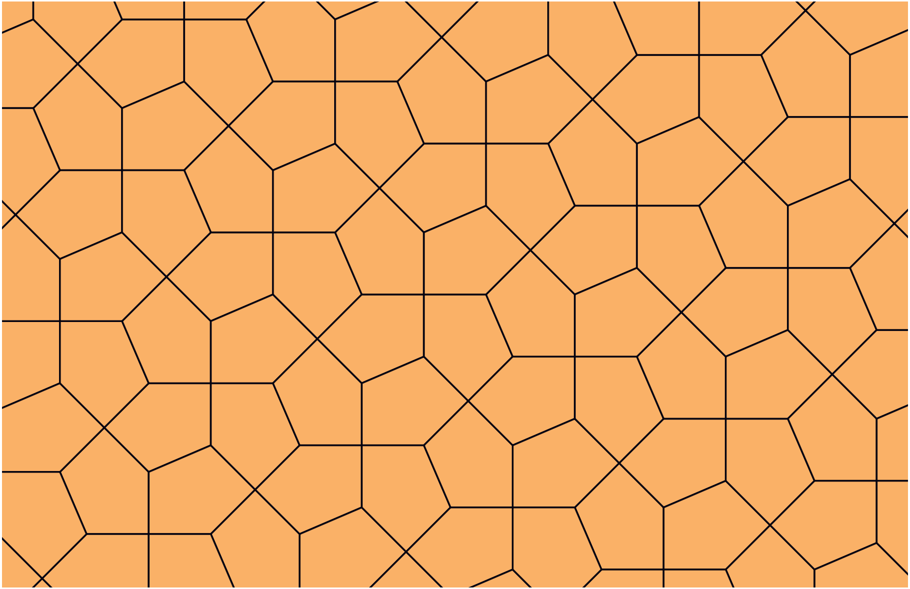
Open problems in the mathematics of tilings

Dirk Frettlöh

Technische Fakultät
Universität Bielefeld

Ateneo de Manila University
10. Jan. 2014

Tilings



Here: **infinite tilings**. Focus on shape and symmetry.

Definitions:

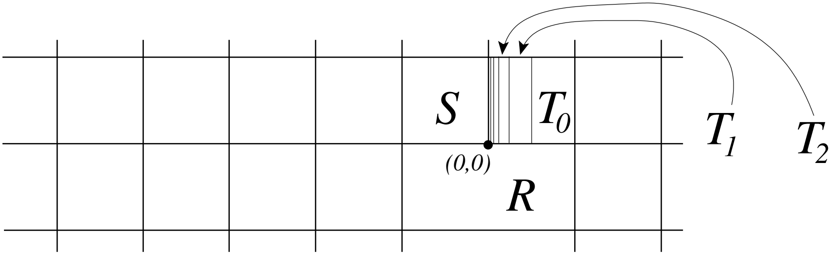
- *tiling*: covering of $\mathbb{R}^2$ (resp. $\mathbb{R}^d$) by *tiles* without overlaps
- *tile*: compact set T with $\overline{\mathring{T}} = T$.
  Often a (convex) polygon (resp. polytope).
- *vertex* of a tile: vertex of a polygon.
- *vertex* of a tiling: isolated point of the intersection of ≥ 3 (resp. $\geq d+1$) tiles.

Here: **infinite tilings**. Focus on shape and symmetry.

