# Highly symmetric fundamental cells for planar lattices

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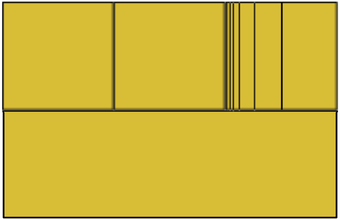
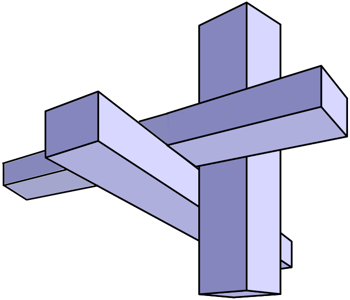
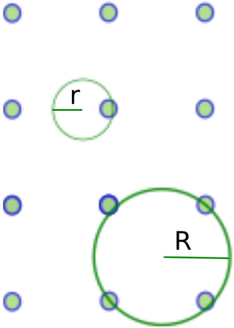
- Lonely vertex theorem
- Bilipschitzequivalence of Delone sets
- Perfect colourings
- Highly symmetric FCs for planar lattices

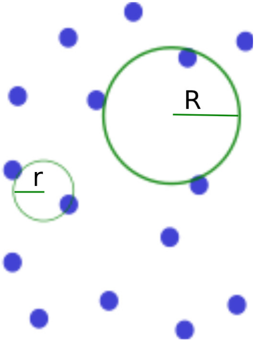
 

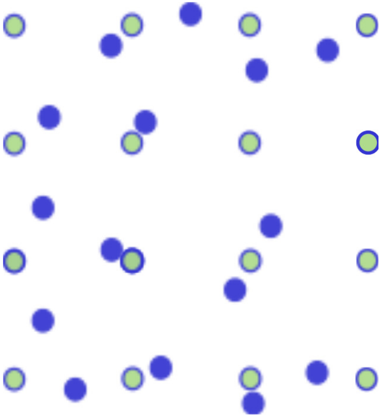
There is no **lonely vertex** in a locally finite tiling of $\mathbb{R}^d$ by convex polytopes (F-Glazyrin 2008)

(I.e., no $x \in \mathbb{R}^d$ is vertex of exactly one tile)

Bilipschitzequivalence of **Delone sets**

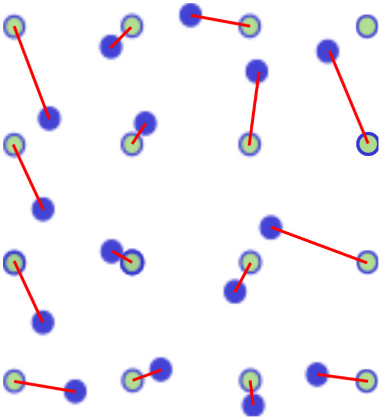




Delone sets D, E are called *bilipschitz-equiv*, if there is a bijection $f : D \to E$, such that $f$ and $f^{-1}$ are both Lipschitz continuous.

There are Delone sets in $\mathbb{R}^2$ which are not bilipsch-equiv (Burago-Kleiner, C. McMullen 1997)



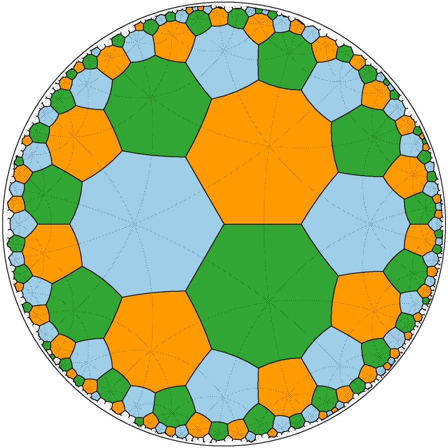
D, E are in *bounded distance*, if there is a bijection $f : D \to E$, $C > 0$, such that $d(x, f(x)) < C$ for all $x \in D$.

Bounded distance implies bilipschitz equivalence.

Certain Delone sets (particular "model sets") are bilipschitz equiv to $\mathbb{Z}^2$.

