Bounded distance equivalence in repetitive tiling hulls

Dirk Frettlöh

Joint work with Lorenzo Sadun (Univ. Texas)
and Alexey Garber (Univ. Texas)

Technische Fakultät
Universität Bielefeld

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Summary:

On a conference in Leiden Jamie Walton asked:
"How many equivalence classes wrt bounded distance equivalence does a hull have?"

Answer: for a repetitive Delone set:

One or uncountably many.

For instance: hull of a cut-and-project set, or the hull of a primitive FLC substitution.

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each open ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each closed ball of radius R contains at least one point of Λ (*relatively dense*)

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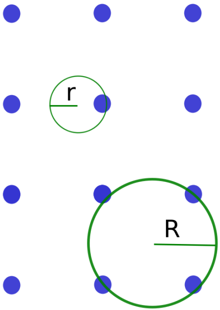
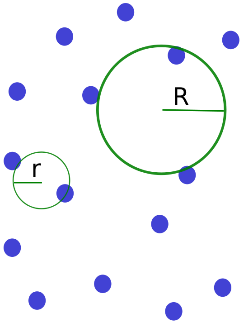


periodic crystal



disordered

An equivalence relation for Delone sets:

Definition

Two Delone sets Λ, Λ' are bounded distance equivalent (bde), if there is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad \|x - g(x)\| < C$$

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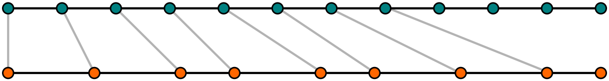

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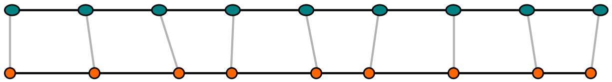

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Of course, density matters: $\mathbb{Z}^2 \overset{\text{bd}}{\not\sim} 2\mathbb{Z}^2$.

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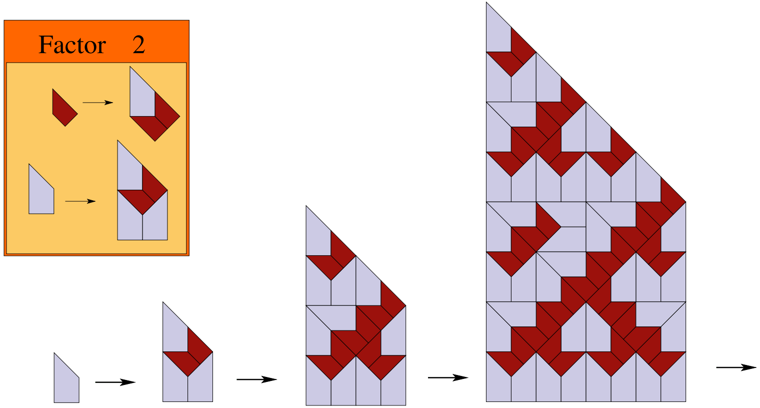
Hence interesting examples are non-periodic.

A simple way to generate interesting (non-periodic, but highly ordered) Delone sets goes via *substitution tilings*.

Substitution tiling with *substitution factor* 2, and two *prototiles*:



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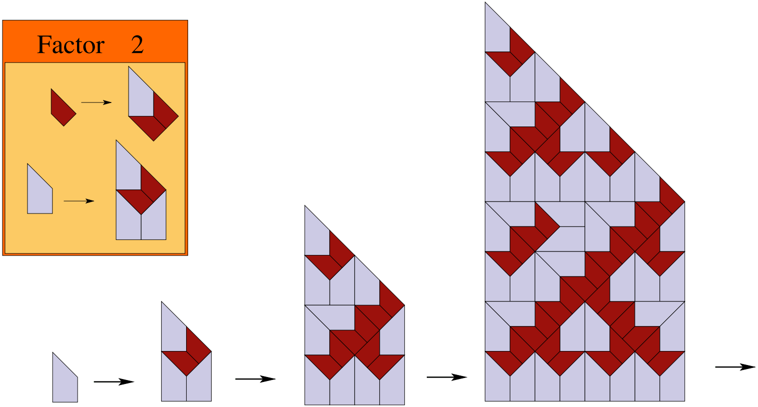
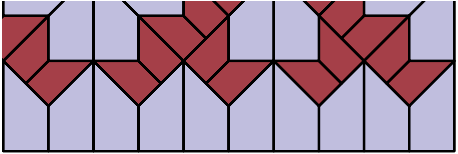
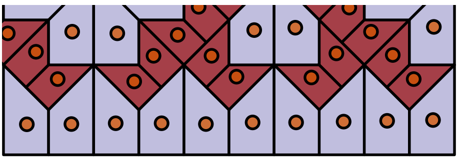


Substitution matrix here $M_\sigma = \left(\begin{smallmatrix} 2 & 2 \\ 1 & 3 \end{smallmatrix}\right)$.

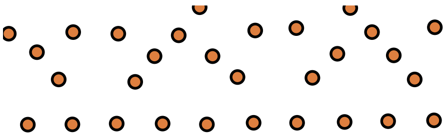
In order to obtain a Delone set from a tiling...



...put one point in each tile....



... and forget the tiles.



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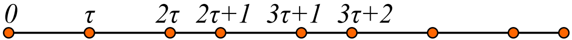
In particular, if we start with a patch P with 7 tiles of type 1 and 3 tiles of type 2, then $M_\sigma^n \binom{7}{3}$ counts the tiles in $\sigma^n(P)$.

Translate the Fibonacci tiling into a Delone set Λ_{Fib}:



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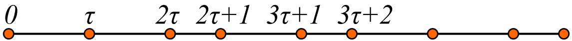

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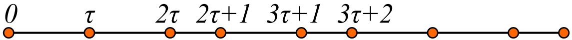

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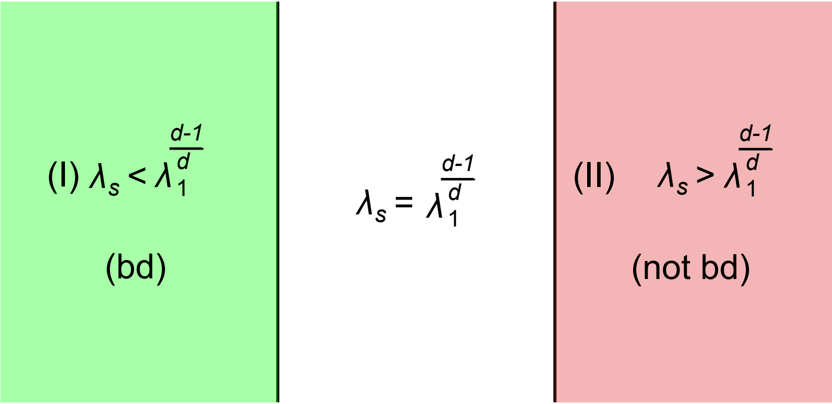
Hence for *all* Delone sets Λ from tilings $\mathcal{T} \in \mathbb{X}_{Fib}$ we have $\Lambda \overset{\mathrm{bd}}{\sim} \mathbb{Z}$.
Hence $\mathbb{X}_{Fib}$ contains only one equivalence class.