Definitions:

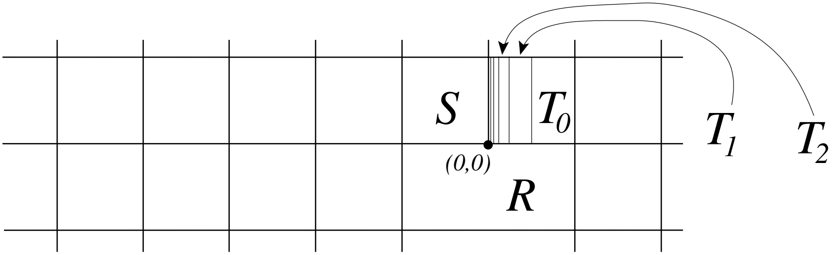
- *tiling*: covering of $\mathbb{R}^2$ (resp. $\mathbb{R}^d$) by *tiles* without overlaps
- *tile*: compact set T with $\overline{\mathring{T}} = T$.
  Often a (convex) polygon (resp. polytope).
- *vertex* of a tile: vertex of a polygon.
- *vertex* of a tiling: isolated point of the intersection of ≥ 3 (resp. $\geq d+1$) tiles.
- *k-periodic* tiling $\mathcal{T}$: there are k linearly independent $v_1, \ldots v_k$ with $\mathcal{T} + v = \mathcal{T}$. ($k = 2$ (resp. $k = d$): *crystallographic*)
- $\mathcal{T}$ is *non-periodic*: $\mathcal{T} + v = \mathcal{T}$ implies $v = 0$.
- *locally finite*: each ball intersect only finitely many tiles

Not locally finite:



Problem 0: In a locally finite tiling by convex polygons, is there a point that is vertex of exactly one tile?

Not locally finite:

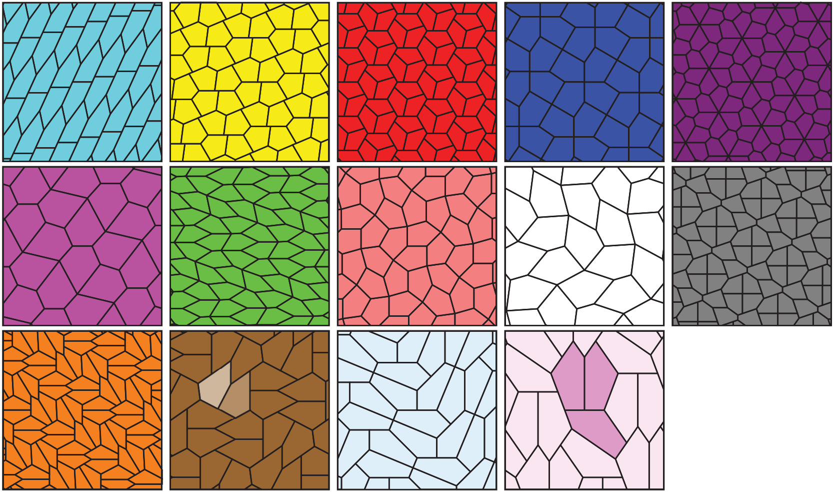


Problem 0: In a locally finite tiling by convex polygons, is there a point that is vertex of exactly one tile?

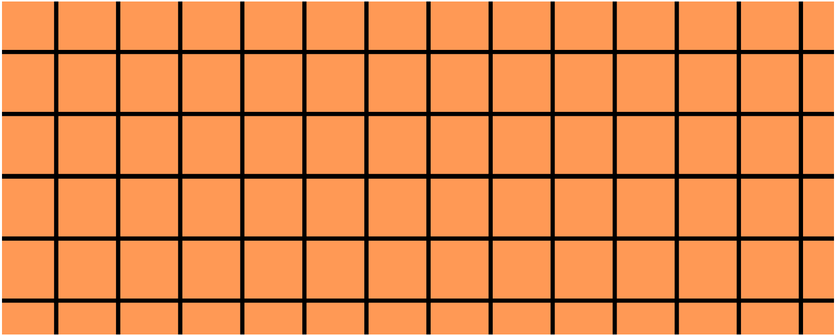
Solved: No! “The lonely vertex problem” (in any dimension)

Problem 1: (Danzer's problem) In a locally finite tiling by convex polygons, are there always triangles of arbitrary large area containing no vertex?