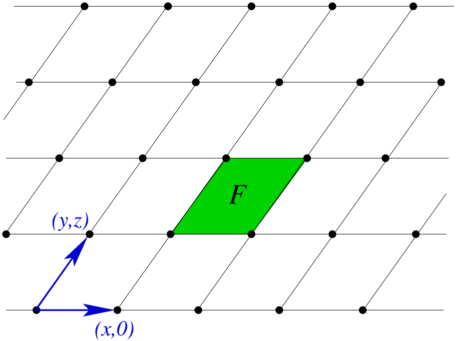
(Dolbilin, F, Garber, work in progress)

**Perfect colourings** of hyperbolic tilings (F 2008)



*Point lattice* $\Gamma$ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of $d$ linearly independent vectors.

*Fundamental cell* of $\Gamma$: $\overline{\mathbb{R}^d/\Gamma}$.



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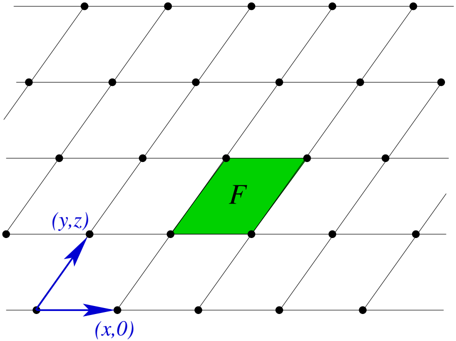
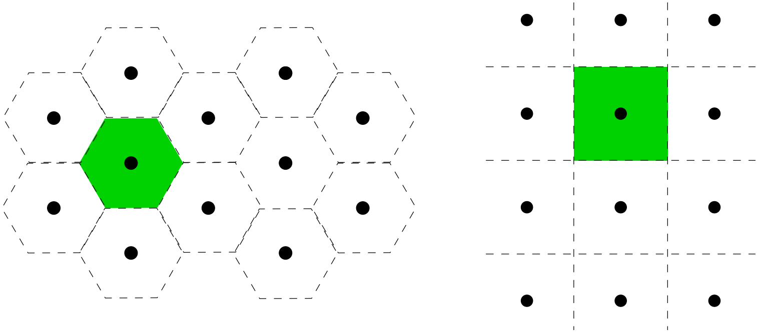



*Point group* $P(\Gamma)$ of $\Gamma$: All linear isometries $f$ with $f(\Gamma) = \Gamma$.

Trivially, each lattice $\Gamma$ has a fundamental cell whose symmetry group is $P(\Gamma)$.

For instance, take the Voronoi cell of a lattice point $x$. (That is the set of points closer to $x$ than to each other lattice point.)



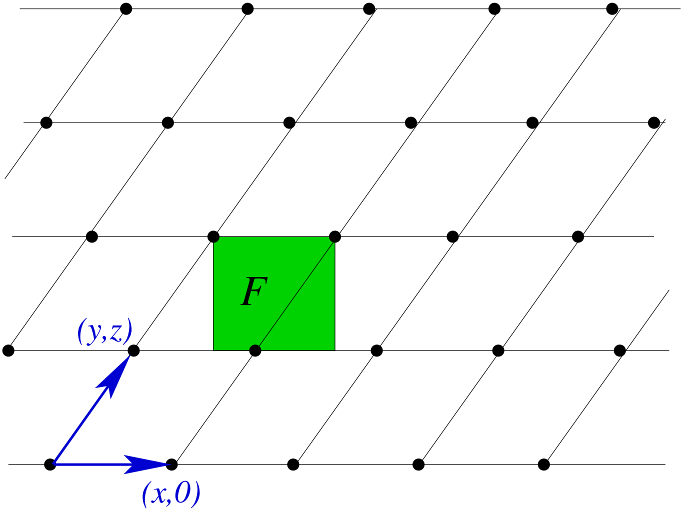
# Main result

Theorem (Elser, F)

*Let* $\Gamma \subset \mathbb{R}^2$ *be a lattice, but not a rhombic lattice. Then there is a fundamental cell* $F$ *of* $\Gamma$ *whose symmetry group* $S(F)$ *is strictly larger than* $P(\Gamma)$*:* $\quad [S(F) : P(\Gamma)] = 2$.

'Rhombic lattice' means here: one with basis vectors of equal length, but neither a square lattice nor a hexagonal lattice.

