Highly symmetric fundamental cells for lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$

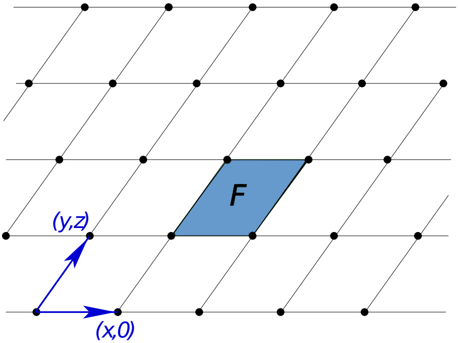
Dirk Frettlöh

Arbeitsgruppe Diskrete Geometrie
Institut für Mathematik
FU Berlin

12. Dec 2011

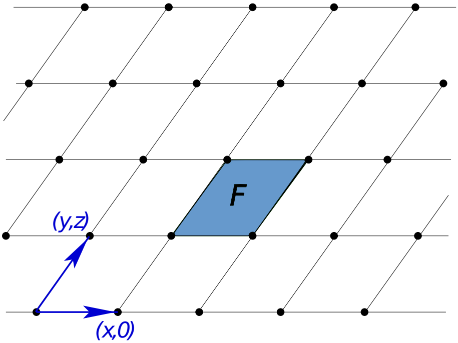
Point lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

Fundamental cell of Γ: $\overline{\mathbb{R}^d/\Gamma}$.



Point lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

Fundamental cell of Γ: $\overline{\mathbb{R}^d/\Gamma}$.



Point group $P(\Gamma)$ of Γ: All $g \in O(d,\mathbb{R})$ with $g\Gamma = \Gamma$.

A point group of a lattice is finite. Its elements are

- rotations and reflections ($d = 2$)
- rotations, reflections and rotoreflections ($d = 3$)

Crystallographic point group: A subgroup of a lattice point group. In other words: a subgroup of $O(n, \mathbb{R})$ fixing some lattice.

How many lattice point groups are there?

A point group of a lattice is finite. Its elements are

- rotations and reflections ($d = 2$)
- rotations, reflections and rotoreflections ($d = 3$)

Crystallographic point group: A subgroup of a lattice point group. In other words: a subgroup of $O(n, \mathbb{R})$ fixing some lattice.

How many lattice point groups are there?

Crystallographic restriction: Rotational symmetries of 2-dim and 3-dim lattices are either 2-fold, 3-fold, 4-fold, or 6-fold.

$d = 2$: 10 candidates: $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_6$

$d = 3$: 32 candidates.

$d = 2$: 10 candidates: $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_6$

$d = 3$: 32 candidates.

Only 4 lattice point groups* in $\mathbb{R}^2$:

$$\mathcal{C}_2, \mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6 \qquad (2, *2, *4, *6 \text{ in orbifold notation})$$

Only 7 lattice point groups* in $\mathbb{R}^3$:

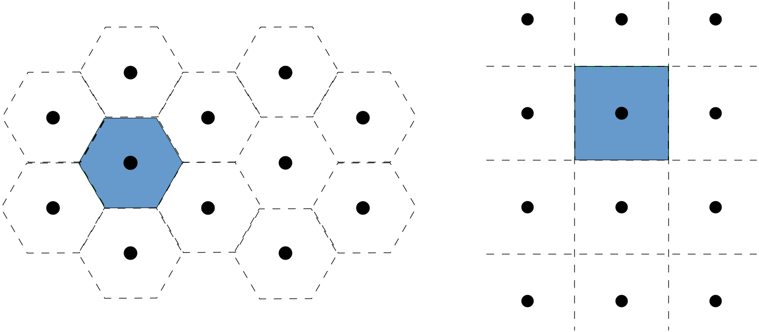
$$\mathcal{C}_2, \mathcal{D}_2, \mathcal{D}_2 \times \mathcal{C}_2, \mathcal{D}_3 \times \mathcal{C}_2, \mathcal{D}_4 \times \mathcal{C}_2, \mathcal{D}_6 \times \mathcal{C}_2, \text{cube group}$$

$$(2, *2, *222, 2*3, *422, *622, *432 \text{ in orbifold notation})$$

(*: since, for instance, $x \mapsto -x$ is symmetry of any lattice)

Trivially, each lattice Γ has a fundamental cell whose symmetry group is $P(\Gamma)$.

