Perfect colourings of regular graphs

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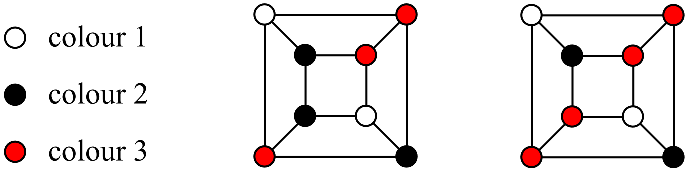
A *perfect colouring* of the vertices of a graph $G = (V, E)$ with m colours:

Colour V such that for all $v \in V$ with colour i holds: v is adjacent to a_{i1} vertices of colour 1, v is adjacent to a_{i2} vertices of colour 2, ... v is adjacent to a_{im} vertices of colour m.

I.e., all white vertices have the same number of white neighbours, of black neighbours, ...

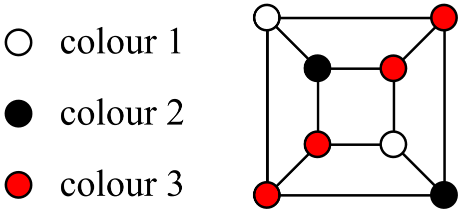


not perfect

perfect

(Note that adjacent vertices are allowed to have the same colour)

All white vertices are adjacent to the same number a_{11} of white vertices, a_{12} of black vertices...



So here: $a_{11} = 0$, $a_{12} = 1$, $a_{13} = 2$, ... altogether:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Questions:

- Find necessary and sufficient conditions on M to be the matrix of a perfect colouring.
- Find all perfect colourings of a given (class of) graph(s).

Three simple necessary criteria:

Lemma (weak symmetry)

$a_{ij} = 0$ *iff* $a_{ji} = 0$.

Clear: if each red vertex has a white neighbour then each white vertex has a red neighbour.

OK: $\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$ Not OK: $\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$

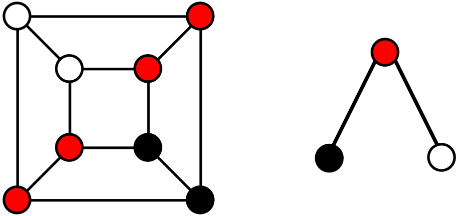
From now on let G be simple, connected, without loops.

Lemma (connected colour graph)

The corresponding colour graph

$$G' = (\{1, \ldots, m\}, \{\{i, j\} \mid a_{ij} \neq 0\})$$

is connected.



G G'

Clear: Since G is connected, there is a path from each colour to any colour.

Lemma (consistent counting)

For each cycle $v_1, v_2, \ldots, v_k, v_1$ $(k \geq 2)$ holds:

$$a_{v_1,v_2} \cdot a_{v_2,v_3} \cdots a_{v_k,v_1} = a_{v_1,v_k} \cdot a_{v_k,v_{k-1}} \cdots a_{v_2,v_1}$$

Simple: n_1 white vertices are adjacent to a_{12} black vertices, n_2 black vertices are adjacent to a_{21} white vertices, hence: $n_1 a_{12} = n_2 a_{21}$.

Ditto $n_2 a_{23} = n_3 a_{32}$ and $n_1 a_{13} = n_3 a_{31}$. Hence

$$n_1 a_{12} a_{23} a_{31} = n_2 a_{21} a_{23} a_{31} = n_3 a_{21} a_{32} a_{31} = n_1 a_{21} a_{32} a_{13},$$

hence $a_{12} a_{23} a_{31} = a_{13} a_{32} a_{21}$ and so on.

Theorem

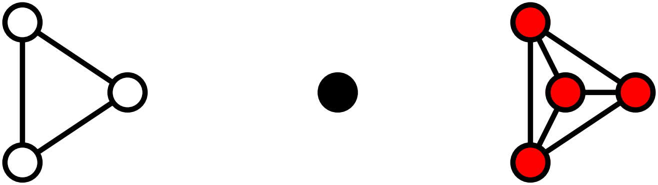
Lemmas 1-3 are necessary and sufficient. I.e., a matrix $M \in \mathbb{N}^{m \times m}$ is the colouring matrix of a connected graph iff it has the properties weak symmetry, connected colour graph and consistent counting.

- Necessary: see above.
- Sufficient: construct graphs for each instance.

For instance, for $M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$.

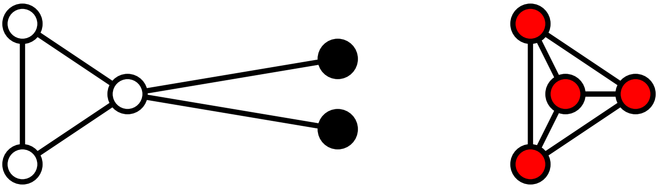
$$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

The vertices of one color are a_{ii}-regular graph:



$$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$a_{12} = 2$, $a_{21} = 1$:



$$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

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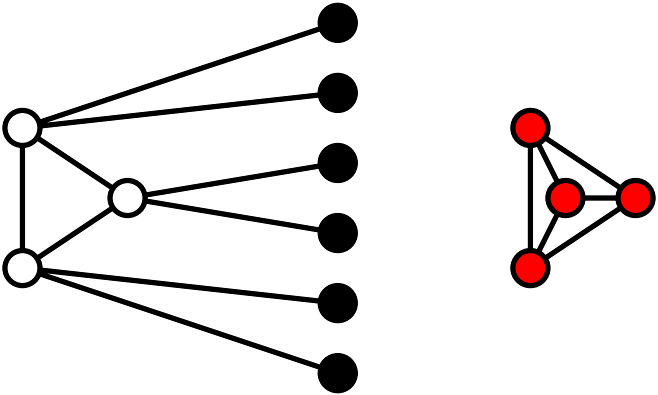
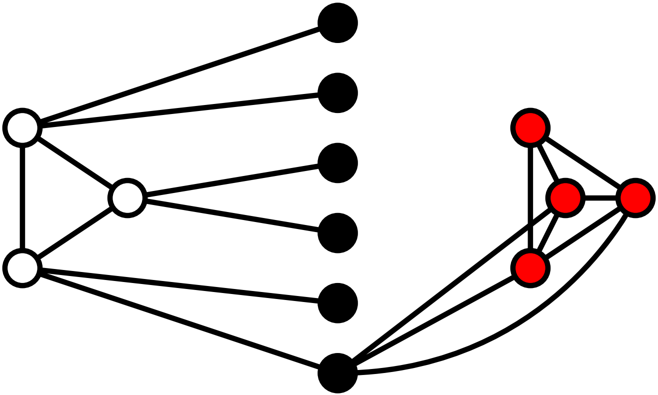