Generic lattice:



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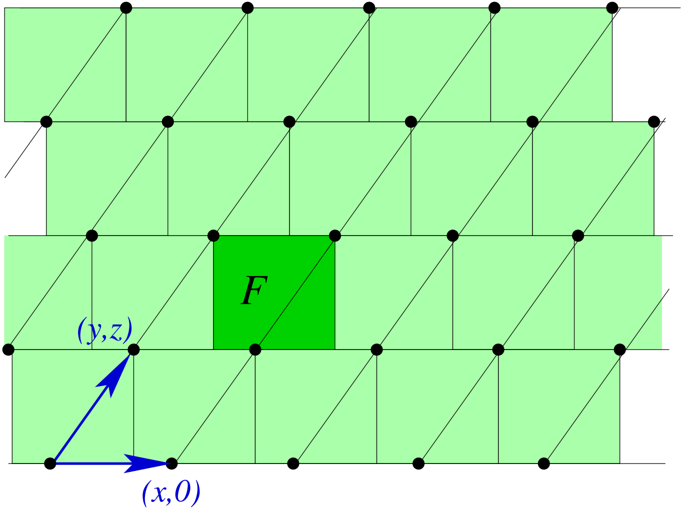
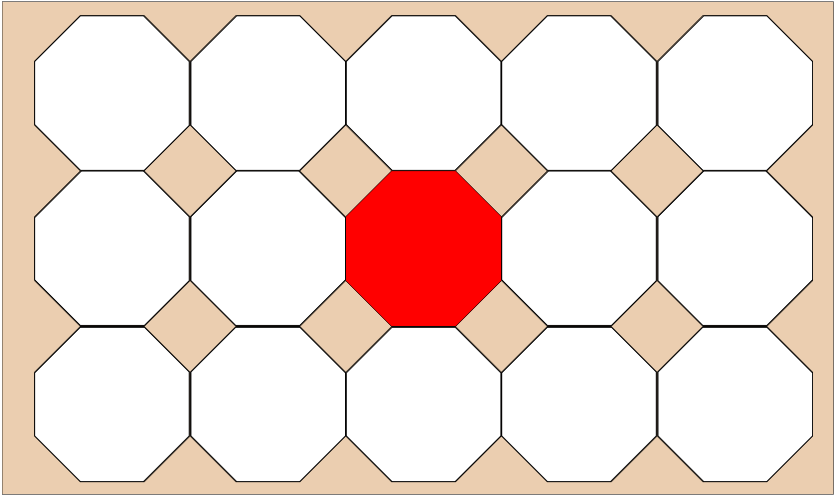
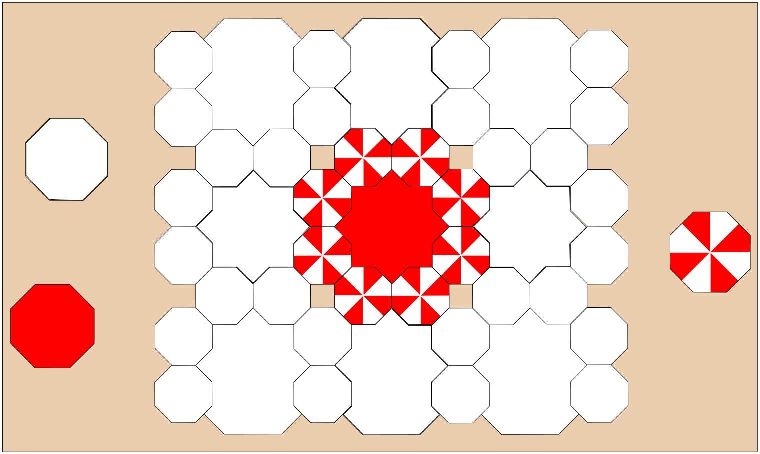
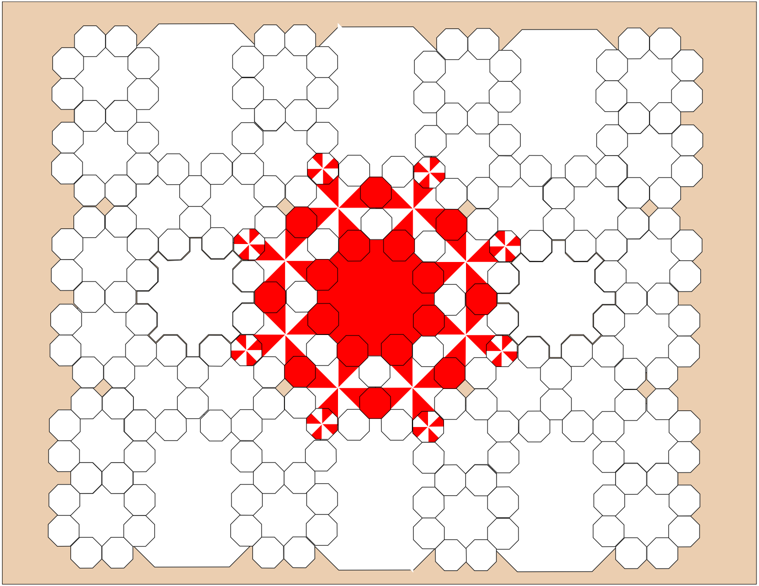
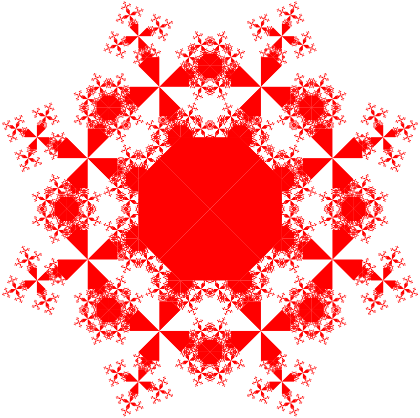