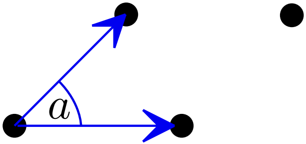
For instance, take the Voronoi cell of a lattice point x. (That is the set of points closer to x than to each other lattice point.)



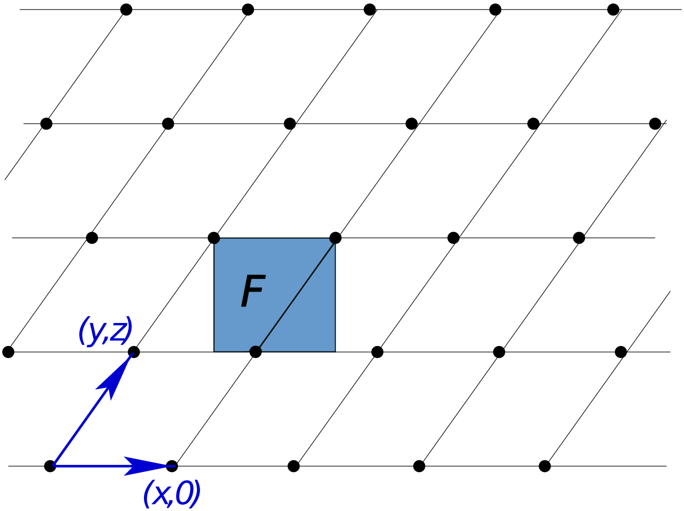
Theorem (Elser, F)

Let $\Gamma \subset \mathbb{R}^2$ be a lattice, but not a rhombic lattice. Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $\quad [S(F) : P(\Gamma)] = 2.$

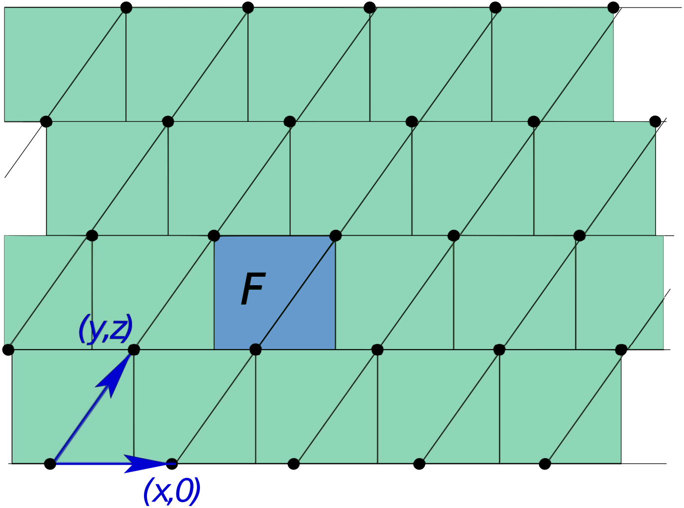
'Rhombic lattice' means: one with basis vectors of equal length, but neither a square lattice nor a hexagonal lattice.



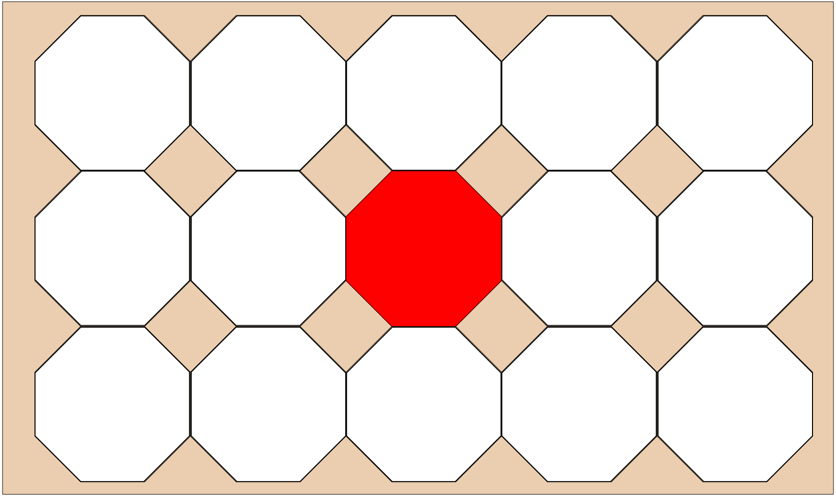
Proof: Case 1: Generic lattice ($\mathcal{C}_2$):