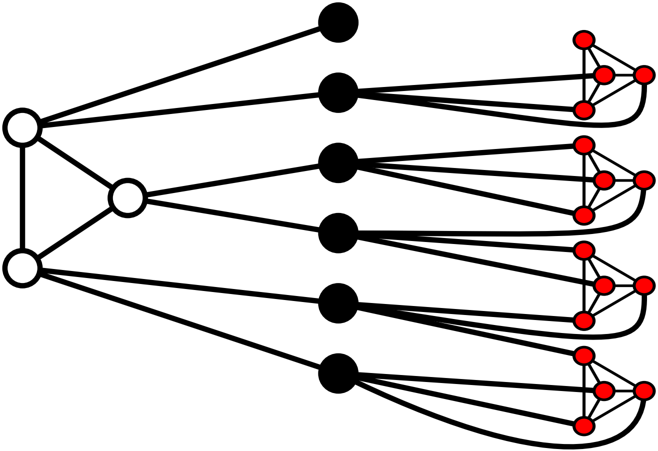
$$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$a_{23} = 3$, $a_{32} = 1$:



$$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$a_{23} = 3,\ a_{32} = 1$:



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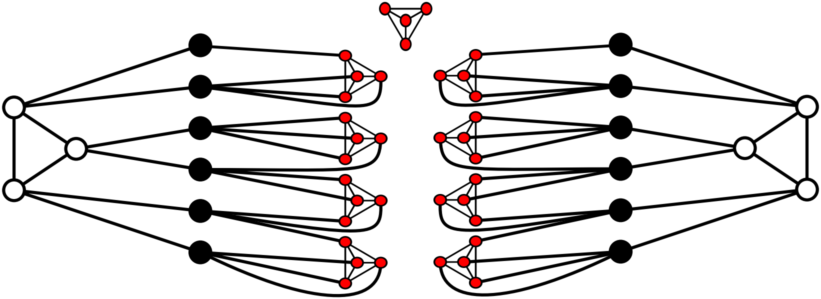
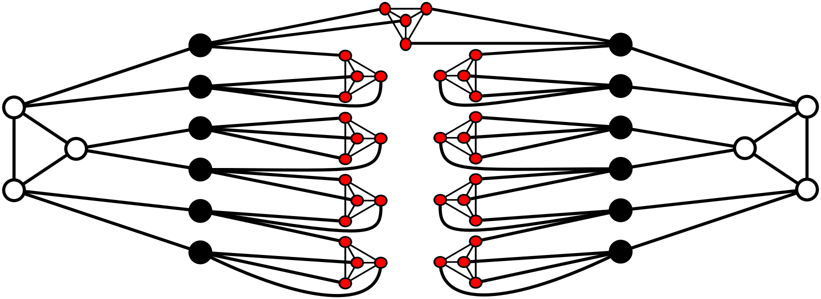


$$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$a_{23} = 3$, $a_{32} = 1$:



The consistent counting condition ensures that this works.

Application: List all m-colouring matrices of k-regular graphs.
k-regular: row sum equals k.

Numbers of colouring matrices among all possible matrices (nonnegative integer entries, all row sums $= k$.)

$m \setminus k$	3	4	5
2	6 of 16	10 of 25	15 of 36
3	18 of 1000	64 of 3375	153 of 9261
4	72 of 16 000	485 of 1 500 625	2042 of 9 834 496

Counting is up to permutation of colours. (This is the computationally most expensive part)

(Computations both in `SageMath` and `scilab`)

Application: List all m-colouring matrices of k-regular graphs.
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Numbers of colouring matrices among all possible matrices (nonnegative integer entries, all row sums $= k$.)

$m \setminus k$	3	4	5
2	< 1 sec	< 1 sec	< 1 sec
3	< 1 sec	2 sec	12 sec
4	3 min	55 min	one night

Counting is up to permutation. (This is the computationally most expensive part)

(Times for computations in `SageMath`)

All matrices for perfect 2-colorings...

...of 3-regular graphs

$\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

...of 4-regular graphs

$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

...of 5-regular graphs

$\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 4 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$
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All matrices for perfect 3-colorings...