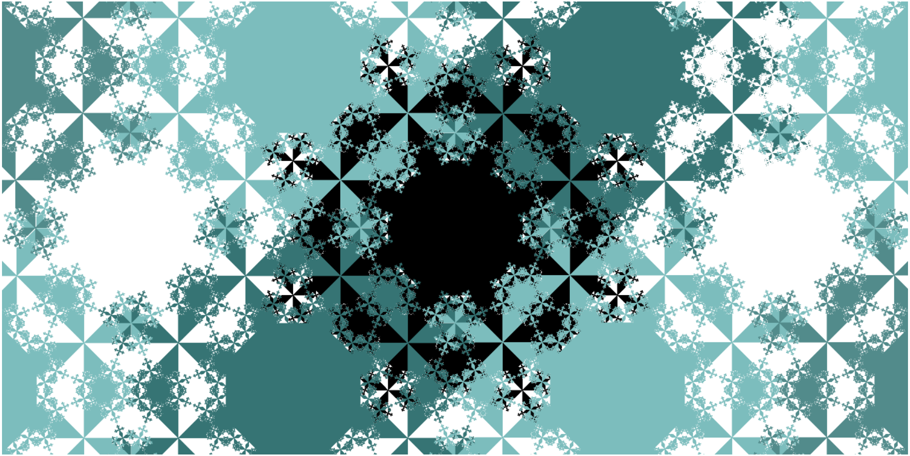
Square lattice (Veit Elser):



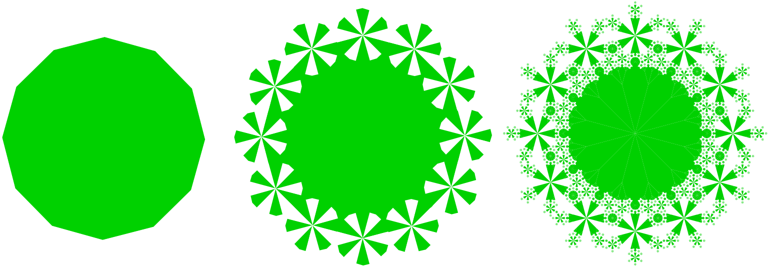




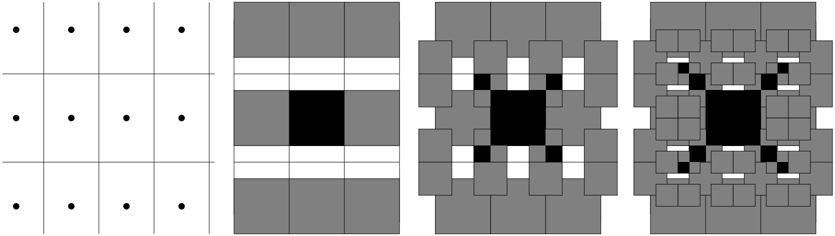


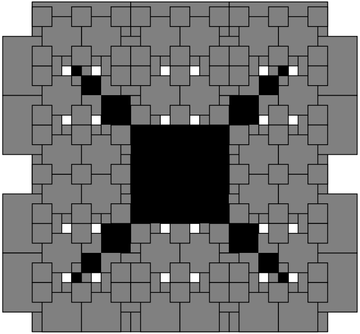


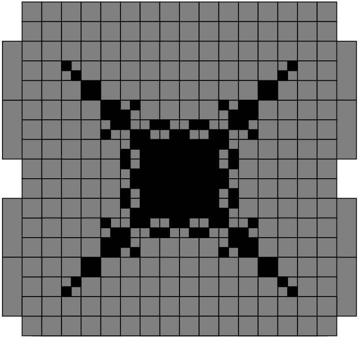
Hexagonal lattice (Elser-Cockayne, Baake-Klitzing-Schlottmann):