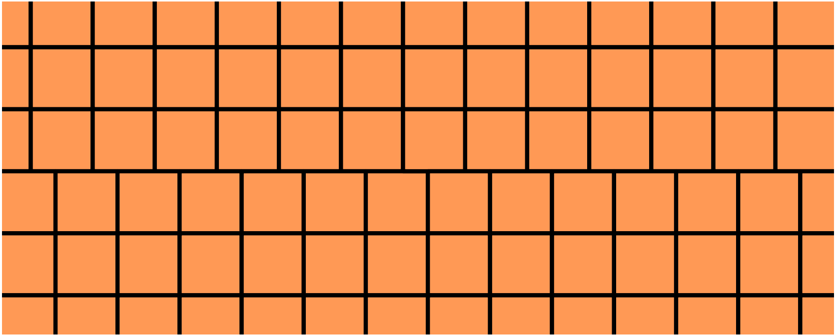
Problem 2: Which pentagons do tile the plane?



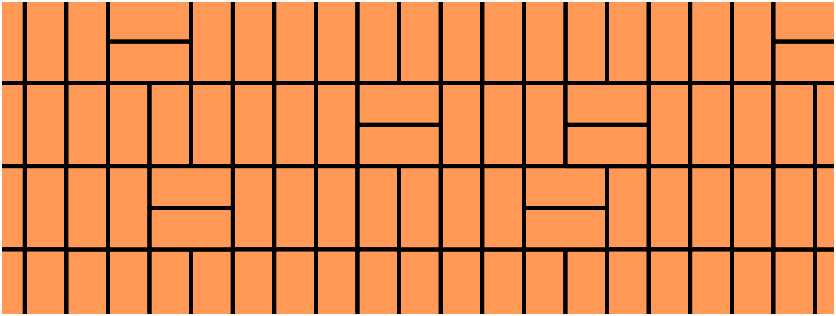
2-periodic (= crystallographic) tiling:



1-periodic tiling:



non-periodic tiling:



Problem 3: (Aperiodic Monotile) Is there a single tile shape that allows only non-periodic tilings of $\mathbb{R}^2$?

(Hard. See Socolar-Taylor 2011)

Problem 3: (Aperiodic Monotile) Is there a single tile shape that allows only non-periodic tilings of $\mathbb{R}^2$?

(Hard. See Socolar-Taylor 2011)

Problem 4: (Periodic Monotile) Which (convex) tile shapes allow only 2-periodic tilings of $\mathbb{R}^2$?

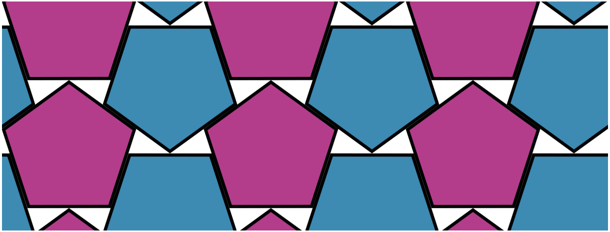
(~~Square~~, ~~rectangle~~, regular hexagon, ... full list?)

Problem 5: (Danzer-Grünbaum-Shephard) Is there a locally finite tiling by tiles such that the tiles

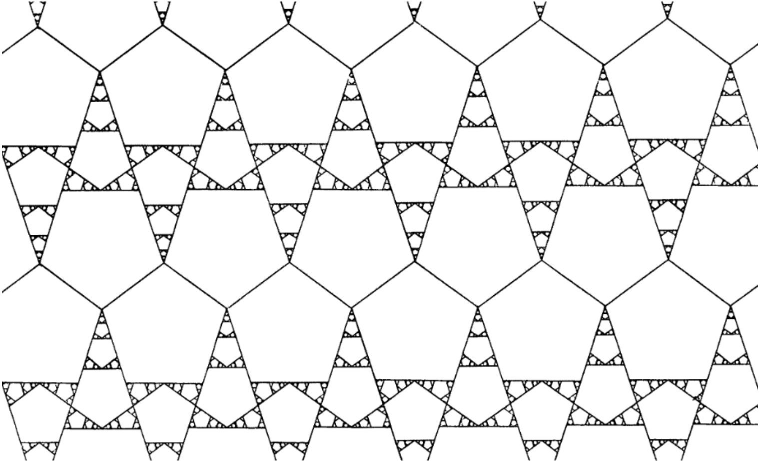
- have 5-fold rotational symmetry, and
- have diameter at most 1, and
- are topological disks?