...of 3-regular graphs

$$\begin{pmatrix}0&0&3\\0&0&3\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&3\\0&0&3\\1&2&0\end{pmatrix}\begin{pmatrix}0&0&3\\0&1&2\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&3\\0&1&2\\1&2&0\end{pmatrix}\begin{pmatrix}0&0&3\\0&2&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&3\\0&2&1\\2&1&0\end{pmatrix}\begin{pmatrix}0&1&2\\1&0&2\\1&1&1\end{pmatrix}\begin{pmatrix}0&1&2\\1&1&1\\2&1&0\end{pmatrix}\begin{pmatrix}0&1&2\\1&2&0\\1&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&1&2\\1&2&0\\2&0&1\end{pmatrix}\begin{pmatrix}0&3&0\\1&0&2\\0&1&2\end{pmatrix}\begin{pmatrix}1&0&2\\0&0&3\\1&2&0\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&2\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&2\\1&2&0\end{pmatrix}\begin{pmatrix}1&0&2\\0&2&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&2&0\\1&0&2\end{pmatrix}\begin{pmatrix}1&2&0\\1&0&2\\0&1&2\end{pmatrix}$$

...of 4-regular graphs

$$\begin{pmatrix}0&0&4\\0&0&4\\1&1&2\end{pmatrix}\begin{pmatrix}0&0&4\\0&0&4\\1&2&1\end{pmatrix}\begin{pmatrix}0&0&4\\0&0&4\\1&3&0\end{pmatrix}\begin{pmatrix}0&0&4\\0&0&4\\2&2&0\end{pmatrix}\begin{pmatrix}0&0&4\\0&1&3\\1&1&2\end{pmatrix}\begin{pmatrix}0&0&4\\0&1&3\\1&2&1\end{pmatrix}\begin{pmatrix}0&0&4\\0&1&3\\1&3&0\end{pmatrix}\begin{pmatrix}0&0&4\\0&1&3\\2&2&0\end{pmatrix}\begin{pmatrix}0&0&4\\0&2&2\\1&1&2\end{pmatrix}$$

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$$\begin{pmatrix}2&2&0\\1&1&2\\0&1&3\end{pmatrix}$$

...of 5-regular graphs

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$$\begin{pmatrix}1&0&4\\0&4&1\\1&1&3\end{pmatrix} \begin{pmatrix}1&0&4\\0&4&1\\2&1&2\end{pmatrix} \begin{pmatrix}1&0&4\\0&4&1\\3&1&1\end{pmatrix} \begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix} \begin{pmatrix}1&1&3\\1&1&3\\2&2&1\end{pmatrix} \begin{pmatrix}1&1&3\\1&3&1\\3&1&1\end{pmatrix} \begin{pmatrix}1&1&3\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}1&1&3\\1&4&0\\2&0&3\end{pmatrix} \begin{pmatrix}1&1&3\\1&4&0\\3&0&2\end{pmatrix} \begin{pmatrix}1&2&2\\1&0&4\\1&4&0\end{pmatrix} \begin{pmatrix}1&2&2\\1&1&3\\1&3&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&2\\1&2&2\\1&2&2\end{pmatrix} \begin{pmatrix}1&2&2\\1&3&1\\1&1&3\end{pmatrix} \begin{pmatrix}1&2&2\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}1&2&2\\2&1&2\\1&1&3\end{pmatrix} \begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix} \begin{pmatrix}1&2&2\\2&2&1\\2&1&2\end{pmatrix} \begin{pmatrix}1&2&2\\2&3&0\\1&0&4\end{pmatrix} \begin{pmatrix}1&2&2\\2&3&0\\2&0&3\end{pmatrix} \begin{pmatrix}1&4&0\\1&0&4\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\1&0&4\\0&2&3\end{pmatrix} \begin{pmatrix}1&4&0\\1&0&4\\0&3&2\end{pmatrix}$$

$$\begin{pmatrix}1&4&0\\1&1&3\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\1&1&3\\0&2&3\end{pmatrix} \begin{pmatrix}1&4&0\\1&2&2\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\2&0&3\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\2&0&3\\0&2&3\end{pmatrix} \begin{pmatrix}1&4&0\\2&1&2\\0&1&4\end{pmatrix} \begin{pmatrix}1&4&0\\3&0&2\\0&1&4\end{pmatrix} \begin{pmatrix}2&0&3\\0&0&5\\1&2&2\end{pmatrix} \begin{pmatrix}2&0&3\\0&0&5\\1&3&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&0&5\\1&4&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\1&2&2\end{pmatrix}$$

$$\begin{pmatrix}2&0&3\\0&1&4\\1&3&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\1&4&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&1&4\\2&3&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\1&1&3\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\1&2&2\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\1&3&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\2&2&1\end{pmatrix} \begin{pmatrix}2&0&3\\0&2&3\\2&3&0\end{pmatrix} \begin{pmatrix}2&0&3\\0&3&2\\1&1&3\end{pmatrix} \begin{pmatrix}2&0&3\\0&3&2\\1&2&2\end{pmatrix} \begin{pmatrix}2&0&3\\0&3&2\\2&2&1\end{pmatrix}$$

$$\begin{pmatrix}2&0&3\\0&4&1\\1&1&3\end{pmatrix} \begin{pmatrix}2&0&3\\0&4&1\\2&1&2\end{pmatrix} \begin{pmatrix}2&1&2\\1&2&2\\1&1&3\end{pmatrix} \begin{pmatrix}2&1&2\\1&3&1\\2&1&2\end{pmatrix} \begin{pmatrix}2&1&2\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}2&1&2\\1&4&0\\2&0&3\end{pmatrix} \begin{pmatrix}2&3&0\\1&0&4\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\1&0&4\\0&2&3\end{pmatrix} \begin{pmatrix}2&3&0\\1&0&4\\0&3&2\end{pmatrix} \begin{pmatrix}2&3&0\\1&1&3\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\1&1&3\\0&2&3\end{pmatrix}$$

$$\begin{pmatrix}2&3&0\\1&2&2\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\2&0&3\\0&1&4\end{pmatrix} \begin{pmatrix}2&3&0\\2&0&3\\0&2&3\end{pmatrix} \begin{pmatrix}2&3&0\\2&1&2\\0&1&4\end{pmatrix} \begin{pmatrix}3&0&2\\0&0&5\\1&3&1\end{pmatrix} \begin{pmatrix}3&0&2\\0&0&5\\1&4&0\end{pmatrix} \begin{pmatrix}3&0&2\\0&1&4\\1&3&1\end{pmatrix} \begin{pmatrix}3&0&2\\0&1&4\\1&4&0\end{pmatrix} \begin{pmatrix}3&0&2\\0&2&3\\1&2&2\end{pmatrix} \begin{pmatrix}3&0&2\\0&2&3\\1&3&1\end{pmatrix} \begin{pmatrix}3&0&2\\0&3&2\\1&1&3\end{pmatrix}$$

