V. Since the substitution M_σ is an integer matrix, its eigenvalues λ and λ' are two conjugate quadratic irrational numbers. (The case where λ is an integer requires that the internal space H is non-Euclidean [4, 41]. Since our substitutions will always be unimodular in the sequel, this case cannot occur here.) Let $v_\lambda = (1, \ell_\lambda)$ be the left eigenvector associated with the dominant eigenvalue λ. This eigenvector yields the 'natural' lengths of the prototiles. Thus let $T_1 = [0, 1]$, $T_2 = [0, \ell_\lambda]$.

Note that 1 and ℓ are rationally independent. Now, let

$$\Lambda = \langle v, w\rangle_{\mathbb{Z}} = \{\alpha\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) + \beta\left(\begin{smallmatrix}\ell_\lambda\\\ell'_\lambda\end{smallmatrix}\right) \mid \alpha,\beta\in\mathbb{Z}\}, \tag{9}$$

where ℓ'_λ denotes the algebraic conjugate of ℓ_λ. The projections π_1 and π_2 correspond to the canonical projections. Now, consider the set V of endpoints of the intervals in a tiling in $\mathbb{X}_s$. Without loss of generality, let one endpoint be 0. Then all other endpoints are of the form $\alpha+\beta\ell_\lambda \in \mathbb{Z}[\ell_\lambda] \subset \mathbb{Q}(\lambda)$. Any point $\alpha+\beta\ell_\lambda \in V$ has a unique preimage in Λ (since 1, ℓ_λ are rationally independent), namely, $\alpha\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) + \beta\left(\begin{smallmatrix}\ell_\lambda\\\ell'_\lambda\end{smallmatrix}\right)$. Thus, each point $\alpha+\beta\ell_\lambda$ in V has a unique counterpart in the internal space H, namely $\pi_2\circ\pi_1^{-1}(\alpha+\beta\ell_\lambda) = \alpha+\beta\ell'_\lambda$. The map $\pi_2\circ\pi_1^{-1}$ is called *star map* and will be abbreviated by $\star$.

Since $\pi_2(\Lambda)$ is dense in H and W is the closure of its interior, the window W is obtained as the closure of $\pi_2\circ\pi_1^{-1}(V)$. Note that the fact that V *is* a cut and project tiling guarantees that W is a bounded set, since being compact. With the help of the star map we can write shortly $W = \mathrm{cl}(V^\star)$. In our context, the star map has a very simple interpretation: by construction of the lattice Λ, the star map is just mapping an element of $\mathbb{Q}(\lambda)$ to its algebraic conjugate

$$(\alpha+\beta\ell_\lambda)^\star = \alpha+\beta\ell'_\lambda. \tag{10}$$

Nevertheless, in general it is more complicated, and one should keep in mind that the star map maps G to H. In general, G and H can be very different from each other, for instance of different dimension.

Now recall that any self-similar tile-substitution is uniquely defined by its digit set matrix $\mathcal{D}$ (Remark 3.3). This allows us to define the star-dual of a tile-substitution by applying the star map to the transpose $\mathcal{D}^T$ of the matrix $\mathcal{D}$, as performed in [44, 19, 18] where this 'Galois' duality for tile-substitutions is developed.

Definition 3.6. *Let s be a primitive invertible self-similar tile-substitution yielding cut and project tilings, with digit set matrix $\mathcal{D}$. Then the* star-dual *substitution $s^\star$ of s is the unique tile-substitution defined by $(\mathcal{D}^T)^\star$, with the star map being defined in (10).*

Here $X^\star$ means the application of the star map to each element of some set $X \subset \mathbb{Q}(\lambda)$ separately. This definition together with Definition 3.2 defines the *star-dual* tiling space $\mathbb{X}_{s^\star}$.

Example 3.7. The star-dual of the tile-substitution associated with the squared Fibonacci substitution in Example 3.5 is easily obtained by applying the star map to the transpose of the digit set matrix in (8). We obtain

$$(\mathcal{D}^T)^\star = \begin{pmatrix}\{0,-\tau^{-1}\} & \{1\}\\ \{0\} & \{1\}\end{pmatrix}.$$

3.3. **Natural decomposition method and Rauzy fractals.** We now explain how to associate a candidate window W with any primitive two-letter substitution σ in such a way that the corresponding tile-substitution s yields a cut and project set. This candidate is the so-called Rauzy fractal associated with the substitution σ that is introduced below.

In this section we follow [42], originally defined on a d-letter alphabet. We restrict here to the case $d=2$. As above, let σ be a primitive unimodular substitution on the two-letter alphabet $\mathcal{A}=\{a,b\}$. Let $\lambda>1$ be its inflation factor and λ' the other eigenvalue of the substitution matrix M_σ. Let $u=(u_n)_{n\in\mathbb{N}}$ be an infinite word such that $\sigma(u)=u$. (Note that we use one-sided infinite words here, in accordance with [42]. The extension to bi-infinite words is straightforward.) Since σ is primitive, it suffices to replace σ by a suitable power of σ for such a fixed point word to exist.

The set $\mathcal{X}_\sigma$ is mapped into $\mathbb{R}$ via a valuation (compare with the valuation map δ introduced in Remark 3.4), which is a map $\Delta : \mathcal{A}^* \to \mathbb{R}$, satisfying $\Delta(vw) = \Delta(v)+\Delta(w)$ and $\Delta(\sigma(w)) = \lambda'\Delta(w)$ for all $v, w \in \mathcal{A}^*$. Hence, Δ satisfies for all $w \in \mathcal{A}^*$

$$\Delta(w) = |w|_a + |w|_b \ell_{\lambda'} = \langle A(w), v_{\lambda'} \rangle,$$

where $v_{\lambda'} = (1, \ell_{\lambda'})$ is the normalised left eigenvector associated with the eigenvalue λ' of the substitution matrix M_σ. According to [37], the set

$$\mathfrak{R} := \overline{\{\Delta(u_0 u_1 \dots u_m) \mid m \geq 0\}}$$

is called the *Rauzy fractal* associated with the one-sided fixed point u of the substitution σ. Since σ is unimodular, $|\lambda'| < 1$ and one deduces that $\mathfrak{R}$ is a compact set. For more about Rauzy fractals, see [35], [8] or else [10] and the references therein.

Let

$$\mathfrak{R}_i := \overline{\{\Delta(u_0 u_1 \dots u_m) \mid u_{m+1} = i,\ m \geq 0\}}$$

where $i \in \mathcal{A}$. Clearly $\mathfrak{R} = \mathfrak{R}_a \cup \mathfrak{R}_b$. We shall call $(\mathfrak{R}_a, \mathfrak{R}_b)$ the *natural decomposition* of the Rauzy fractal $\mathfrak{R}$. The natural decomposition of $\mathfrak{R}$ is the attractor of a graph directed IFS (cf. [30] and also [22]) in the following way, with the notation of Remark 3.4: $(\mathfrak{R}_a, \mathfrak{R}_b)$ satisfies

$$\mathfrak{R}_i = \bigcup_{(j,k)\in F_i} (\lambda' \mathfrak{R}_j + \Delta([\sigma(j)]_{k-1}), \quad \text{for } i \in \mathcal{A}, \tag{11}$$

with F_i being defined in Remark 3.4. To prove it, we use the fact that $\sigma(u) = u$.

Note that the sets $\mathfrak{R}_i$ are the closure of their interior and their boundary has zero measure, as proved in [42] in general case of a d-letter substitution. For more properties, see e.g. [35, 8, 10]. Furthermore, the sets $\mathfrak{R}_i$ are not necessarily intervals. In fact it can be proved that they are intervals if and only if the two-letter primitive substitution σ is invertible (see also Theorem 1.1). For more details, see e.g. [26, 13, 11] for proofs of this folklore result.