Problem 5: (Danzer-Grünbaum-Shephard) Is there a locally finite tiling by tiles such that the tiles

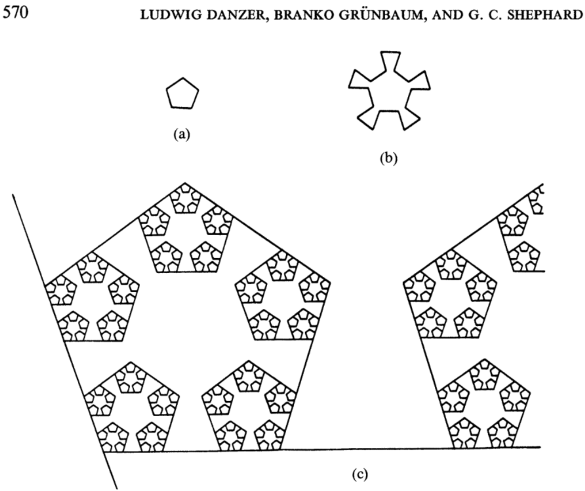
- have 5-fold rotational symmetry, and
- have diameter at most 1, and
- are topological disks?



Not a tiling!



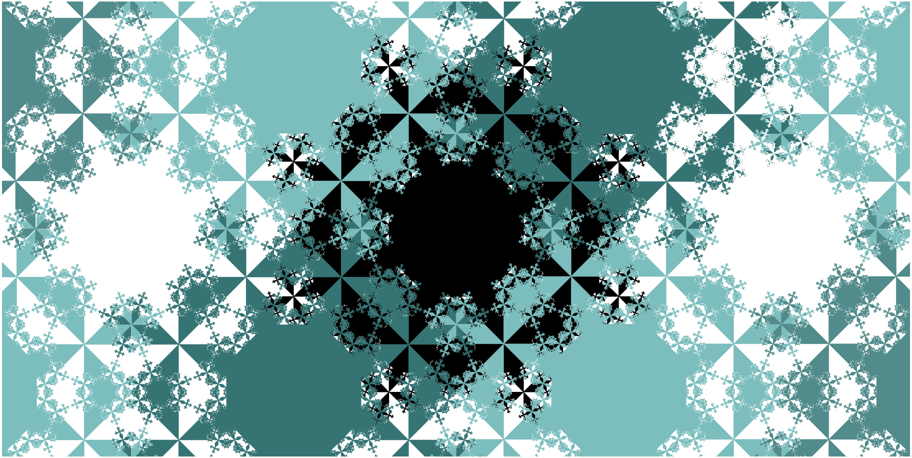
Not locally finite!