$$\begin{pmatrix}3&0&2\\0&3&2\\1&2&2\end{pmatrix} \begin{pmatrix}3&0&2\\0&4&1\\1&1&3\end{pmatrix} \begin{pmatrix}3&1&1\\1&3&1\\1&1&3\end{pmatrix} \begin{pmatrix}3&1&1\\1&4&0\\1&0&4\end{pmatrix} \begin{pmatrix}3&2&0\\1&0&4\\0&1&4\end{pmatrix} \begin{pmatrix}3&2&0\\1&0&4\\0&2&3\end{pmatrix} \begin{pmatrix}3&2&0\\1&0&4\\0&3&2\end{pmatrix} \begin{pmatrix}3&2&0\\1&1&3\\0&1&4\end{pmatrix} \begin{pmatrix}3&2&0\\1&1&3\\0&2&3\end{pmatrix} \begin{pmatrix}3&2&0\\1&2&2\\0&1&4\end{pmatrix}$$

All matrices for perfect 4-colorings...

...of 3-regular graphs

$$\begin{pmatrix}0&0&0&3\\0&0&0&3\\0&0&0&3\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&0&3\\0&0&1&2\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&0&3\\0&0&2&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&1&2\\0&1&0&2\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&1&2\\0&1&2&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&1&2\\0&1&2&0\\1&2&0&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&2&1\\0&2&0&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&2&1\\0&2&1&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&2&1\\0&2&1&0\\2&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&3\\0&0&3&0\\0&1&0&2\\1&0&2&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&3&0\\0&1&1&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&3&0\\0&2&0&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&0&3&0\\0&2&0&1\\2&0&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&1&0&2\\0&0&1&2\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&1&0&2\\0&0&2&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&1&1&1\\0&1&2&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&1&1&1\\0&1&2&0\\2&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&3\\0&1&2&0\\0&1&0&2\\1&0&2&0\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&1&2&0\\0&1&1&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&3\\0&2&0&1\\0&0&2&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&0&1&2\\1&1&1&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&0&2&1\\1&2&0&0\\2&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&1&0&2\\1&0&0&2\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&1&0&2\\1&0&2&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&1&1&1\\1&1&1&0\\2&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&1&2&0\\1&2&0&0\\1&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&2\\0&1&2&0\\1&2&0&0\\2&0&0&1\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&2&0&1\\1&0&2&0\\2&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&2&1&0\\1&1&0&1\\2&0&1&0\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&2&1&0\\1&1&1&0\\1&0&0&2\end{pmatrix} \begin{pmatrix}0&0&1&2\\0&2&1&0\\1&1&1&0\\2&0&0&1\end{pmatrix} \begin{pmatrix}0&0&3&0\\0&0&0&3\\1&0&0&2\\0&1&1&1\end{pmatrix} \begin{pmatrix}0&0&3&0\\0&0&1&2\\1&1&1&0\\0&1&0&2\end{pmatrix} \begin{pmatrix}0&0&3&0\\0&0&1&2\\2&1&0&0\\0&1&0&2\end{pmatrix} \begin{pmatrix}0&0&3&0\\0&1&0&2\\1&0&0&2\\0&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&2\\1&0&0&2\\0&0&2&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&1&0&2\\1&0&2&0\\0&1&0&2\\1&0&2&0\end{pmatrix} \begin{pmatrix}0&1&0&2\\1&0&2&0\\0&1&1&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}0&1&0&2\\1&0&2&0\\0&1&2&0\\1&0&0&2\end{pmatrix} \begin{pmatrix}0&1&0&2\\1&2&0&0\\0&0&0&3\\1&0&2&0\end{pmatrix} \begin{pmatrix}0&1&0&2\\1&2&0&0\\0&0&1&2\\1&0&2&0\end{pmatrix} \begin{pmatrix}0&1&0&2\\1&2&0&0\\0&0&2&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&1\\1&1&0&1\\1&0&2&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}0&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{pmatrix} \begin{pmatrix}0&1&2&0\\1&2&0&0\\1&0&0&2\\0&0&1&2\end{pmatrix} \begin{pmatrix}0&3&0&0\\1&0&0&2\\0&0&0&3\\0&1&2&0\end{pmatrix} \begin{pmatrix}0&3&0&0\\1&0&0&2\\0&0&1&2\\0&1&2&0\end{pmatrix} \begin{pmatrix}0&3&0&0\\1&0&0&2\\0&0&2&1\\0&1&1&1\end{pmatrix} \begin{pmatrix}0&3&0&0\\1&0&2&0\\0&1&0&2\\0&0&1&2\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&0&1&2\\0&1&2&0\\1&2&0&0\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&0&2&1\\0&2&0&1\\1&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&2\\0&0&2&1\\0&2&1&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&0&3&0\\0&2&0&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&1&0&2\\0&0&1&2\\1&1&1&0\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&1&0&2\\0&0&2&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&1&1&1\\0&1&2&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&1&2&0\\0&1&0&2\\1&0&2&0\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&1&2&0\\0&1&1&1\\1&0&1&1\end{pmatrix} \begin{pmatrix}1&0&0&2\\0&2&0&1\\0&0&2&1\\1&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}1&0&1&1\\0&2&0&1\\1&0&2&0\\1&1&0&1\end{pmatrix} \begin{pmatrix}1&0&2&0\\0&0&0&3\\1&0&0&2\\0&1&1&1\end{pmatrix} \begin{pmatrix}1&0&2&0\\0&0&1&2\\1&1&1&0\\0&1&0&2\end{pmatrix} \begin{pmatrix}1&0&2&0\\0&1&0&2\\1&0&0&2\\0&1&1&1\end{pmatrix} \begin{pmatrix}1&2&0&0\\1&0&0&2\\0&0&0&3\\0&1&2&0\end{pmatrix} \begin{pmatrix}1&2&0&0\\1&0&0&2\\0&0&1&2\\0&1&2&0\end{pmatrix} \begin{pmatrix}1&2&0&0\\1&0&0&2\\0&0&2&1\\0&1&1&1\end{pmatrix} \begin{pmatrix}1&2&0&0\\1&0&2&0\\0&1&0&2\\0&0&1&2\end{pmatrix}$$