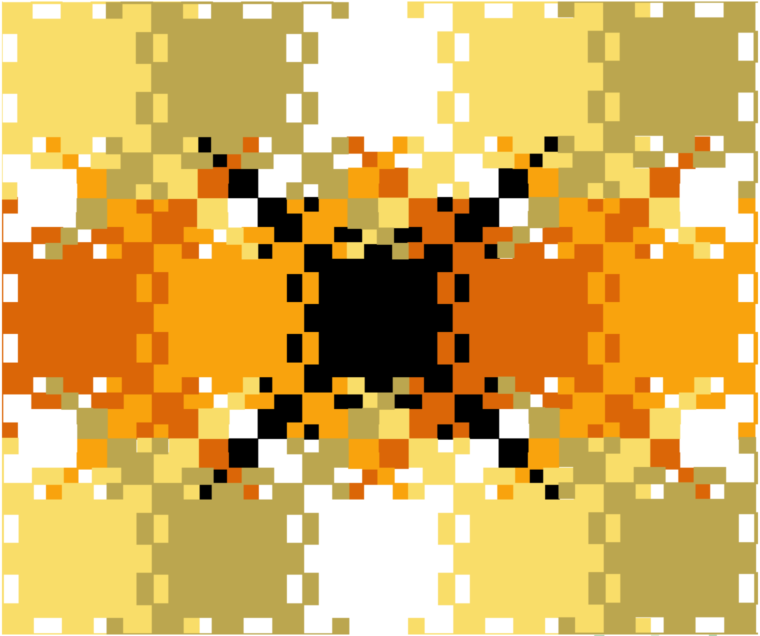


# Rectangular lattice









# Application: Minimal matchings

Consider the square lattice $\mathbb{Z}^2$, and $R_{45}\mathbb{Z}^2$, the square lattice rotated by $45^\circ$.

**Problem:** Find a perfect matching between $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$ with maximal distance not larger than $C > 0$. How small can $C$ be?

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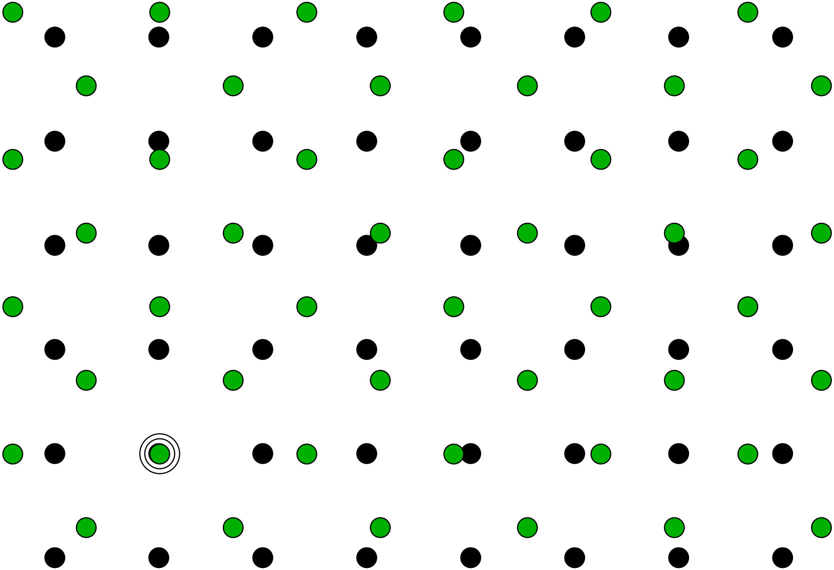
**Problem:** Find a perfect matching between $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$ with maximal distance not larger than $C > 0$. How small can $C$ be?