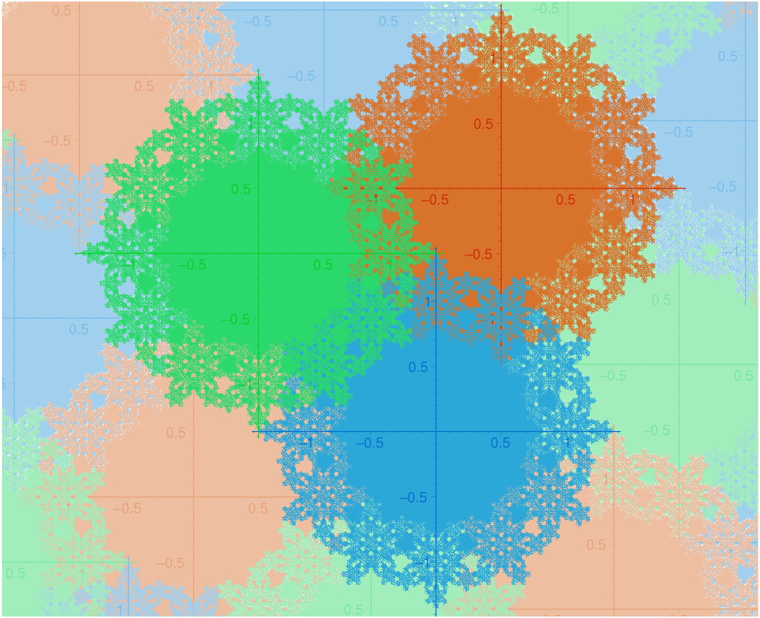


Diameter not bounded!

Problem 6: Same question for 7-fold, 8-fold, ... symmetry.

If we drop "topological disk" then 8-fold and 12-fold is possible.



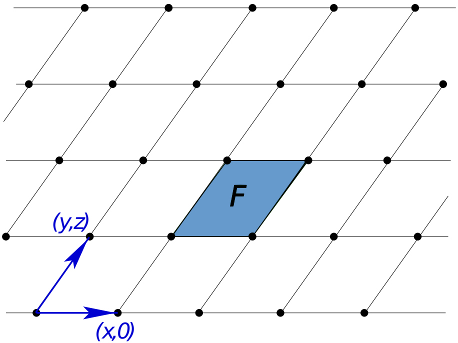


Problem 7: Can a fundamental cell of a lattice have more symmetry than the point group?

Problem 7: Can a fundamental cell of a lattice have more symmetry than the point group?

Lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

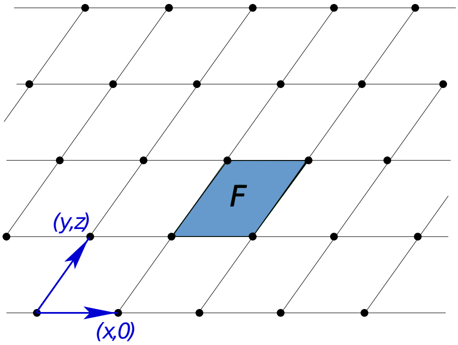
Fundamental cell of Γ: $\overline{\mathbb{R}^d/\Gamma}$.



Problem 7: Can a fundamental cell of a lattice have more symmetry than the point group?

Lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

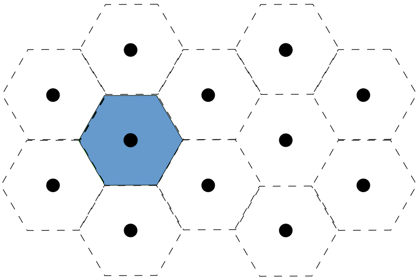
Fundamental cell of Γ: $\overline{\mathbb{R}^d/\Gamma}$.

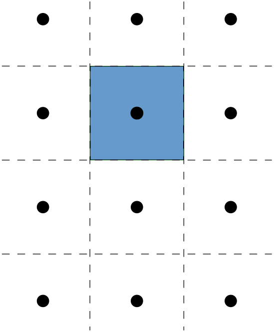


Point group $P(\Gamma)$ of Γ: All $g \in O(d,\mathbb{R})$ with $g\Gamma = \Gamma$.

Trivially, each lattice Γ has a fundamental cell whose symmetry group is $P(\Gamma)$.

For instance, take the Voronoi cell of a lattice point x. (That is the set of points closer to x than to each other lattice point.)





Theorem (Elser, F)

Let $\Gamma \subset \mathbb{R}^2$ be a lattice, but not a rhombic lattice. Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $\quad [S(F) : P(\Gamma)] = 2$.

Theorem (F)

Let $\Gamma \subset \mathbb{R}^3$ be a lattice, but not a cubic lattice. Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $\quad [S(F) : P(\Gamma)] = 2$.