A moment of thought yields that Δ maps the m-th letter of u, which corresponds to the right endpoint $|u|_a + |u|_b \ell_\lambda$ of the m-th tile in the tiling in G (corresponding to the word u) to $|u|_a + |u|_b \ell'_\lambda$ in H. In other words, $\mathfrak{R}$ is a right candidate for the window W for the tiling in G, with $V = \delta(\{u_0 \dots u_m \mid m \geq 0\})$ (see Remark 3.4 for the definition of the map δ):

$$\mathfrak{R} = \overline{\pi_2 \circ \pi_1^{-1}\{\delta(u_0 \dots u_m) \mid m \geq 0\}},$$

with π_1 and π_2 being defined in Section 3.2. In particular $\pi_2 \circ \pi_1^{-1}\delta(u_0 \dots u_m) = \Delta(u_0 \dots u_m)$, for any $m \geq 0$. It remains to check that $\{\pi_1(x) \mid x \in \Lambda,\ \pi_2(x) \in \mathfrak{R}\} = V$, i.e., that we do not have V strictly included in $\{\pi_1(x) \mid x \in \Lambda,\ \pi_2(x) \in \mathfrak{R}\}$. This comes from the following result.

Theorem 3.8. [5, 23, 21] *Let σ be a primitive unimodular substitution on two letters. Then σ yields the cut and project set $V = \{\delta(u_0 \dots u_m) \mid m \geq 0\}$ with associated window $\mathfrak{R}$.*

Example 3.9. We consider the square of the Fibonacci substitution studied in Example 3.5: $\varrho : a \to aba,\ b \to ab$. Its inflation factor is $\lambda = \tau^2$, where $\tau = \frac{1+\sqrt{5}}{2}$. We define the valuation Δ with respect to the left eigenvector $(1, \tau' - 1) = (1, \tau'^{-1})$ associated with the eigenvalue $\lambda'(= 1/\tau^2)$.

The natural decomposition $(\mathfrak{R}_a, \mathfrak{R}_b)$ of its Rauzy fractal $\mathfrak{R}$ is given by the solution of the equation

$$\begin{array}{rcl}\mathfrak{R}_a & = & \lambda'\mathfrak{R}_a + \Delta([\sigma(a)]_0) \cup (\lambda'\mathfrak{R}_a + \Delta([\sigma(a)]_2)) \cup (\lambda'\mathfrak{R}_b + \Delta([\sigma(b)]_0)) \\ \mathfrak{R}_b & = & (\lambda'\mathfrak{R}_a + \Delta([\sigma(a)]_1)) \cup (\lambda'\mathfrak{R}_b + \Delta([\sigma(b)]_1)).\end{array}$$

So the previous equation can be written as

$$\begin{array}{rcl}\mathfrak{R}_a & = & \lambda'\mathfrak{R}_a \cup (\lambda'\mathfrak{R}_a + \tau') \cup \lambda'\mathfrak{R}_b = \lambda'\mathfrak{R}_a \cup (\lambda'\mathfrak{R}_a - \tau^{-1}) \cup \lambda'\mathfrak{R}_b \\ \mathfrak{R}_b & = & (\lambda'\mathfrak{R}_a + 1) \cup (\lambda'\mathfrak{R}_b + 1).\end{array}$$

The intervals $\mathfrak{R}_a = [-1, \tau^{-1}]$ and $\mathfrak{R}_b = [\tau^{-1}, \tau]$ satisfy this equation.

Moreover, the natural decomposition of $\mathfrak{R}$ yields a tile-substitution in $\mathbb{R}$, see Definition 3.1, in the following way: from (11) we get

$$(\lambda')^{-1}\mathfrak{R}_i = \bigcup_{(j,k)\in F_i} \big(\mathfrak{R}_j + \lambda'^{-1}\Delta([\sigma(j)]_{k-1})\big), \text{ for } i \in \mathcal{A}.$$

Note that we assume that $\det(M_\sigma) = 1$, thus $\lambda\lambda' = 1$, which yields

$$\lambda\mathfrak{R}_i = \bigcup_{j\in\mathcal{A}} (\mathfrak{R}_j + \mathcal{E}_{ji}), \tag{12}$$

where

$$\mathcal{E}_{ji} := \{\lambda\Delta([\sigma(j)]_{k-1}) \mid (j,k) \in F_i\}. \tag{13}$$

Example 3.10. We continue with the square of the Fibonacci substitution studied in Example 3.5 and 3.9: $\varrho : a \to aba$, $b \to ab$. The intervals $\mathfrak{R}_a = [-1, \tau^{-1}]$ and $\mathfrak{R}_b = [\tau^{-1}, \tau]$ generate a dual tiling in $H = \mathbb{R}$ with the digit set

$$\mathcal{E} = \begin{pmatrix} \{0, -\tau\} & \{\tau^2\} \\ \{0\} & \{\tau^2\} \end{pmatrix}.$$

The tiling obtained by (12) is the dual tiling of the tiling described below, where duality is in the sense of the cut and project scheme. Indeed, we consider as in Remark 3.4 the tile-substitution s associated with σ with prototiles the intervals T_i of length l_i, for $i \in \mathcal{A}$, by recalling that the vector $v_\lambda = (l_a, l_b)$ is the positive left λ-eigenvector of M_σ such that $l_a = 1$. Observe that we can associate with the two-sided fixed point $u = \ldots u_{-1}.u_0u_1u_2\ldots$ of the substitution σ the tiling

$$\{\ldots T_{u_{-1}} - l_{u_{-1}}, T_{u_0}, T_{u_1} + l_{u_0}, T_{u_2} + l_{u_0} + l_{u_1}, \ldots\}.$$

According to Section 3.2, we define the $\star$ map as:

$$\star : \mathbb{Q}(\lambda) \to \mathbb{R},\ \lambda \mapsto \lambda'.$$

Since $\delta(vw) = \delta(v) + \delta(w)$ and $\delta(\sigma(w)) = \lambda\delta(w)$, for any word $v, w \in \mathcal{A}^*$, it follows that $\Delta(w) = (\delta(w))^\star$, for all $w \in \mathcal{A}^*$. Hence, by (13), one gets

$$(\mathcal{D}^\star_{ij})^T_{ij} = \lambda^{-1}(\mathcal{E}_{ij})_{ij}.$$

Example 3.11. One checks that $(\mathcal{D}^\star_{ij})^T_{ij} = \lambda^{-1}(\mathcal{E}_{ij})_{ij}$ for the matrices of Example 3.7 and 3.10.

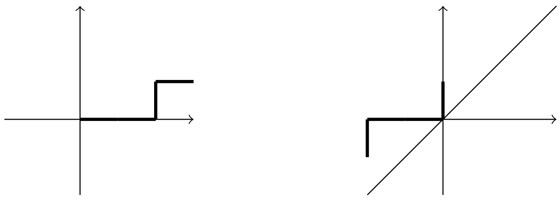

FIGURE 3. The path associated with $aaba$ (left), an example of a finite strand with coding word $baab$ (right).

4. Dual maps of substitutions

In this section, we present a notion of substitution whose production rule is a formal translation of (12).

4.1. **Generalised substitutions.**

We follow here the formalism introduced in [2, 39] defined originally on a d-letter alphabet. We restrict ourselves here to the case $d = 2$. Let $\mathcal{A}$ be the finite alphabet $\{a, b\}$.

Finite strand Let (e_a, e_b) stand for the canonical basis of $\mathbb{R}^2$. One associates with each finite word $w = w_1 w_2 \dots w_n$ on the two-letter alphabet $\mathcal{A}$ a path in the two-dimensional space, starting from 0 and ending in $A(w)$, with vertices in $\{A(w_1 \dots w_i) \mid i = 1 \dots n\}$: we start from 0, advance by e_i if the first letter is i, and so on. For an illustration, see Figure 3 (left).

More generally, we define the notion of strand by following the formalism of [6]. A *finite strand* is a subset of $\mathbb{R}^2$ defined as the image by a piecewise isometric map $\gamma\colon [i, j] \to \mathbb{R}^2$, where $i, j \in \mathbb{Z}$, which satisfies the following: for any integer $k \in [i, j)$, there is a letter $x \in \{a, b\}$ such that $\gamma(k+1) - \gamma(k) = e_x$. If we replace $[i, j]$ by $\mathbb{Z}$, we get the notion of *bi-infinite strand.* A strand is thus a connected union of unit segments with integer vertices which projects orthogonally in a one-to-one way onto the line $x = y$ (see Figure 3). In particular, the path associated with a finite word w such as defined in the previous paragraph is a finite strand.

We introduce the following notation for *elementary strands*: for $W \in \mathbb{Z}^2$ and $i \in \mathcal{A}$, we set $(W, i) = \{W + \lambda e_i \mid 0 \le \lambda \le 1\}$.