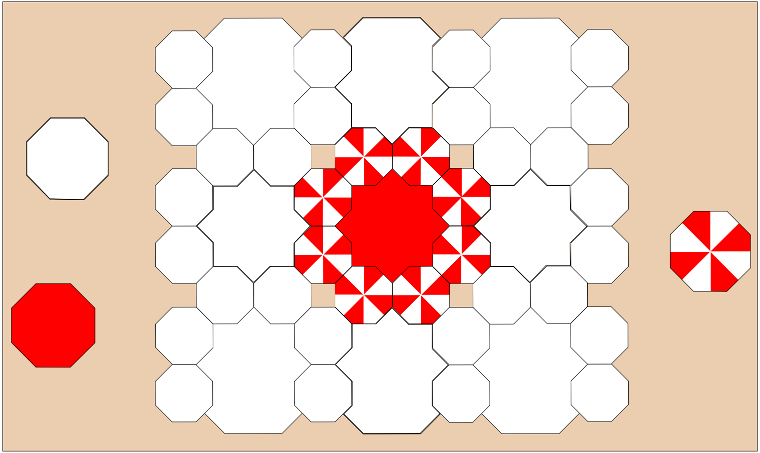


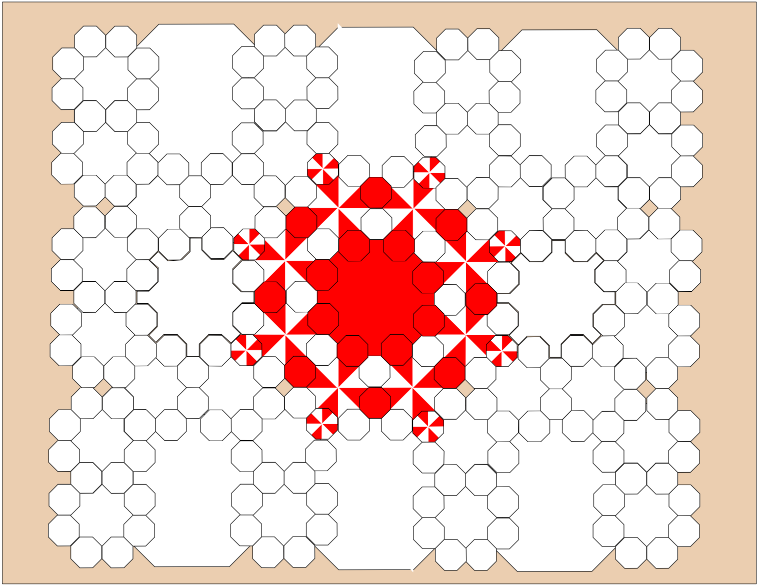
Case 1: Generic lattice:

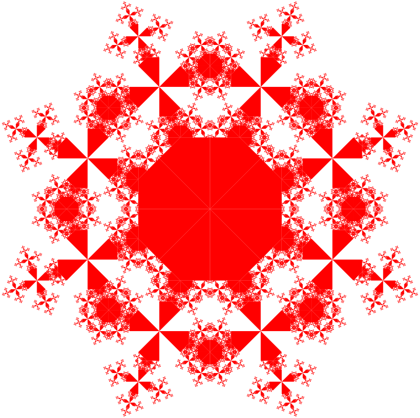


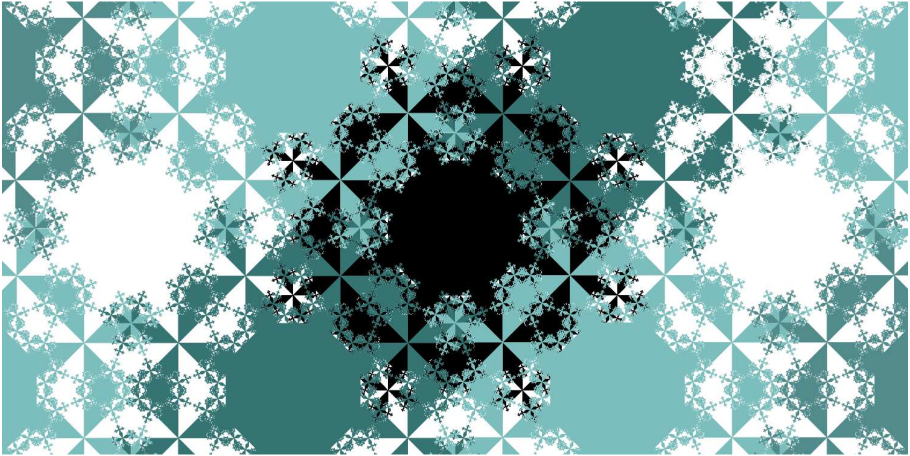
Case 2: Square lattice ($\mathcal{D}_4$) (V. Elser):