...of 4-regular graphs

$$\begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&0&4\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&0&4\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&1&3\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&1&3\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&2&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&2&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&2&2\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&3&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&0&4\\0&0&3&1\\1&2&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&0&3\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\1&2&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\1&3&0&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&1&3\\0&1&3&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&1&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&2&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&2&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&3&0\\1&1&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&2&2\\0&1&3&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&0&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&0&2\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&1&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\1&2&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&2&2\\0&2&2&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&0&1\\1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&0&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&1&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&1&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&3&1\\0&3&1&0\\3&1&0&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&1&0&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&1&1&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&1&2&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&0&2\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&0&2\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&0&2\\2&0&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&1&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&2&1&1\\2&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&3&0&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&3&0&1\\2&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&0&4&0\\0&3&0&1\\3&0&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&0&4\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&1&3\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&1&3\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&2&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&2&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&3&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&0&3\\0&0&3&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&1&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&1&2\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&2&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\1&2&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&1&2\\0&1&3&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&1&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&1&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&2&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&2&1\\0&2&2&0\\3&1&0&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&1&0&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&1&1&2\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&1&3&0\\0&1&2&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&0&2\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&0&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&0&2\\2&0&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&1&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&1&3&0\\0&2&1&1\\2&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&1&3\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&2&2\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&2&2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&2&2\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&3&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&3&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&0&2\\0&0&3&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&2&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&2&1\\2&1&1&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&3&0\\2&1&0&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&1&1\\0&1&3&0\\3&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&0&3\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&0&3\\1&0&3&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&1&2\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&1&2\\1&0&2&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&1&2\\2&0&2&0\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&2&1\\1&0&1&2\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&2&2&0\\0&1&2&1\\2&0&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&3&0&1\\0&0&3&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&0&4\\0&3&0&1\\0&0&3&1\\2&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&3\\0&0&1&3\\1&1&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&0&1&3\\1&1&2&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&0&3&1\\1&3&0&0\\3&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&1&0&3\\1&0&0&3\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&1&0&3\\1&0&3&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&1&0&3\\1&0&3&0\\2&2&0&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&1&2&1\\1&2&1&0\\3&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&1&3&0\\1&3&0&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&1&3&0\\1&3&0&0\\2&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&3\\0&1&3&0\\1&3&0&0\\3&0&0&1\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&2&1&1\\1&1&2&0\\3&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&0&1\\3&0&1&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&1&0\\1&0&0&3\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&1&0\\2&0&0&2\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&2&2&0\\1&2&1&0\\3&0&0&1\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&3&0&1\\1&0&3&0\\3&1&0&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&1&1\\3&0&1&0\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&2&0\\1&0&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&2&0\\2&0&0&2\end{pmatrix} \begin{pmatrix}0&0&1&3\\0&3&1&0\\1&1&2&0\\3&0&0&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&0&4\\1&0&3&0\\1&3&0&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&1&3\\1&1&2&0\\1&3&0&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&2&2\\1&1&0&2\\1&1&2&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&2&2\\1&1&1&1\\1&1&1&1\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&2&2\\1&1&2&0\\1&1&0&2\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&2&2\\1&2&0&1\\1&2&1&0\end{pmatrix} \begin{pmatrix}0&0&2&2\\0&0&2&2\\1&2&1&0\\1&2&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&0&2&2\\2&2&0&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&0&2&2\\2&2&0&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&0&3\\1&0&2&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&0&3\\1&0&3&0\\1&2&0&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&0&3\\1&0&3&0\\1&3&0&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&1&2\\1&1&1&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&1&2\\1&1&2&0\\1&2&0&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&1&2\\2&1&1&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&1&1&2\\2&1&1&0\\2&2&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&1&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&2&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&2&1\\1&2&1&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\1&0&3&0\\1&2&0&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\2&0&0&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\2&0&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&0&2\\2&0&2&0\\2&2&0&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&1&1\\1&1&0&2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&2&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&1&1\\1&1&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&1&1\\2&1&0&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&1&1\\2&1&1&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&2&2&0\\2&2&0&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&3&0&1\\1&0&1&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&3&0&1\\1&0&2&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&3&0&1\\1&0&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&3&0&1\\2&0&1&1\\2&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&2\\0&3&0&1\\2&0&2&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&0&2&2\\0&3&1&0\\2&1&1&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&0&3\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&0&3\\0&1&2&1\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&0&3\\0&2&2&0\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&1&2\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&0&4\\1&0&1&2\\0&2&1&1\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&0&4\\2&0&0&2\\0&2&1&1\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&1&3\\1&1&2&0\\0&1&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&4&0\\0&0&1&3\\1&1&2&0\\0&2&0&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&1&3\\2&1&1&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&1&3\\2&1&1&0\\0&2&0&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&1&3\\3&1&0&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&1&3\\3&1&0&0\\0&2&0&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&2&2\\1&2&1&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&2&2\\2&2&0&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&0&4&0\\1&1&0&2\\0&0&1&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&0&3\\0&1&1&2\end{pmatrix}$$