Any bi-infinite strand defines a bi-infinite word $w = (w_k)_{k \in \mathbb{Z}} \in \{a, b\}^{\mathbb{Z}}$ that satisfies $\gamma(k + 1) - \gamma(k) = e_{w_k}$. The corresponding map which sends bi-infinite strands on bi-infinite words is called *strand coding.*

This allows us to define a map on strands, coming from the word substitution, by mapping the strand for w to the strand for $\sigma(w)$. In fact, this map can be made into a linear map, in the following way. Let σ be a substitution on $\mathcal{A}$. Let us recall the notation for $i \in \{a, b\}$:

$$\sigma(i) = \sigma(i)_1 \cdots \sigma(i)_{|\sigma(i)|} = \sigma(i)[k-1]\sigma(i)_k \sigma(i)_{k+1} \cdots \sigma(i)_{|\sigma(i)|}.$$

Definition 4.1. [2, 39] *We let $\mathcal{G}$ denote the real vector space generated by elementary strands. Let $E_1(\sigma)$ be the linear map defined on $\mathcal{G}$ by:*

$$E_1(\sigma)(W, i) = \sum_{k=1}^{|\sigma(i)|} (M_\sigma . W + A(\sigma(i)[k-1]), \sigma(i)_k).$$

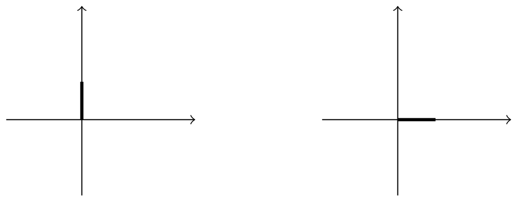

FIGURE 4. The segment $(0, a^*)$ (left) and the segment $(0, b^*)$ (right).

We call $E_1(\sigma)$ *the* one-dimensional extension *of* σ.

Definition 4.2. *Let* σ *be a primitive substitution. The* strand space $\mathbf{X}_\sigma$ *is the set of bi-infinite strands* η *such that each finite substrand* ξ *of* η *is a substrand of some* $E_1(\sigma)^n(W, x)$, *for* $W \in \mathbb{Z}^2$, $n \in \mathbb{N}$ *and* $x \in \{a, b\}$.

4.2. **Dual maps.** From now on, we suppose that σ is a unimodular substitution. In the sequel we will assume that σ has determinant $+1$. In view of Proposition 2.2 this is no restriction: if σ has determinant -1, we will consider σ^2 instead.

We want to study the dual map $E_1^*(\sigma)$ of $E_1(\sigma)$, as a linear map on $\mathcal{G}$. We thus denote by $\mathcal{G}^*$ the dual space of $\mathcal{G}$, i.e., the space of dual maps with finite support (that is, dual maps that give value 0 to all but a finite number of the vectors of the canonical basis).

The space $\mathcal{G}^*$ has a natural basis (W, i^*), for $i = a, b$, defined as the map that gives value 1 to (W, i) and 0 to all other elements of $\mathcal{G}$. It is possible to give a geometric meaning to this dual space: for $i = a, b$, we represent the element (W, i^*) as the *lower* unit segment perpendicular to the direction e_i of the unit square with lowest vertex W. By a slight abuse of notation, (W, i^*) will stand both for the corresponding dual map and for the segment, i.e.,

$$(W, a^*) = \{W + \lambda e_b \mid 0 \le \lambda \le 1\}, \ (W, b^*) = \{W + \lambda e_a \mid 0 \le \lambda \le 1\}.$$

For an illustration, see Figure 4. Such a segment is called an *elementary dual strand*.

The map $E_1(\sigma)$ has a dual map, which is easily computed:

Theorem 4.3. [2] *Let* σ *be a unimodular substitution. The dual map* $E_1^*(\sigma)$ *is defined on* $\mathcal{G}^*$ *by*

$$E_1^*(\sigma)(W, i^*) = \sum_{j,k:\ \sigma(j)[k]=i} \big(M_\sigma^{-1}(W + A(\sigma(j)_{k+1} \cdots \sigma(j)_{|\sigma(j)|}), j^*\big).$$

Furthermore, if τ *is also a unimodular substitution, then*

$$E_1^*(\sigma \circ \tau) = E_1^*(\tau) \circ E_1^*(\sigma).$$

Example 4.4. We consider the square of the Fibonacci substitution. Let $\varrho\colon a \mapsto aba, b \mapsto ab$. One has $M_\varrho = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $M_\varrho^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. We thus get

$$\begin{cases} E_1^*(\varrho)(0, a^*) = (0, a^*) + (e_b, a^*) + (-e_a + 2e_b, b^*) \\ E_1^*(\varrho)(0, b^*) = (e_a - e_b, a^*) + (0, b^*). \end{cases}$$

To be more precise, the definition of the map $E_1^*(\sigma)$ in [2] involves prefixes instead of suffixes, whereas, for $i = a, b$, the element (W, i^*) is represented by the *upper* face perpendicular to

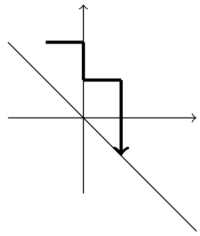

FIGURE 5. An example of a finite dual strand coded by the word *babaa*. The arrow at the end of the path indicates that we read the letters from left to right.

the direction $\vec{e}_i$ of the unit square with lowest vertex W. Nevertheless, an easy computation shows that both formulas coincide.

Dual strands. We can also define a notion of strand associated with this dual formalism. A *finite dual strand* is a subset of $\mathbb{R}^2$ defined as the image by a piecewise isometric map $\gamma\colon [i,j] \to \mathbb{R}^2$, where $i,j \in \mathbb{Z}$, which satisfies the following: for any integer $k \in [i,j)$, there is a letter $x \in \{a,b\}$ such that

$$\gamma(k+1) - \gamma(k) = e_a \text{ if } x = a,\ \gamma(k+1) - \gamma(k) = -e_b,\ \text{otherwise.}$$

Segments (W, x^*), for $W \in \mathbb{Z}^2, x \in \{a,b\}$, are in particular dual strands. If we replace $[i,j]$ by $\mathbb{Z}$, we get the notion of *bi-infinite dual strand.* A dual strand is a connected union of segments with integer vertices which projects orthogonally in a one-to-one way onto the line $x+y=0$ (see Figure 5).

Any bi-infinite dual strand defines a bi-infinite word $(w_k)_{k\in\mathbb{Z}} \in \{a,b\}^{\mathbb{Z}}$ that satisfies for all k

$$\gamma(k+1) - \gamma(k) = e_a \text{ if } w_k = b,\ \text{and } \gamma(k+1) - \gamma(k) = -e_b \text{ otherwise.}$$

Similarly, any finite dual strand s defines a finite word w. The map ψ^* that sends finite dual strands on words in $\mathcal{A}^*$ is called *dual coding.* In particular the word coding (W, a^*) is the letter b, and the word coding (W, b^*) is a.

4.3. **Dual strands.** Before being able to define the notion of dual strand space in Section 4.4, we need to recall several facts on the behaviour of $E_1^*(\sigma)$ on finite dual strands.

One bi-infinite dual strand plays here a particular role. Recall that v_λ stands for a positive left eigenvector of the substitution matrix M_σ of the primitive substitution σ associated with the inflation factor λ. Let α be the frequency of σ. We define $\mathcal{S}_\alpha$ as the union of segments (W, i^*), for $i = a, b$, that satisfy

$$0 \le \langle W, v_\lambda \rangle < \langle e_i, v_\lambda \rangle.$$

One checks that $\mathcal{S}_\alpha$ is a bi-infinite dual strand. For more details, see e.g. [11]. One key property is that this bi-infinite dual strand is preserved under the action of $E_1^*(\sigma)$.

Theorem 4.5. [2] *Let σ be a unimodular primitive two-letter substitution with frequency α. The map $E_1^*(\sigma)$ maps any elementary dual strand of $\mathcal{S}_\alpha$ on a finite union of elementary dual strands of $\mathcal{S}_\alpha$. Furthermore, if (V, i^*) and (W, j^*) are two distinct segments included in some $\mathcal{S}_\alpha$, for $\alpha \in (0,1)$, then the intersection of their images by $E_1^*(\sigma)$ is either empty, or reduced to a point.*

Note that if σ is not invertible, the image by $E_1^*(\sigma)$ of a finite dual strand might not be connected. However, if σ is invertible, connectedness is preserved: finite dual substrands of $\mathcal{S}_\alpha$ (i.e., connected unions of segments) are preserved under $E_1^*(\sigma)$. Note that a proof different from the following one can be found in [17].