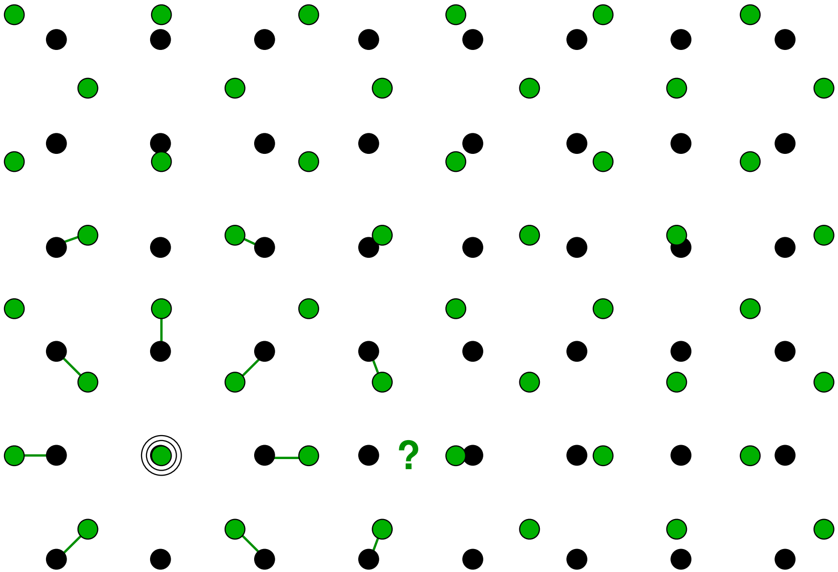
That is, find $f : \mathbb{Z}^2 \to R_{45}\mathbb{Z}^2$, where $f$ is bijective and

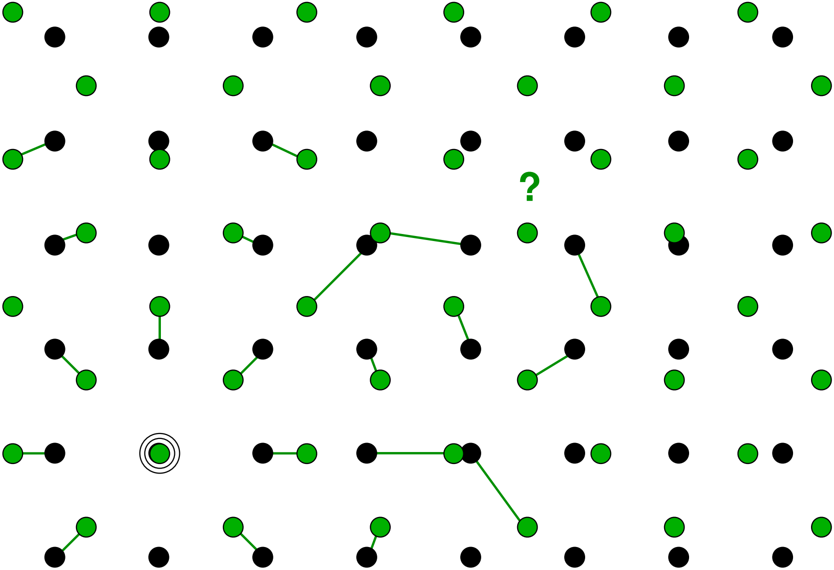
$$\forall x \in \mathbb{Z}^2 : \quad d(x, f(x)) \le C$$

for $C$ as small as possible.

(It is easy to see that $C \ge \frac{\sqrt{2}}{2} = 0.7071\ldots$.)







Naively: difficult.

Using the 8-fold fundamental cell $F$ yields a matching with $C = 0.92387\ldots$.

How?

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Using the 8-fold fundamental cell $F$ yields a matching with $C = 0.92387...$.

How?

- Consider $\mathbb{Z}^2 + F$. Each $x + F$ $(x \in \mathbb{Z}^2)$ contains exactly one point of $\mathbb{Z}^2$ in its centre.
- $F$ is also fundamental cell for $R_{45}\mathbb{Z}^2$. Thus each $x + F$ $(x \in \mathbb{Z}^2)$ contains exactly one point $x' \in R_{45}\mathbb{Z}^2$.
- Let $f(x) = x'$.

This (and its analogues) yields good matchings for

- $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$: $C = 0.92387\ldots.$
- The hexagonal lattice $H$ and $R_{30}H$: $C = 0.78867\ldots$
- A rectangular lattice $P$ and $R_{90}P$: $C \le \frac{1}{\sqrt{2}} \frac{\sqrt{5}+1}{2} b.$

($b$ is the length of the longer lattice basis vector of $P$.)

# What next?

- Rhombic lattices?
- Discrete Geometry:
  - Higher dimensions, root lattices
  - Hyperbolic spaces
- Fractals:
  - Iterated function systems
  - Dimension of the boundaries
  - Connectivity
- Optimization:
  - Better matchings
  - Finite subsets of $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$


Thank you.