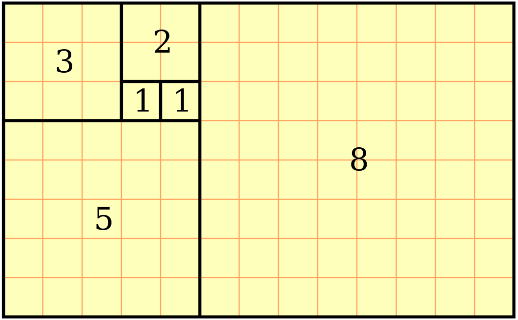
Problem 7 still open for:

- rhombic lattices in $\mathbb{R}^2$
- ~~cubic lattices in~~ $\cancel{\mathbb{R}^3}$
- anything in $\mathbb{R}^d$, $d \geq 4$
- even more symmetry: $[S(F) : P(\Gamma)] > 2$

Spiral tilings:

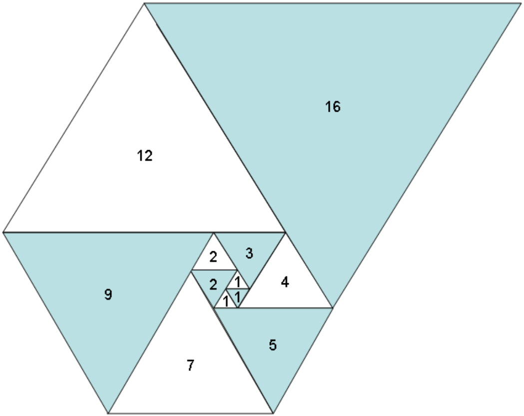
Fibonacci numbers: $F_n = F_{n-1} + F_{n-2},\ F_1 = 1,\ F_2 = 1.$

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233, \ldots$$



Padovan numbers: $P_0 = P_1 = P_2 = 1,\ P_n = P_{n-2} + P_{n-3}.$

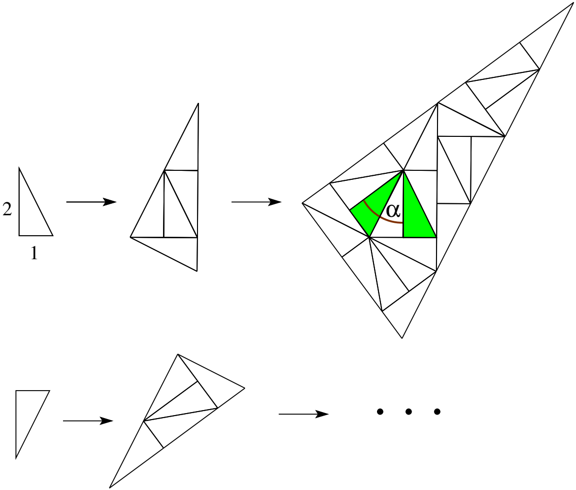
1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, ...