Proposition 4.6. *Let σ be a primitive invertible two-letter substitution. The map $E_1^*(\sigma)$ maps every finite strand onto a finite strand.*

Proof. We first check that the generators $E, L, \tilde{L}$ of the monoid of invertible two-letter substitutions (see (3)) map every finite dual strand made of two adjacent segments to some finite dual strand:

$$\begin{cases} E_1^*(L)((x,a^*)+(x,b^*)) = (x,a^*)+(x,b^*)+(x-e_a+e_b,b^*) \\ E_1^*(L)((x,a^*)+(x+e_b,a^*)) = (x,a^*)+(x-e_a+e_b,b^*)+(x-e_a+e_b,a^*)+(x-2e_a+2e_b,b^*) \\ E_1^*(L)((x,b^*)+(x+e_a,b^*)) = (x,b^*)+(x+e_a,b^*) \\ E_1^*(L)((x,b^*)+(x+e_a-e_b,a^*)) = (x,b^*)+(x+e_a,b^*)+(2e_a-e_b,a^*) \end{cases}$$

$$\begin{cases} E_1^*(\tilde{L})((x,a^*)+(x,b^*)) = (x,a^*)+(x,b^*)+(x+e_a,b^*) \\ E_1^*(\tilde{L})((x,a^*)+(x+e_b,a^*)) = (x,a^*)+(x,b^*)+(x-e_a+e_b,a^*)+(x-e_a+e_b,b^*) \\ E_1^*(\tilde{L})((x,b^*)+(x+e_a,b^*)) = (x+e_a,b^*)+(x+2e_a,b^*) \\ E_1^*(\tilde{L})((x,b^*)+(x+e_a-e_b,a^*)) = (x+e_a,b^*)+(x+2e_a-e_b,a^*)+(2e_a-e_b,b^*) \end{cases}$$

$$\begin{cases} E_1^*(E)((x,a^*)+(x,b^*)) = (x,a^*)+(x,b^*) \\ E_1^*(E)((x,a^*)+(x+e_b,a^*)) = (x,b^*)+(x+e_a,b^*) \\ E_1^*(E)((x,b^*)+(x+e_a,b^*)) = (x,a^*)+(x+e_b,a^*) \\ E_1^*(E)((x,b^*)+(x+e_a-e_b,a^*)) = (x,a^*)+(x-e_a+e_b,b^*). \end{cases}$$

Hence these generators map finite dual strands to connected unions of unit segments with integer vertices. It remains to check that these unions are indeed dual strands (i.e., that they can be projected orthogonally in a one-to-one way to $x+y=0$). By Theorem 4.5, they all are substrands of $\mathcal{S}_\alpha$. We deduce that the generators map finite dual strands to finite dual strands.

Let us prove now that invertible substitutions map finite dual strands to finite dual strands. Let τ be a two-letter substitution that maps every finite dual strand to a finite dual one. Now, if $\sigma = \tau \circ L$, then we deduce from $E_1^*(\sigma) = E_1^*(L) \circ E_1^*(\tau)$, that the map $E_1^*(\sigma)$ also maps every finite dual strand to a connected union of unit segments, and hence by Theorem 4.5, to a finite dual strand. The same holds true for the other generators. We thus conclude by induction on the length of a decomposition on the generators $E, L, \tilde{L}$. □

4.4. **Dual strand space and dual substitution.** We now can introduce the notion of dual strand space for an invertible substitution.

Definition 4.7. *Let σ be a primitive invertible substitution over a two-letter alphabet. The* dual strand space $\mathbf{X}_\sigma^*$ *is the set of bi-infinite dual strands η such that each finite substrand ξ of η is a substrand of some $E_1^*(\sigma)^n(W,x^*)$, for $W \in \mathbb{Z}^2$, $n \in \mathbb{N}$ and $x \in \{a,b\}$.*

According to Proposition 4.6, since the image by $E_1^*(\sigma)$ of a finite strand is a finite strand, it can be coded as a substitution via the dual coding ψ^* (introduced in Section 4.2).

Definition 4.8. *Let σ be a primitive invertible word substitution over $\{a,b\}$ whose substitution matrix has determinant* 1*. The* dual substitution σ^* *is defined on the alphabet $\{a,b\}$*

as

$$\sigma^*(x) = \psi^*(E_1^*(\sigma)(0, x^*)) \text{ for } x = a, b.$$

One has $L^*\colon a \mapsto ba,\, b \mapsto b$, $\tilde{L}^*\colon a \mapsto ab,\, b \mapsto b$ and $E^*\colon a \mapsto b,\, b \mapsto a$. The substitution L^* is usually denoted as R. (we will use it in Appendix B).

Example 4.9. Let $\varrho\colon a \mapsto aba,\ b \mapsto ab$ be the square of the Fibonacci substitution. One has $\varrho^*\colon a \mapsto baa,\ b \mapsto ba$.

Remark 4.10. Let us observe that the substitution matrix of σ^* is the transpose of the substitution matrix of σ. Indeed, the dual strand coded by $\sigma^*(0, a^*)$ is located on the left of the dual strand coded by $\sigma^*(0, b^*)$ (we use the fact that M_σ has determinant $+1$). Furthermore, one checks that σ^* is invertible (it suffices to check it on the generators $E, L, \tilde{L}$). Furthermore, the inflation factor of σ^* is equal to the inflation factor λ of σ. Furthermore, σ and τ are conjugate if and only if σ^* and τ^* are conjugate.

This allows us to define a notion of dual frequency.

Definition 4.11. *The dual frequency α^* of σ is defined as the frequency of σ^*.*

The dual frequency is closely related to the algebraic conjugate of the frequency. Indeed, one has the following result.

Theorem 4.12. *Let σ be a primitive two-letter invertible substitution whose substitution matrix has determinant* 1*. Let α be its frequency and α' its algebraic conjugate. One has*

$$\alpha^* = \frac{\alpha' - 1}{2\alpha' - 1}$$

Proof. According to Remark 4.10, the substitution matrix of σ^* is the transpose of the substitution matrix of σ. The eigenvector of frequencies $(\alpha^*, 1 - \alpha^*)$ of σ^* is thus orthogonal to the eigenvector $(\alpha', 1 - \alpha')$ of M_σ. Indeed, they have distinct eigenvalues which are respectively λ and λ'. This yields $\alpha^*\alpha' + (1 - \alpha^*)(1 - \alpha') = 0$, which gives the desired conclusion. $\square$

Remark 4.13. There exist several codings for dual strands and dual substitutions that are possible [16, 11, 9]. One could indeed code strands from right to left, or exchange the roles played by a and b. In particular, the coding developed by [16] yields $\widetilde{\sigma^{-1}}$, where $\tilde{\tau}$ is the substitution deduced from τ by reading the letters in the images of letters by τ in reverse order. The substitution matrices of these substitutions are either equal or transpose of the substitution matrix of σ. Furthermore, their frequency belongs to $\{\alpha^*, 1 - \alpha^*\}$. Note also that a similar adequate coding can be introduced if the substitution matrix M_σ has determinant -1.

5. Relations between distinct concepts of ‘dual substitution’

In this section we will compare the various notions of duality we have introduced so far. For this purpose we will introduce the appropriate equivalence relations among hulls, tiling spaces etc. But first we will have a closer look on the inverse of a word substitution.

5.1. **Equivalences.** If σ is a primitive invertible substitution on $\{a,b\}$, then it is clear that σ^{-1} cannot be a substitution on the original letters a,b, since for the Abelianisation would follow: $M_\sigma M_{\sigma^{-1}} = M_\sigma (M_\sigma)^{-1} = \mathrm{id}$, but this is impossible for primitive $M_\sigma \geq 0$ and $(M_\sigma)^{-1} \geq 0$. But we have the following:

Proposition 5.1. *Let σ be an invertible substitution over $\{a,b\}$. Assume that its substitution matrix M_σ has determinant $+1$. Then its inverse is a substitution on the two-letter alphabet $\{a^{-1},b\}$. Furthermore if σ is primitive, then its inverse is also primitive.*

This is a point where we need $\det(M_\sigma) = 1$. For $\det(M_\sigma) = -1$, things are more difficult (σ^{-1} would mix a^{-1}, b with a, b^{-1}). In such a case we consider σ^2.