Yaar Solomon gave a pretty general version (for $\mathbb{R}^d$, essentially: the second largest eigenvalue λ_2 determines bde or not: $\lambda_2 \overset{?}{\lesseqgtr} \lambda^{\frac{d-1}{d}}$)



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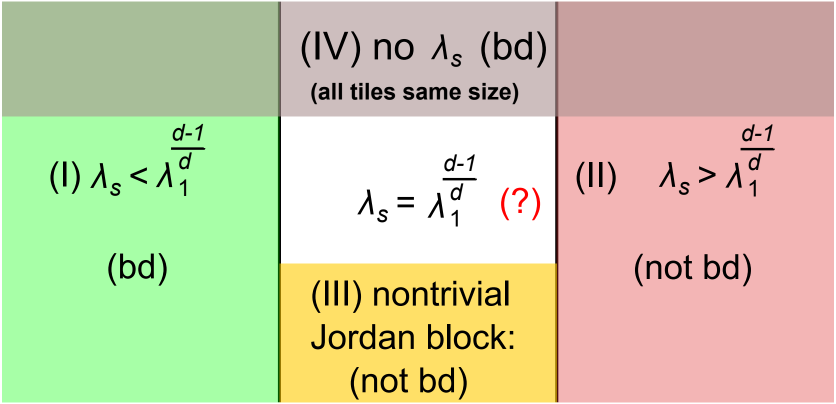

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Idea of the proof:

The Master Theorem (used in Solomon, F-Garber, ...):

Theorem (Laczkovich 1992)

Let Λ be a Delone set in $\mathbb{R}^d$. $\Lambda \overset{\mathrm{bd}}{\sim} \alpha\mathbb{Z}^d$ if and only there is $c > 0$ such that for all unions P of lattice cubes holds:

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Note that Laczkovich deals with $\Lambda \overset{\mathrm{bd}}{\sim} \alpha\mathbb{Z}^d$ only.
Yaar, Yotam and me showed a sufficient version for $\Lambda \overset{\mathrm{bd}}{\not\sim} \Lambda'$.

Still, the idea of the proof:

- Sort the eigenvalues of M according to their size:

$$\lambda > |\lambda_2| > \cdots > |\lambda_m|$$

- Take an appropriate patch P with numbers of tile types $p = (p_1, \ldots, p_m)^T$.
- Consider $M_\sigma^n v$. Assume M_σ has m linearly independent eigenvectors $v, v_2, \ldots, v_m$.
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For any p the term $\frac{n(P)}{\mathrm{vol}_d(P)}$ tends to the same value.
If $|\lambda_2| < 1$ then the error term is small.

If $|\lambda_2|^n$ grows faster than the boundary term then different patches P yield values with different error terms.

Hence there are "rich" patches R and "poor" patches P of arbitrary size. Use Laczkovich resp. F-Smilansky-Solomon..

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So much about history.

Situation for substitution tiling hulls:

- If $|\lambda_2| < 1$: for all $\Lambda \in \mathbb{X}_\sigma$ holds $\Lambda \overset{\mathrm{bd}}{\sim} \alpha\mathbb{Z}^d$. That is, one bde class.
- If $|\lambda_2| > 1$: for all $\Lambda \in \mathbb{X}_\sigma$ holds $\Lambda \overset{\mathrm{bd}}{\nsim} \alpha\mathbb{Z}^d$. How many bde classes?

New result by Yaar Solomon (`arXiv:2004:07387`.)

Theorem

Let σ be a primitive substitution such that $\lambda_2 > \lambda^{\frac{d-1}{d}}$. Let $\mathbb{X}_\sigma$ contain patches P, Q with $\mathrm{supp}(P) = \mathrm{supp}(Q) + t$
and [some technical condition on the population vectors $p(P)$ and $p(Q)$ required for the approach above to work: $p(P) - p(Q) \neq 0$ and ...].
Then the hull $\mathbb{X}_\sigma$ contains uncountably many bde classes.

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Then the hull $\mathbb{X}_\sigma$ *contains uncountably many bde classes.*

Similar approach: the error term (here: difference of tiles in $\sigma^n(P)$ and $\sigma^n(Q)$) grows faster than the boundary term.

Because of primitivity he can construct nested sequences of larger and larger patches:

$$Q \text{ in } \sigma^{n_1}(P) \text{ in } \sigma^{n_2}(P) \text{ in } \sigma^{n_3}(Q) \text{ in} \ldots$$

Any word in $\{P, Q\}$ gives rise to a tiling in $\mathbb{X}_\sigma$.