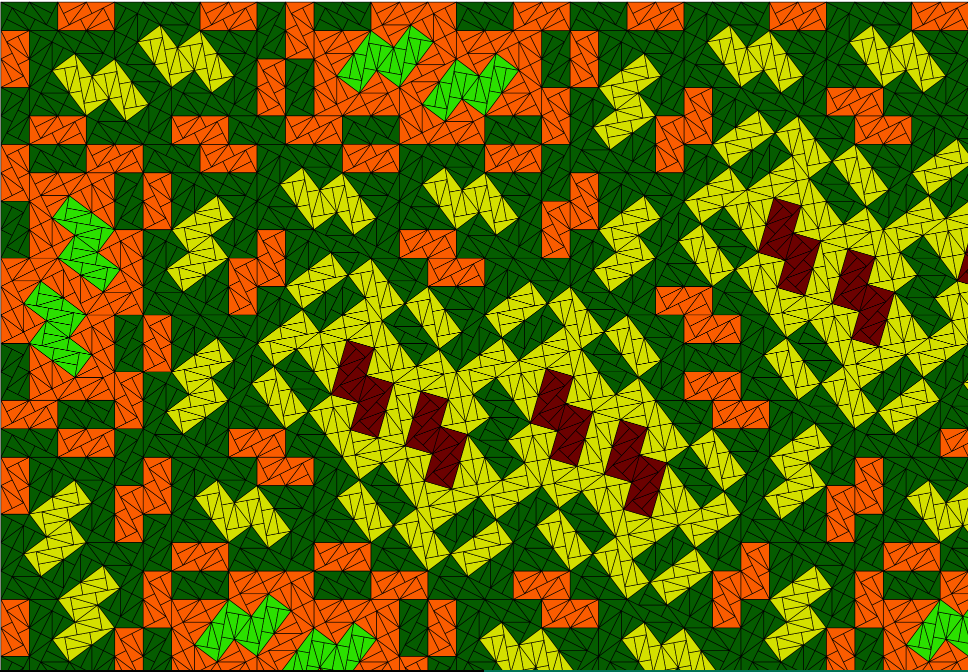
Problem 8: Which recursive integer sequences can be realised as spiral tilings?

(And what means "realised"?)

Pinwheel substitution tiling:



The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



Which tile shapes do forbid infinitely many orientations?

Theorem (F.-Harriss, 2013)

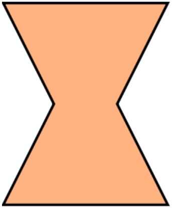
Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many tile shapes, all centrally symmetric convex polygons (i.e., $P = -P$). Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

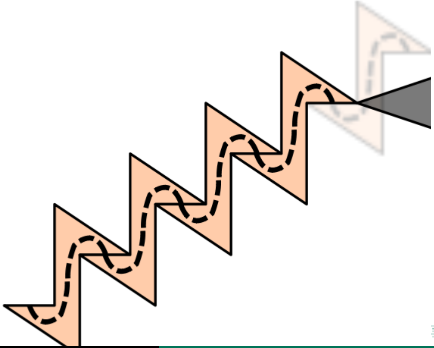
Which tile shapes do forbid infinitely many orientations?

Theorem (F.-Harriss, 2013)

Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many tile shapes, all centrally symmetric convex polygons (i.e., $P = -P$). Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

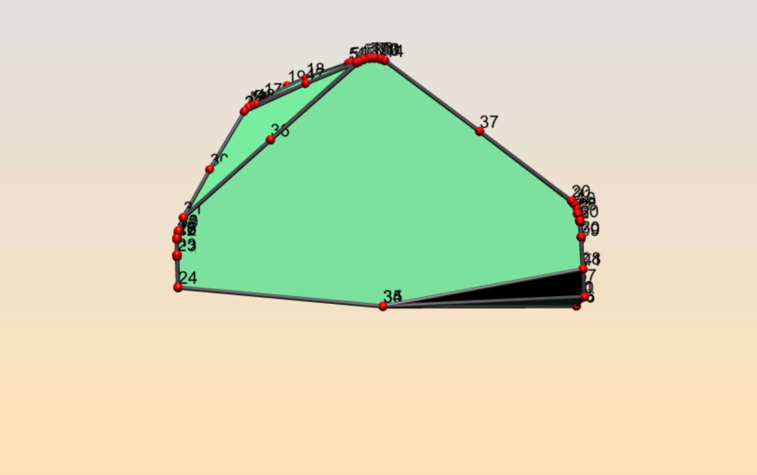
- **Problem 9:** The same for non-convex
- **Problem 10:** The same for higher dimensions





Literature:

- Grünbaum & Shephard: *Tilings and Patterns*
- Croft, Falconer & Guy: *Unsolved Problems in Geometry*
- Baake & Grimm: *Aperiodic Order*



THANK YOU!