$$\begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&0&3\\0&1&2&1\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&0&3\\0&2&2&0\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&0&3\\1&0&1&2\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&0&3\\2&0&0&2\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&1&2\\1&1&0&2\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&1&2\\1&1&2&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&1&2\\2&1&1&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&1&2\\3&1&0&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&1&3&0\\1&1&0&2\\0&0&1&3\end{pmatrix}$$

$$\begin{pmatrix}0&0&4&0\\0&2&0&2\\1&0&0&3\\0&1&2&1\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&2&0&2\\1&0&1&2\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&2&0&2\\2&0&0&2\\0&1&1&2\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&2&2&0\\1&1&0&2\\0&0&1&3\end{pmatrix}\begin{pmatrix}0&0&4&0\\0&3&1&0\\1&1&0&2\\0&0&1&3\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&0&4\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&1&3\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&2&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&2&2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&3\\1&0&0&3\\0&0&3&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&0&3\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&1&2\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&2&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&1&3&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&0&2\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&0&2\\2&0&2&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&1&1\\2&0&1&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&2&0\\1&0&0&3\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&3\\1&0&3&0\\0&2&2&0\\2&0&0&2\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&1&2&0\\0&1&0&3\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&1&2&0\\0&1&3&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&0&4\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&0&4\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&1&3\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&1&3\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&2&2\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&2&2\\1&0&2&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&2&2\\2&0&2&0\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&3&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&1&0&3\\1&3&0&0\\0&0&3&1\\2&0&1&1\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&0&1&2\\1&1&0&2\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&0&1&2\\1&1&2&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&0&2&1\\1&2&0&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&0&2&1\\1&2&1&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&0&3&0\\1&3&0&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&1&0&2\\1&0&1&2\\1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&2\\1&1&0&2\\1&0&3&0\\1&1&0&2\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&1&1&1\\1&1&1&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&1&1&1\\1&1&2&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&1&2&0\\1&2&1&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&1&2&0\\1&2&1&0\\2&0&0&2\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&2&0&1\\1&0&2&1\\2&1&1&0\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&2&0&1\\1&0&3&0\\2&1&0&1\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&2&1&0\\1&1&2&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&2&1&0\\1&1&2&0\\2&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&2\\1&3&0&0\\1&0&3&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&1&1&2\\1&3&0&0\\1&0&3&0\\2&0&0&2\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&0&3&0\\1&1&0&2\\0&0&1&3\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&1&0&2\\2&0&2&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&1&0&2\\3&0&1&0\\0&1&0&3\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&3&0&0\\1&0&0&3\\0&0&1&3\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&3&0&0\\1&0&0&3\\0&0&2&2\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&3&0&0\\1&0&1&2\\0&0&1&3\end{pmatrix}\begin{pmatrix}0&1&3&0\\1&3&0&0\\2&0&0&2\\0&0&1&3\end{pmatrix}$$

$$\begin{pmatrix}0&2&0&2\\2&0&0&2\\0&0&1&3\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&0&0&2\\0&0&2&2\\1&1&2&0\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&0&0&2\\0&0&3&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&0&3\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&1&2\\1&0&2&1\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&2&1\\1&0&1&2\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&0&2&0\\0&1&3&0\\1&0&0&3\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&0&4\\1&0&3&0\end{pmatrix}\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&1&3\\1&0&2&1\end{pmatrix}$$

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\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&1&3\\1&0&3&0\end{pmatrix}
\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&2&2\\1&0&2&1\end{pmatrix}
\begin{pmatrix}0&2&0&2\\2&2&0&0\\0&0&3&1\\1&0&1&2\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&0&3\\0&1&1&2\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&0&3\\0&2&2&0\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&3&0\\0&1&0&3\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&0&0&3\\1&0&3&0\\0&2&0&2\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&0&1&2\\1&1&0&2\\0&1&1&2\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&0&1&2\\1&1&2&0\\0&1&0&3\end{pmatrix}
$$

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\begin{pmatrix}0&2&2&0\\1&1&0&2\\1&0&1&2\\0&1&1&2\end{pmatrix}
\begin{pmatrix}0&2&2&0\\1&1&0&2\\1&0&3&0\\0&1&0&3\end{pmatrix}
\begin{pmatrix}0&2&2&0\\2&0&2&0\\1&1&0&2\\0&0&1&3\end{pmatrix}
\begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&0&3\\0&0&1&3\end{pmatrix}
\begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&0&3\\0&0&2&2\end{pmatrix}
\begin{pmatrix}0&2&2&0\\2&2&0&0\\1&0&1&2\\0&0&1&3\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&0&4\\0&1&2&1\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&0&4\\0&1&3&0\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&1&3\\0&1&2&1\end{pmatrix}
$$

$$
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&1&3\\0&1&3&0\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&1&1&2\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&1&2&1\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&2&2\\0&2&2&0\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&3&1\\0&1&1&2\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&0&3\\0&0&3&1\\0&2&1&1\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&0&3\\0&0&1&3\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&0&3\\0&0&2&2\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&0&3&0\\0&1&1&2\\0&0&1&3\end{pmatrix}
$$

$$
\begin{pmatrix}0&4&0&0\\1&0&3&0\\0&2&0&2\\0&0&1&3\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&0&4\\0&1&3&0\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&1&3\\0&1&2&1\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&1&3\\0&1&3&0\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&2&2\\0&1&2&1\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&0&2\\0&0&3&1\\0&1&1&2\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&2&0\\0&1&0&3\\0&0&1&3\end{pmatrix}
\begin{pmatrix}0&4&0&0\\1&1&2&0\\0&1&0&3\\0&0&2&2\end{pmatrix}
$$

...and of 5-regular graphs:

[2042 matrices]

Perfect colourings seem to be not well-studied. But in

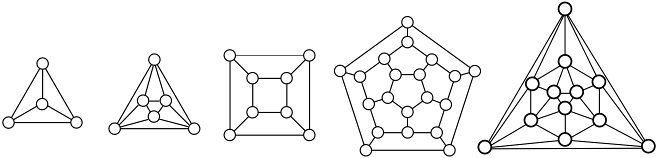
C. Godsil, G. Royle: *Algebraic graph theory*, Springer (2001)

one can find a result due to H. Sachs (1966):

Theorem

Let M be the adjacency matrix of some graph G and let A be the matrix of the colour graph of some perfect colouring of G. Then each eigenvalue of A is an eigenvalue of M (w.r.t multiple counting).