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Problem: $\text{supp}(P) = \text{supp}(Q) + t$.

Think of Fibonacci or Ammann-Beenker or Penrose or Buffalo or...: Support of P determines the number of tiles in P!

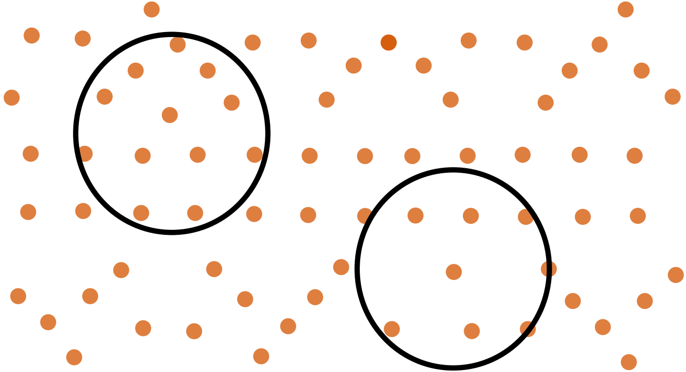
With Lorenzo and Alexey:

Theorem

Let Λ be a Delone set in $\mathbb{R}^d$ that is repetitive and FLC. If $\Lambda \overset{\text{bd}}{\not\sim} \alpha\mathbb{Z}^d$ then the hull of Λ consists of uncountably many bde classes.

(preprint under construction.)

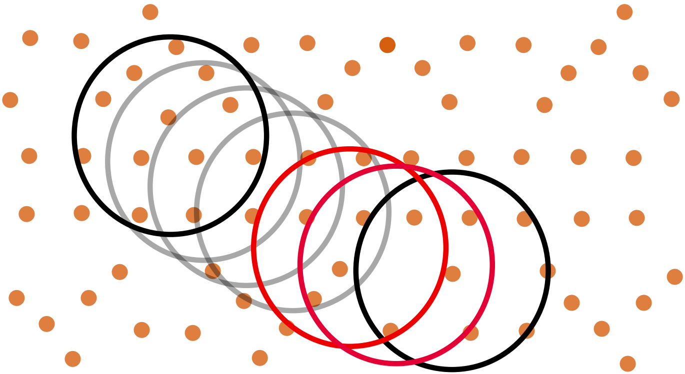
Ideas: Use "rich" and "poor" regions (arbitrary shape)



Problem: we need them to be in the same spot.

Ideas: Lemma:

- close to rich still rich (in a quantified way)
- somewhere between rich and poor it switches (in a correct way)



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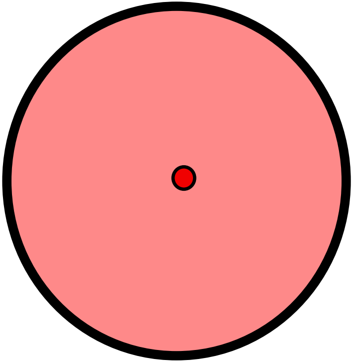

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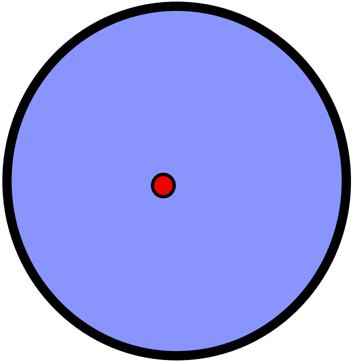

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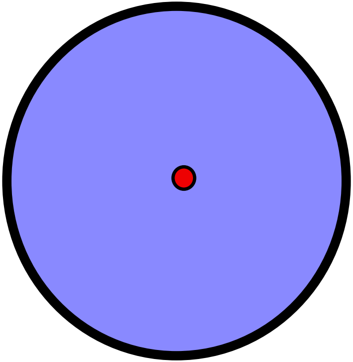