Proof. That σ^{-1} is indeed a substitution over $\{a^{-1},b\}$ with Abelianisation $M_{\sigma^{-1}}$ can easily be checked by an induction on the length of a decomposition of σ on the set of generators $\{E, L, \tilde{L}\}$ (see (3)). It suffices to consider L^{-1} and $\tilde{L}^{-1}$, they are substitutions over $\{a^{-1},b\}$. □

In Section 6 we will address the question in which cases the substitution σ^{-1} yields the same hull as σ (i.e., the same set of bi-infinite words), up to renaming letters. We know already that σ^{-1} is always a substitution on $\{a^{-1},b\}$. This leaves two possibilities for renaming the letters, namely $a \mapsto a^{-1}$, $b \mapsto b$, or $a \mapsto b^{-1}$, $b \mapsto a$. Both cases will occur: see Example 5.4 and 5.5 below where two examples of substitutions are given whose respective inverses yield the same hull as the original substitutions, up to renaming letters. Thus we define the 'inverse' substitution (that we call reciprocal) as $\tau_a \sigma^{-1} \tau_a$, where τ_a exchanges a with a^{-1}, and later (see Definition 6.1) we will regard σ as selfdual, if its reciprocal is either conjugate to σ itself, or to $E\sigma E$, with $E : a \to b,\ b \to a$. This notion coincides with the notion of duality introduced in [9], see also in the same flavour [24].

Definition 5.2. *Let σ be a substitution over $\mathcal{A} = \{a,b\}$ with determinant* 1, *and let $\tau_a \in Aut(F_2)$, $\tau_a : a \to a^{-1},\ b \to b$. If σ is invertible, then we call $\overline{\sigma} := \tau_a \sigma^{-1} \tau_a^{-1}$ the* reciprocal substitution *of σ.*

Remark 5.3. If $\det M_\sigma = 1$, then $M_E M_{\overline{\sigma}} M_E = M_\sigma^T$. Let $M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Then, since $\det M_\sigma = 1$, one has $M_{\sigma^{-1}} = \left(\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right)$, and the result follows from $M_{\overline{\sigma}} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) M_\sigma^{-1} \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In particular, the substitutions σ and $\overline{\sigma}$ have the same inflation factor.

Example 5.4. The square of the Fibonacci substitution (see Example 3.5) is the substitution ϱ: $a \to aba$, $b \to ab$. Then, ϱ^{-1}: $a^{-1} \to a^{-1}b$, $b \to a^{-1}bb$, and

$$\overline{\varrho}(a) = \tau_a \varrho^{-1} \tau_a^{-1}(a) = \tau_a \varrho^{-1}(a^{-1}) = \tau_a(a^{-1}b) = ab.$$

$$\overline{\varrho}(b) = \tau_a \varrho^{-1} \tau_a^{-1}(b) = \tau_a \varrho^{-1}(b) = \tau_a(a^{-1}bb) = abb.$$

Thus the reciprocal substitution of ϱ is $\overline{\varrho} : a \to ab,\ b \to abb$.

Now we can see that ϱ and $\overline{\varrho}$ are conjugate, after renaming the letters according to $E : a \to b,\ b \to a$. Indeed, the word $w = a$ yields a conjugation between ϱ and $E\overline{\varrho}E$.

$$aE\overline{\varrho}E(a)a^{-1} = abaaa^{-1} = aba = \varrho(a),$$
$$aE\overline{\varrho}E(b)a^{-1} = abaa^{-1} = ab = \varrho(b).$$

In particular, it follows that the hulls $\mathcal{X}_\varrho$ and $\mathcal{X}_{\overline{\varrho}}$ are equal (up to renaming letters). This is a case where the letters change their role: in $\mathcal{X}_\varrho$ the letter a is more frequent, whereas in $\mathcal{X}_{\overline{\varrho}}$ the letter b is more frequent.

Example 5.5. Consider $\sigma : a \to abaab,\ b \to ababaab$. Then, $\sigma^{-1}\colon a^{-1} \to ba^{-1}a^{-1}ba^{-1},\ b \to ba^{-1}a^{-1}ba^{-1}ba^{-1}$. Thus $\overline{\sigma}\colon a \to baaba,\ b \to baababa$. Here, $w = b^{-1}a^{-1}a^{-1}b^{-1}$ yields a conjugation between σ and $\overline{\sigma}$:

$$\begin{aligned} w\overline{\sigma}(a)w^{-1} &= b^{-1}a^{-1}a^{-1}b^{-1}baababaab = abaab = \sigma(a) \\ w\overline{\sigma}(b)w^{-1} &= b^{-1}a^{-1}a^{-1}b^{-1}baabababaab = ababaab = \sigma(b) \end{aligned}$$

This is an example where the letters do not change their role.

Now we want to compare hulls with tiling spaces and strand spaces. Thus we introduce suitable equivalences of these spaces.

Definition 5.6. *Two hulls $\mathcal{X}_\sigma, \mathcal{X}_\varrho$ over $\{a,b\}$ are* equivalent*, short $\mathcal{X}_\sigma \cong \mathcal{X}_\varrho$, if there is a letter-to-letter morphism τ, such that $\mathcal{X}_\sigma = \tau(\mathcal{X}_\varrho) = \{\tau(u) \mid u \in \mathcal{X}_\varrho\}$.*

In other words, $\mathcal{X}_\sigma$ and $\mathcal{X}_\varrho$ are equivalent either if $\mathcal{X}_\sigma = \mathcal{X}_\varrho$, or if $\mathcal{X}_\sigma = E(\mathcal{X}_\varrho)$, where $E : a \to b,\ b \to a$.

Definition 5.7. *Two tilings $\mathcal{T}, \mathcal{T}'$ are called* equivalent*, short: $\mathcal{T} \cong \mathcal{T}'$, if they are* similar*, i.e., there are $c > 0, t \in \mathbb{R}$ such that $c\mathcal{T} + t = \mathcal{T}'$. Such a map is called* similarity*. Let s, s' be primitive tile-substitutions. The tiling spaces $\mathbb{X}_s$ and $\mathbb{X}_t$ are called equivalent, short: $\mathbb{X}_s \cong \mathbb{X}_{s'}$, if there is a one-to-one similarity mapping $\mathbb{X}_s$ to $\mathbb{X}_{s'}$.*

By the analogue of Prop. 2.3 for tiling space, $\mathbb{X}_s \cong \mathbb{X}_{s'}$ whenever there are $\mathcal{T} \in \mathbb{X}_s, \mathcal{T}' \in \mathbb{X}_{s'}$ such that $\mathcal{T} \cong \mathcal{T}'$, with s, s' being primitive.

We have seen in Section 3.1 that a word substitution σ yields a one-dimensional tile-substitution in a canonical way: the substitution matrix M_σ yields the inflation factor λ, and the left eigenvector of λ (unique up to scaling) yields the tile lengths. This, together with the order of the tiles in σ, yields the digit set matrix $\mathcal{D}$. Vice versa, a one-dimensional tile-substitution (where the tiles are intervals) yields a unique word substitution: just replace the tiles by symbols. Whenever a word substitution σ is linked with a tile-substitution in this manner, we say $\sigma \cong s$. If one of them (thus both) are primitive, then we write $\mathcal{X}_\sigma \cong \mathbb{X}_s$ whenever $\sigma \cong s$.

Using the coding maps (strand coding, dual strand coding), we can define equivalence between strand spaces and dual strand spaces as follows.

Definition 5.8. *Let σ, σ' be two unimodular primitive substitutions over a two-letter alphabet. We define $\mathcal{X}_\sigma \cong \mathbf{X}_{\sigma'}$ if $\psi(\mathbf{X}_{\sigma'}) = \mathcal{X}_\sigma$, and $\mathcal{X}_\sigma \cong \mathbf{X}^*_{\sigma'}$ if $\psi^*(\mathbf{X}_{\sigma'}) = \mathcal{X}_\sigma$. Furthermore, one has $\mathbf{X}_\sigma \cong \mathbf{X}_{\sigma'}$ if $\psi(\mathbf{X}_{\sigma'}) = \psi(\mathbf{X}_\sigma)$.*

Note that

$$\mathbf{X}^*_\sigma \cong \mathbf{X}_{\sigma^*} \cong \mathcal{X}_{\sigma^*}.$$

5.2. **Equivalence theorem.** Let us summarise these considerations.