Application of the application: find all 2-, 3-, 4-colourings of the Platonic graphs



The eigenvalues of the adjacency matrices of the Platonic graphs:

G	tetrahedron	cube	octahedron
	$-1^3, 3$	$-3, -1^3, 1^3, 3$	$-2^2, 0^3, 4$

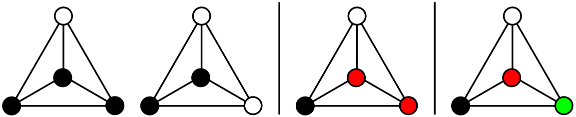
G	dodecahedron	icosahedron
	$-\sqrt{5}^3, -2^4, 0^4, 1^5, \sqrt{5}^3, 3$	$-\sqrt{5}^3, -1^5, \sqrt{5}^3, 5$

In order to find candidates for perfect colourings: browse the lists, collect all matrices with the correct eigenvalues.

Candidates for perfect 2-, 3- and 4-colourings of the tetrahedron:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
2. 3 colours: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$
3. 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

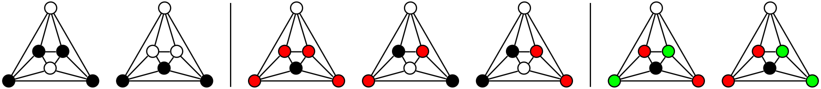
All are matrices for perfect colourings of the tetrahedron:



Candidates for perfect 2-, 3- and 4-colourings of the octahedron:

1. 2 colours: $\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.
3. 4 colours: $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

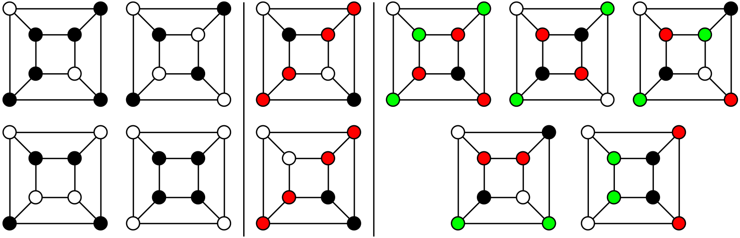
All but one are matrices for perfect colourings of the octahedron::



Candidates for perfect 2-, 3- and 4-colourings of the cube:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.
3. 4 col's: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

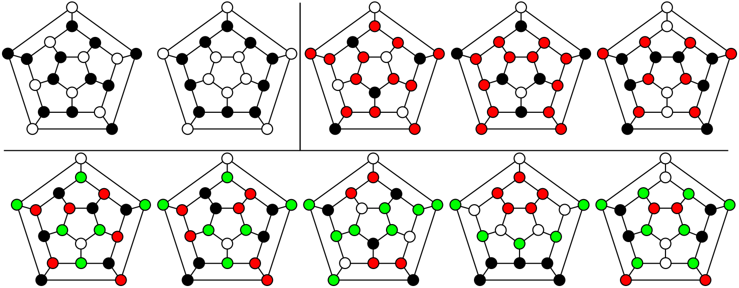
All are matrices for perfect colourings of the octahedron:



Candidates for perfect 2-, 3- and 4-colourings of the dodecahedron:

1. 2 colours: $\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.
3. 4 col's: $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

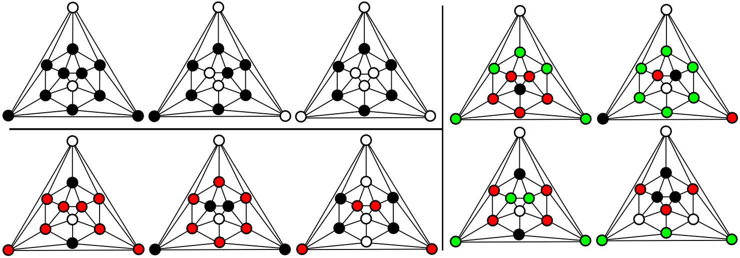
All are matrices for perfect colourings of the dodecahedron:



Candidates for perfect 2-, 3- and 4-colourings of the icosahedron:

1. 2 colours: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.
2. 3 colours: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.
3. 4 colours: $\begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$

All are matrices for perfect colourings of the icosahedron:



More in:

Joseph R.C. Damasco, Dirk Frettlöh:
Perfect colourings of regular graphs,
AMS Contemporary Mathematics, in press;
`arXiv:1804.03552`

Noncongruent equipartitions of the plane

Dirk Frettlöh

Joint work with Christian Richter (Jena)

Technische Fakultät

Universität Bielefeld

International Symposium on Discrete Geometry and Convexity
Shinjiazhuang, 27^{th} August 2019

A rich source of interesting problems:

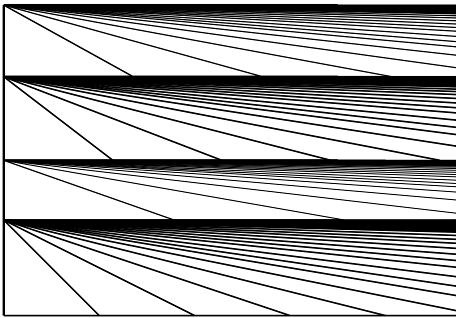
`nandacumar.blogspot.com`

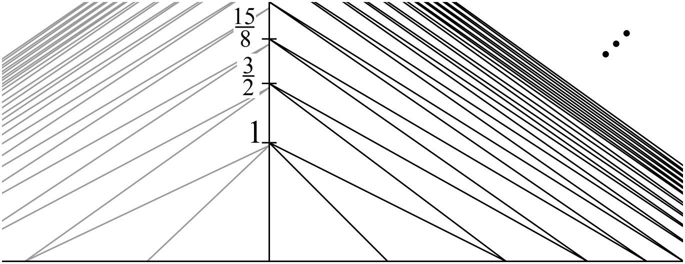
One is: *Is it possible to tile the plane with pairwise noncongruent triangles of the same area and perimeter?*