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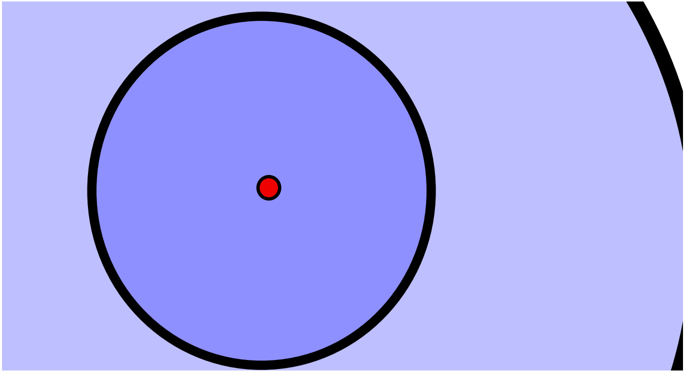

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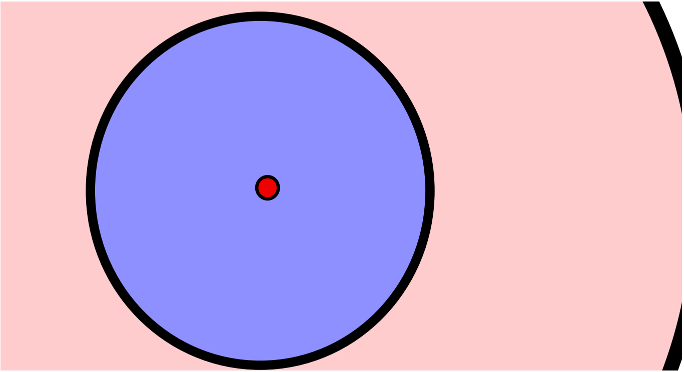

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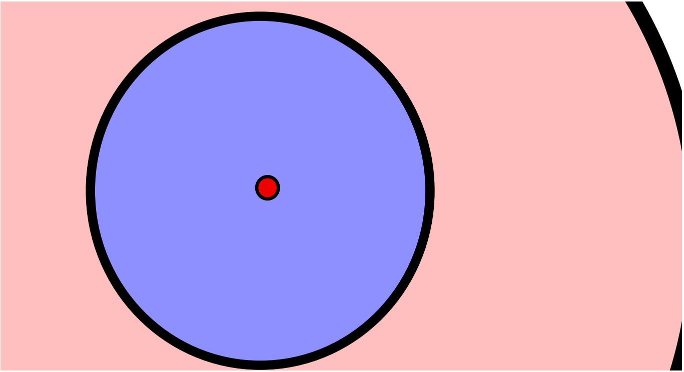

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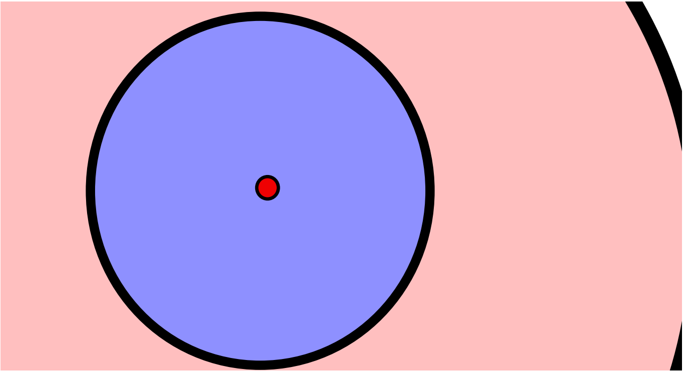

Any two Delone sets arising in this way from P-Q words with different letters in infinitely many positions are not bde to each other.

$\star \quad \star$
$\star$
Thank you! Cheers!