Remark 5.9. Let σ be an invertible substitution on two letters, and let s be some tile-substitution derived from σ as described above. Then, by definition,

$$\mathbf{X}_\sigma \cong \mathcal{X}_\sigma \cong \mathbb{X}_s.$$

Now we can state the main theorem of this section:

Theorem 5.10. *Let σ be a primitive invertible substitution on two letters with determinant 1. Then*

$$\mathbf{X}^*_\sigma \cong \mathbf{X}_{\sigma^*} \cong \mathcal{X}_{\overline{\sigma}} \cong \mathcal{X}_{\sigma^*} \cong \mathbb{X}_{s^\star}.$$

A different proof of the equivalence between σ^* and $E_1^*(\sigma)$ can also be found in [9].

Proof. In order to show the equivalence of $s^\star$ and $E_1^*(\sigma)$, note that it does not matter on which lattice the paths in $E_1(\sigma)$ are defined: all lattices of full rank in $\mathbb{R}^2$ are isomorphic. We use here the notation of Section 3.2. So, let the underlying lattice be $\Lambda = \langle \binom{1}{1}, \binom{\ell_\lambda}{\ell^\star_\lambda} \rangle_{\mathbb{Z}}$, rather than $\mathbb{Z}^2$. The substitution matrix M_σ acts as an automorphism on Λ. Indeed, since the projection π_1 from Λ to the first coordinate is $(1, \ell_\lambda)\binom{\alpha}{\beta} = \alpha + \beta\ell$, and $(1, \ell_\lambda)$ is a left eigenvector of M_σ, multiplication by M_σ in Λ acts as multiplication by λ in $\mathbb{Z}[\ell_\lambda]$:

$$(1, \ell_\lambda) M_\sigma \binom{\alpha}{\beta} = \lambda(1, \ell_\lambda)\binom{\alpha}{\beta} = \lambda(\alpha + \beta\ell_\lambda).$$

The same holds for $\ell^\star_\lambda$. In the next paragraph, the important idea is that M_σ acts thus as an automorphism in $\mathbb{Z}[\ell]$, and that a stepped path considered here is in one-to-one correspondence with a tiling of the line.

We proceed by translating the formal sum

$$E_1^*(\sigma)(W, i^*) = \sum_{j,k:\ \sigma(j)[k]=i} \left(M_\sigma^{-1}(W + A(\sigma(j)_{k+1} \cdots \sigma(j)_{|\sigma(j)|}), j^* \right)$$

into the language of digit sets and tile-substitutions, and into the internal space H. Then, multiplication by M_σ^{-1} in the integers $\mathbb{Z}[\ell_\lambda]$, embedded in G, is just multiplication by λ^{-1} in G. And, by construction of the lattice Λ, multiplication by M_σ^{-1} in the integers $\mathbb{Z}[\ell_\lambda]$, embedded in H, is multiplication by $(\lambda')^{-1} = (\lambda^{-1})^{-1} = \lambda$ in H (we use the fact that σ has determinant 1). Furthermore, the term $M_\sigma^{-1}A(u)$, projected to H, reads in $\mathbb{Z}[\ell_\lambda]$ as $A(u) = |u|_a + |u|_b \ell^\star_\lambda$. The formal sum above translates into

$$x + T_i^\star \mapsto \{\lambda x + T_j^\star - t_{ijn} \mid n, j, \text{ such that the } n\text{-th letter in } \sigma(j) \text{ is } i\},$$

where t_{ijn} denotes the 'prefix' of the n-th letter, this means here: the digit d_{ijn} starred, that is, $d^\star_{jin}$. Here x is the projection of W. In other words, this means

$$x + T_i^\star \mapsto \lambda x + \{T_j^\star - d^\star_{jin} \mid i = 1, \ldots, m, n = 1, \ldots, N\} = \lambda x + \{T_j^\star - \mathcal{D}^\star_{ji} \mid i = 1, \ldots, m\},$$

where $N = N(i,j) = |\{j \mid \sigma(j)_n = i\}$. This shows that $s^\star$ and $E_1^*(\sigma)$ are equivalent.

The equivalence $\mathcal{X}_{\overline{\sigma}} \cong \mathcal{X}_{\sigma^*}$ comes from the fact that $\overline{\sigma}$ and σ^* have equivalent substitution matrices in $SL(2, \mathbb{Z})$. Indeed, the substitution matrix of σ^* is the transpose of M_σ, whereas the substitution matrix of $\overline{\sigma}$ satisfies $M_E M_{\overline{\sigma}} M_E = M_\sigma^T$ according to Remark 5.3. By Theorem 2.6 and 2.7 the claim follows. $\square$

6. Selfduality

With respect to Definition 5.2, it is now natural to ask which substitutions and substitution hulls are selfdual.

Definition 6.1. *Let σ be a primitive two-letter substitution over $\{a,b\}$ with determinant 1. If $\mathcal{X}_\sigma = \mathcal{X}_{\overline{\sigma}}$, or $\mathcal{X}_\sigma = E(\mathcal{X}_{\overline{\sigma}})$, then σ is said to have a* selfdual hull.

If $\sigma \sim \overline{\sigma}$ or $\sigma \sim E\overline{\sigma}E$, then the substitution σ is said to be a selfdual substitution.

By Theorem 5.10, this definition translates immediately to star-duals $s^\star$, and to dual maps of substitution σ^*.

According to Remark 5.3 and Corollary 2.8, we deduce the following:

Proposition 6.2. *A substitution is selfdual if and only if its hull $\mathcal{X}_\sigma$ is selfdual.*

Proposition 6.3. *If σ is selfdual (with determinant 1), then $(M_\sigma)^{-1} = Q^{-1} M_\sigma Q$ for either $Q = M_{\tau_a} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, or $Q = M_{\tau_a} M_E = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; and $M_\sigma^T = P M_\sigma P^T$, with either $P = M_E = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, or $P = \mathrm{id}$.*

Proof. If σ is selfdual, then $M_\sigma = M_{\overline{\sigma}}$, or $M_\sigma = M_E M_{\overline{\sigma}} M_E$. Furthermore, by Remark 5.3, $M_E M_{\overline{\sigma}} M_E = M_\sigma^T$. Hence one has either $M_\sigma^T = M_\sigma$, or $M_\sigma^T = E M_\sigma E$.

Provided that $\det M_\sigma = +1$, one has $M_\sigma^{-1} = \left(\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right)$, with $M_\sigma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$.

If $M_\sigma = M_\sigma^T$, then one gets $(M_{\tau_a} M_E)^{-1} M_\sigma (M_{\tau_a} M_E) = \left(\begin{smallmatrix} s & -r \\ -q & p \end{smallmatrix}\right) = M_{\sigma^{-1}}$, since $r = q$.

If $M_\sigma = M_E M_\sigma^T M_E = \left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$, then $M_{\tau_a} M_\sigma M_{\tau_a} = \left(\begin{smallmatrix} p & -q \\ -r & s \end{smallmatrix}\right)$, since $p = s$. $\square$

The following theorem states that this necessary condition is already sufficient.

Theorem 6.4. *Let P, Q be as above (two possibilities each). If σ is a two-letter primitive invertible substitution with $\det M_\sigma = 1$, then the following are equivalent:*

1. *σ is selfdual;*
2. *M_σ is of the form*
$$M_{m,k} = \begin{pmatrix} m & k \\ \frac{m^2-1}{k} & m \end{pmatrix} \quad or \quad M'_{m,k} = \begin{pmatrix} m & k \\ k & \frac{k^2+1}{m} \end{pmatrix},$$
where $k \geq 1$ divides $m^2 - 1$, respectively $m \geq 1$ divides $k^2 + 1$;
3. *$Q^{-1} M_\sigma Q = (M_\sigma)^{-1}$;*
4. *$P^T M_\sigma P = (M_\sigma)^T$.*

Proof. It is easily seen that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) by simple computation. Indeed, compute the inverse matrices: $M_{m,k}^{-1} = \left(\begin{smallmatrix} m & -\frac{m^2-1}{k} \\ -k & m \end{smallmatrix}\right)$ (resp. $(M'_{m,k})^{-1} = \left(\begin{smallmatrix} \frac{k^2+1}{m} & -k \\ -k & m \end{smallmatrix}\right)$). These are obtained as $Q^{-1} M_{m,k} Q$. Obviously, $Q^T = Q^{-1}$.