Answer: No (Andrey Kupavskii, János Pach, Gábor Tardos: Tilings with noncongruent triangles, *European J. Combin.* 73 (2018))

A weaker question: *Is it possible to tile the plane with pairwise noncongruent triangles of the same area?*

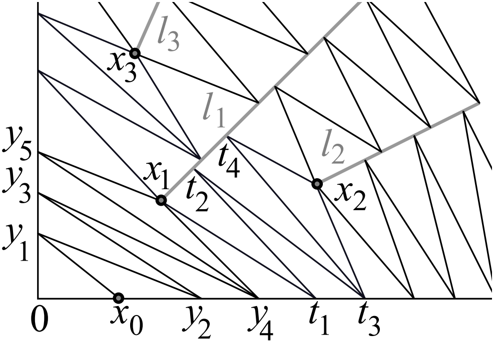
Answer: Yes.




15/8
3/2
1

Slightly harder question: *Is it possible to tile the plane with pairwise noncongruent triangles of the same area and bounded perimeter?*

Answer: Yes.



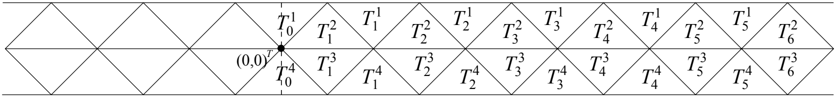
F.: Noncongruent equidissections of the plane, *Discrete Geometry and Symmetry*, Springer (2018)

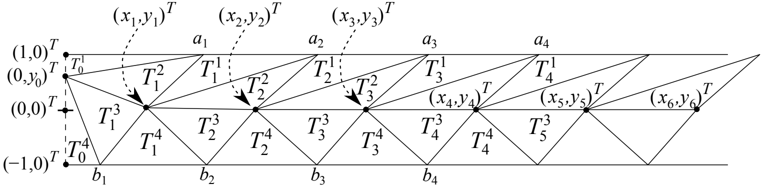
Kupavskii, Pach, Tardos: Tilings of the plane with unit area triangles of bounded diameter, *Acta Math. Hungar.* 155 (2018)

Even harder question: *Is it possible to tile the plane with pairwise noncongruent triangles of the same area and bounded perimeter, such that the triangles are vertex-to-vertex?*

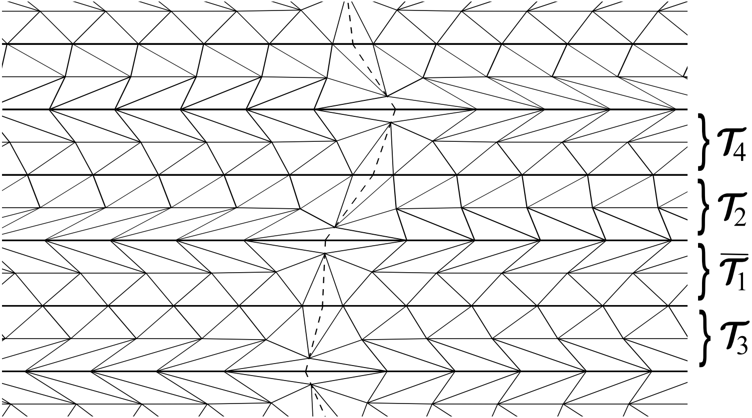
Answer: Yes. (F., Richter: Incongruent equipartitions of the plane, submitted, `arxiv:1905:08144`)

Idea: distort





Stack sheared copies of the strip tiling:



(7 pages of proof show all triangles are indeed incongruent)

Variations of the questions for n-gons ($3 \leq n \leq 6$)

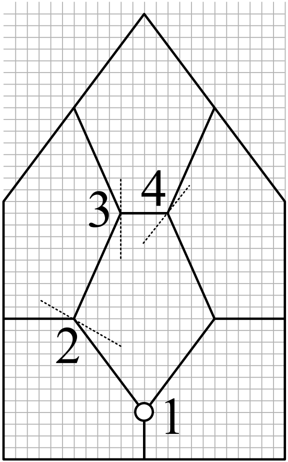
Triangles	vtv	not vtv
bounded perimeter	Yes	Yes
equal perimeter	No	No

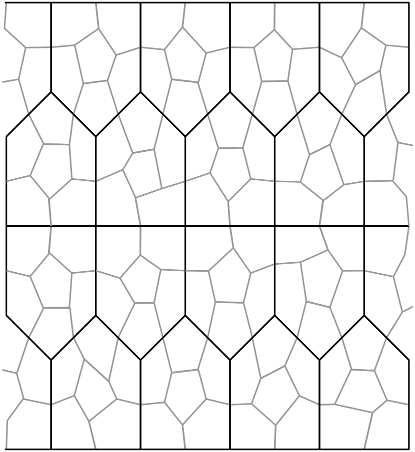
Quadrangles	vtv	not vtv
bounded perimeter	Yes	Yes
equal perimeter	?	?

Pentagons	vtv	not vtv
bounded perimeter	Yes	Yes
equal perimeter	?	?

Hexagons	vtv	not vtv
bounded perimeter	Yes	Yes
equal perimeter	?	?

Quadrangles, pentagons, hexagons are easier. E.g.:





Triangles seem to be the "limiting" case (wrt degrees of freedom)


Thank you.