We know already that (1) $\Rightarrow$ (3). It remains to prove that (3) $\Rightarrow$ (1). Let σ be invertible with substitution matrix of the form (2) in Theorem 6.4. Then, by Proposition 6.2, $M_{\overline{\sigma}} = M_\sigma$, or $M_{\overline{\sigma}} = P M_\sigma P$. Then, by Theorem 2.6, σ is conjugate to $\overline{\sigma}$, or to $E(\overline{\sigma})$, proving the claim. $\square$

Remark 6.5. By Theorem 5.10 the above result immediately transfers to selfdual tile-substitutions (on the line with two tiles) and to dual maps of substitutions.

Remark 6.6. For every matrix $M_{m,k}$ there exists a selfdual substitution having this matrix as substitution matrix, since the map $\sigma \mapsto M_\sigma$ is onto from the set of invertible two-letter substitutions onto $GL(2,\mathbb{Z})$.

The following result gives a necessary condition not only for σ being selfdual, but — slightly stronger — for $\sigma \sim \sigma^*$, in terms of the continued fraction expansion of the frequency α. Note that one can state analogous conditions for $\alpha = 1-\alpha^*$.

Theorem 6.7. *Let σ be a primitive invertible substitution on two letters with frequency α. Let α' stand for the algebraic conjugate of α. The following conditions are equivalent:*

(1) $\alpha = \alpha^*$
(2) $2\alpha\alpha' = \alpha + \alpha' - 1$
(3) $\alpha = [0; 1+n_1, \overline{n_2, \cdots, n_k, n_1}]$ *or* $\alpha = [0; 1, \overline{n_1, \cdots, n_k}]$*, with the word* $n_1 \cdots n_k$ *being a palindrome, i.e.,* $(n_1, \cdots, n_k) = (n_k, \cdots, n_1)$*.*

Proof. Assertion (2) comes from Theorem 4.12. Assertion (3) comes (2) together with the known relation between the continued fraction expansion of α and its conjugate α' for α being quadratic (see [7] Theorem B.1. (see Appendix B below). □

Appendix A. Sturmian words

Sturmian words are infinite words over a binary alphabet that have exactly $n+1$ subwords of length n for every positive integer n. Sturmian words can also be defined in a constructive way as follows. Let $\alpha \in (0,1)$. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x+\alpha$. For a given real number α, we introduce the following two partitions of $\mathbb{T}^1$:

$$\underline{I}_a = [0, 1-\alpha),\quad \underline{I}_b = [1-\alpha, 1);\quad \overline{I}_a = (0, 1-\alpha],\quad \overline{I}_b = (1-\alpha, 1].$$

Tracing the one-sided (resp. two-sided) orbit of $R_\alpha^n(\varrho)$, we define two infinite (resp. bi-infinite) words for $\varrho \in \mathbb{T}^1$:

$$\underline{s}_{\alpha,\varrho}(n) = \begin{cases} a & \text{if } R_\alpha^n(\varrho) \in \underline{I}_1, \\ b & \text{if } R_\alpha^n(\varrho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\varrho}(n) = \begin{cases} a & \text{if } R_\alpha^n(\varrho) \in \overline{I}_1, \\ b & \text{if } R_\alpha^n(\varrho) \in \overline{I}_2. \end{cases}$$

It is well known ([14, 32]) that an infinite word is a Sturmian word if and only if it is equal either to $\overline{s}_{\alpha,\varrho}$ or to $\underline{s}_{\alpha,\varrho}$ for some irrational number α.

The notation c_α stands in all that follows for $\overline{s}_{\alpha,\alpha} = \underline{s}_{\alpha,\alpha}$. This particular Sturmian word is called *characteristic word*.

According to Theorem 1.1, invertible word substitutions on two letters are also called *Sturmian substitutions*. We recall that the set of Sturmian substitutions is a monoid, with one set of generators being $\{E, L, \tilde{L}\}$ [45, 29]. Moreover, we will use in Appendix B the following facts (see [40] and [29]):

Proposition A.1. *Let σ be a primitive Sturmian substitution with frequency α. There exists a Sturmian substitution τ that is conjugate to σ and satisfies $\tau(c_\alpha) = c_\alpha$.*

If σ is a Sturmian substitution that fixes some characteristic word, then σ can be decomposed over the monoid generated by $\{E, L\}$. Conversely, if σ is a primitive word substitution that can be decomposed over the monoid generated by $\{E, L\}$, then σ is Sturmian and satisfies $\sigma(c_\alpha) = c_\alpha$, with α being its frequency.

Note that frequently Sturmian words are defined as one-sided infinite words. For our purposes it is rather natural to consider bi-infinite Sturmian words.

Example A.2. The Fibonacci sequences, that is, the sequences of the hull $\mathcal{X}_\sigma$ of the Fibonacci substitution σ (see (1)) are Sturmian words with frequency parameter $\alpha = \frac{\sqrt{5}-1}{2}$.

A detailed description of Sturmian words can be found in Chapter 2 of [29], see also Chapter 6 in [35].

Appendix B. Arithmetic duality

Let us now express the notion of duality in terms of continued fraction expansion.

Theorem B.1. *Let σ be a primitive invertible substitution over $\{a, b\}$ whose substitution matrix has determinant 1. The continued fraction expansion of the dual frequency α^* of α satisfies the following:*

1. *if $\alpha < 1/2$, then $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_{k+1} + n_1}]$, with $n_{k+1} \geq 0$ and $n_1 \geq 1$*
   - *if $n_{k+1} \geq 1$, then $\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}]$ with k is even*
   - *otherwise, $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_2, n_1, n_k}]$ with k odd*
2. *if $\alpha > 1/2$, then $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k + n_2}]$, with $n_k \geq 0$, $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}]$, and k is even if $n_k \neq 0$, k is odd otherwise.*

The description of the continued fraction expansion of α given in Theorem B.1 is due to [15]; such an α is a called a *Sturm number*; for more details, see [1]. See also in the same flavour [12].

Proof. We follow here the proof of Theorem 3.7 of [7] (see also the proof of Theorem 2.3.25 of [29]).

Let c_α (resp. c^*_α) stand for the characteristic word of frequency α (resp. α^*), such as defined in Appendix A. Without loss of generality (by possibly taking a conjugate of σ by Proposition A.1), we can assume

$$\sigma(c_\alpha) = c_\alpha. \tag{14}$$

Hence, according to Proposition A.1, σ can be decomposed over the monoid $\{E, L\}$, i.e.,

$$\sigma = L^{n_1} E L^{n_2} \cdots E L^{n_{k+1}},$$

with $k \geq 1$, $n_1, n_{k+1} \geq 0$, $n_2, \cdots, n_k \geq 1$. Note that the parity of k yields the sign of the determinant of M_σ.

Furthermore, one has $L^* = R$ and $E^* = E$ (see Example 4.9). Hence the second assertion of Theorem 4.3 yields

$$\sigma^* = R^{n_{k+1}} E \cdots E R^{n_1}.$$

Moreover

$$\sigma^* = E L^{n_{k+1}} \cdots E L^{n_1} E$$

by using $R \circ E = E \circ L$. One deduces from Proposition A.1 that $\sigma^*(c_{\alpha^*}) = c_{\alpha^*}$.

Let us translate (14) on the continued fraction expansion of α. For $m \geq 1$, let

$$\theta_m := L^{m-1} E L.$$

One checks that for every $\beta \in (0,1)$

$$\theta_m(c_\beta) = c_{1/(m+\alpha)} \text{ and } G(c_\beta) = c_{\frac{1}{1+1/\beta}}.$$

We thus obtain that if

$$c_\beta = \theta_{m_1} \circ \theta_{m_2} \circ \cdots \circ \theta_{m_k} \circ L^{m_{k+1}}(c_\gamma), \tag{15}$$

with $k, m_1, \cdots, m_k \geq 1$, $0 < \beta < 1$, $0 < \gamma < 1$, $\gamma = [0; \ell_1, \ell_2, \cdots, \ell_i, \cdots]$, then

$$\beta = [0; m_1, m_2, \cdots, m_k, m_{k+1} - 1 + \ell_1, \ell_2, \cdots, \ell_i, \cdots].$$

With the previous notation,

$$\sigma = \theta_{n_1+1} \circ \theta_{n_2} \circ \cdots \circ \theta_{n_k} \circ L^{n_{k+1}-1}.$$

We distinguish several cases according to the values of n_1 and n_{k+1}.

- We first assume $n_1 > 0$. Hence by (15), one has $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_{k+1} + n_1}]$ if $n_{k+1} > 0$. If $n_{k+1} = 0$, we use the fact that

  $$E \circ \sigma \circ E = E L^{n_1} E \cdots L^{n_k} = \theta_1 \theta_{n_1} \cdots \theta_{n_{k-1}} L^{n_k - 1}.$$

  Since $\sigma(c_\alpha) = c_\alpha$, then $E \circ \sigma \circ E(c_{1-\alpha}) = c_{1-\alpha}$. We deduce from (15) that

  $$1 - \alpha = [0; 1, \overline{n_1, \cdots, n_{k-1}, n_k}].$$

  Since

  $$\alpha = \frac{1}{1 + \frac{1-\alpha}{\alpha}} = \frac{1}{1 + \frac{1-\alpha}{1-(1-\alpha)}} = \frac{1}{1 + \frac{1}{\frac{1}{1-\alpha} - 1}},$$

  we get

  $$\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_1}].$$

  Furthermore,

  $$E \circ \sigma^* \circ E = L^{n_{k+1}} E \cdots E L^{n_1} = \theta_{1+n_{k+1}} \theta_{n_k} \cdots \theta_{n_2} L^{n_1 - 1}.$$

  Since $\sigma^*(c_{\alpha^*}) = c_{\alpha^*}$, then $E \circ \sigma^* \circ E(c_{1-\alpha^*}) = c_{1-\alpha^*}$. We deduce from (15) that

  $$1 - \alpha^* = [0; 1 + n_{k+1}, \overline{n_k, \cdots, n_1 + n_{k+1}}].$$

  From

  $$\alpha^* = \frac{1}{1 + \frac{1}{\frac{1}{1-\alpha^*} - 1}},$$

  we obtain

  $$\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}] \text{ if } n_{k+1} > 0$$

and
$$\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_2, n_1, n_k}] \text{ if } n_{k+1} = 0.$$

- We now assume $n_1 = 0$. One has
$$\sigma^* = R^{n_{k+1}} E \cdots R^{n_2} E = E L^{n_{k+1}} \cdots E L^{n_2}.$$
  - We assume $n_{k+1} > 0$. One has $\alpha = [0; 1, \overline{n_2, \cdots, n_k, n_{k+1}}]$. Moreover
$$\sigma^* = \theta_1 \theta_{n_{k+1}} \cdots \theta_{n_3} L^{n_2 - 1}.$$
  We thus get
$$\alpha^* = [0; 1, \overline{n_{k+1}, \cdots n_3, n_2}].$$
  - We now assume $n_{k+1} = 0$. One has
$$E \circ \sigma \circ E = L^{n_2} E \cdots L^{n_k} = \theta_{n_2+1} \cdots \theta_{n_{k-1}} L^{n_k - 1}.$$
  Since $\sigma(c_\alpha) = c_\alpha$, then $E \circ \sigma \circ E(c_{1-\alpha}) = c_{1-\alpha}$. We deduce from (15) that
$$1 - \alpha = [0; n_2 + 1, \overline{n_3, \cdots, n_{k-1}, n_k + n_2}].$$
  Since
$$\alpha = \frac{1}{1 + \frac{1}{\frac{1}{1-\alpha} - 1}},$$
  we get
$$\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k + n_2}].$$
  Furthermore
$$\sigma^* = E R^{n_k} E \cdots R^{n_2} E = L^{n_k} E \cdots E L^{n_2} = \theta_{1+n_k} \cdots \theta_{n_3} L^{n_2 - 1}.$$
  We thus get
$$\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}].$$

In conclusion, we have proved that

(1) $\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}]$ with k even if $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_{k+1} + n_1}]$, with $n_1 > 0$ and $n_{k+1} > 0$;
(2) $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_2, n_1, n_k}]$ with k odd if $\alpha = [0; 1 + n_1, \overline{n_2, \cdots, n_k, n_1}]$ with $n_1 > 0$;
(3) $\alpha^* = [0; 1, \overline{n_{k+1}, \cdots n_3, n_2}]$ with k odd if $\alpha = [0; 1, \overline{n_2, \cdots, n_k, n_{k+1}}]$ with $n_{k+1} > 0$;
(4) $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}]$ with k even if $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k + n_2}]$.

□

We now give the proof of Theorem 4.12: we want to prove that
$$\alpha^* = \frac{1 - \alpha'}{2\alpha' - 1},$$
where α' is the algebraic conjugate of α.

Proof. (of Theorem 4.12) Our proof is inspired by the proof of Theorem 2.3.26 in [29]. Let α be the frequency of a primitive two-letter substitution σ. Without loss of generality, we assume again $\sigma(c_\alpha) = c_\alpha$. We will use the fact that if $\gamma = [a_1; \overline{a_2, \cdots, a_n, a_1}]$, then
$$-1/\gamma' = [a_n; \overline{a_{n-1}, \cdots, a_1, a_n}].$$

- We first assume $\alpha < 1/2$. According to Theorem B.1, $\alpha = [0; 1+n_1, \overline{n_2, \cdots, n_k, n_{k+1} + n_1}]$, with $n_{k+1} \geq 0$ and $n_1 \geq 1$. Let
$$\gamma = [n_2; \overline{n_3, \cdots, n_k, n_{k+1} + n_1, n_2}].$$
We have $1/\alpha = 1 + n_1 + 1/\gamma$ and $1/\alpha' = 1 + n_1 + 1/\gamma'$. One has
$$-1/\gamma' = [n_{k+1} + n_1; \overline{n_k, \cdots, n_2, n_{k+1} + n_1}].$$
We deduce that $-(1/\gamma' + n_1 + n_{k+1}) = [0; \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}]$.
  - If $n_{k+1} \geq 1$, then $\alpha^* = [0; 1, n_{k+1}, \overline{n_k, \cdots, n_2, n_1 + n_{k+1}}]$, by Theorem B.1. We obtain
  $$\frac{\alpha^*}{1-\alpha^*} == \frac{1}{\frac{1}{\alpha^*} - 1} = n_{k+1} - (1/\gamma' + n_1 + n_{k+1}) = 1 - 1/\alpha',$$
  and thus
  $$\alpha^* = \frac{\alpha' - 1}{2\alpha' - 1}.$$
  - If $n_{k+1} = 0$, then $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_2, n_1, n_k}]$. One gets $-(1/\gamma' + n_1) = [0; \overline{n_k, \cdots, n_2, n_1}]$. We deduce that
  $$\frac{1}{1/\alpha^* - 1} = -1/\gamma' - n_1 = 1 - 1/\alpha'$$
  and thus
  $$\alpha^* = \frac{\alpha' - 1}{2\alpha' - 1}.$$
- If $\alpha > 1/2$, then $\alpha = [0; 1, n_2, \overline{n_3, \cdots, n_{k-1}, n_k + n_2}]$, with $n_k \geq 0$. Let
$$\gamma = [n_3; \overline{n_4, \cdots, n_{k-1}, n_k + n_2, n_3}].$$
We have $\frac{\alpha}{1-\alpha} = n_2 + 1/\gamma$ and $\frac{\alpha'}{1-\alpha'} = n_2 + 1/\gamma'$. One has
$$-1/\gamma' = [n_k + n_2; \overline{n_{k-1}, \cdots, n_3, n_k + n_2}].$$
We deduce that $-(1/\gamma' + n_k + n_2) = [0; \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}]$.
One has $\alpha^* = [0; 1 + n_k, \overline{n_{k-1}, \cdots, n_3, n_2 + n_k}]$. Hence
$$1/\alpha^* = 1 + n_k - (1/\gamma' + n_k + n_2) = 1 - \frac{\alpha'}{1-\alpha'},$$
and thus
$$\alpha^* = \frac{\alpha' - 1}{2\alpha' - 1}.$$

□

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