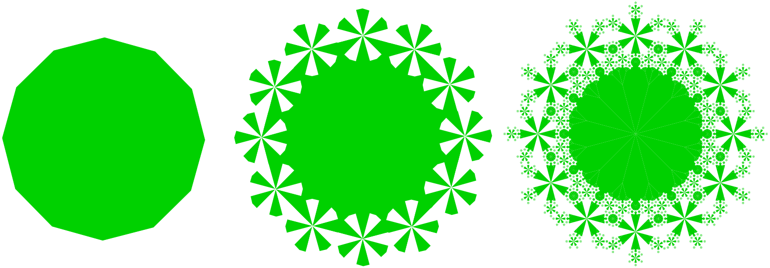




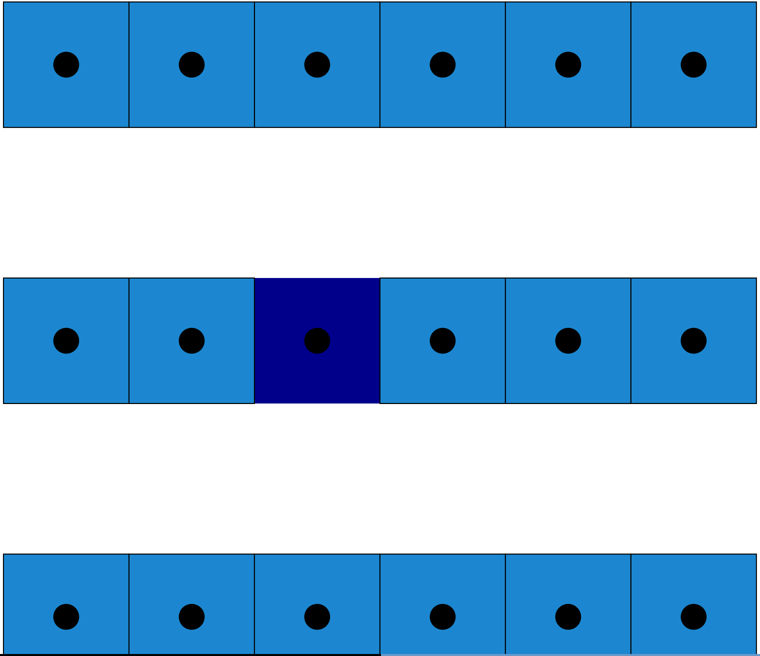


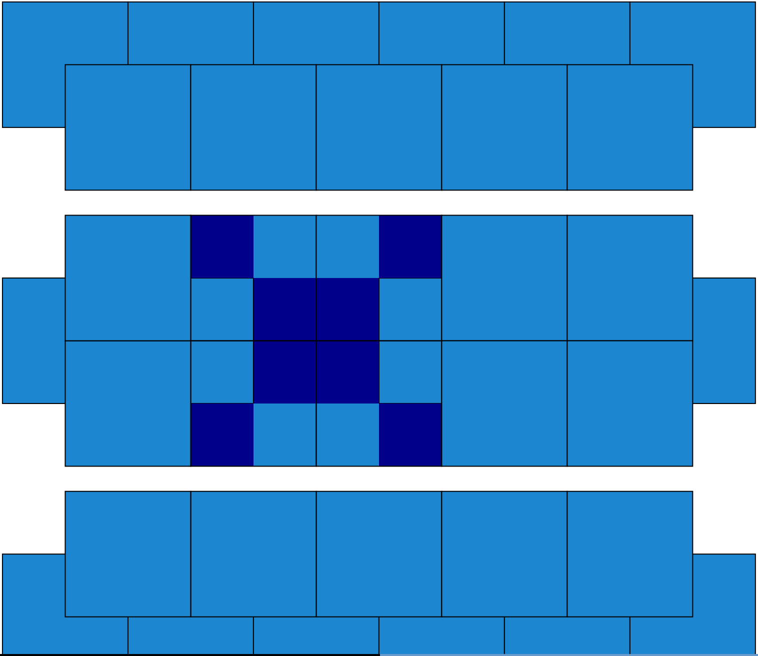


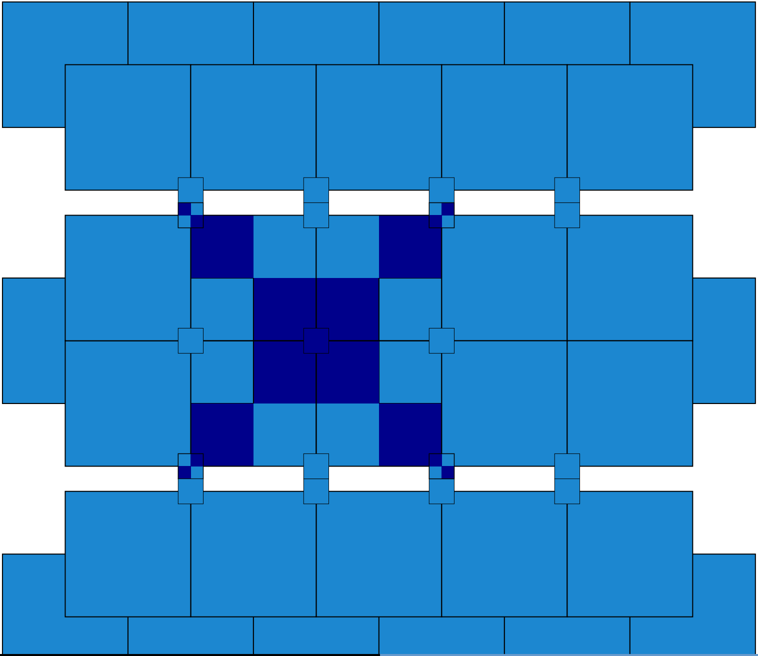
Case 3: Hexagonal lattice ($\mathcal{D}_6$)
(Elser-Cockayne, Baake-Klitzing-Schlottmann):

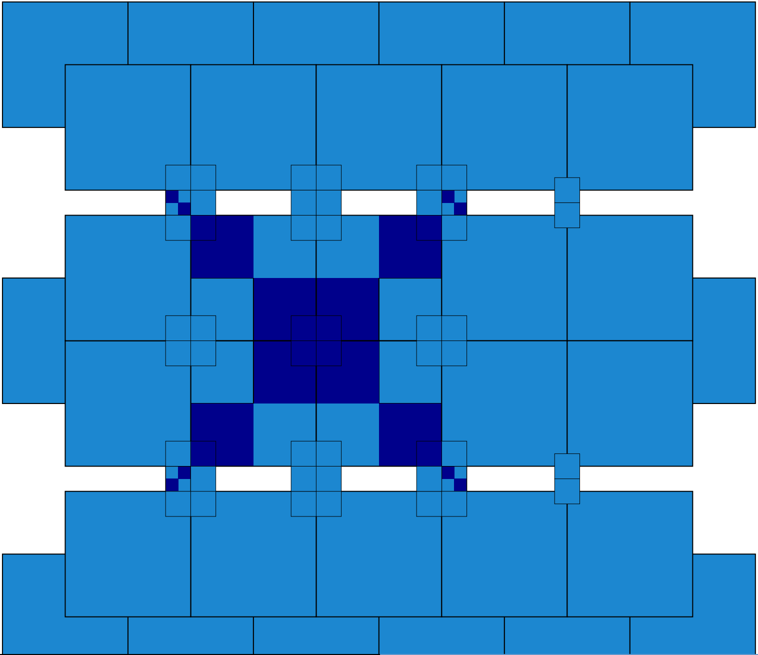


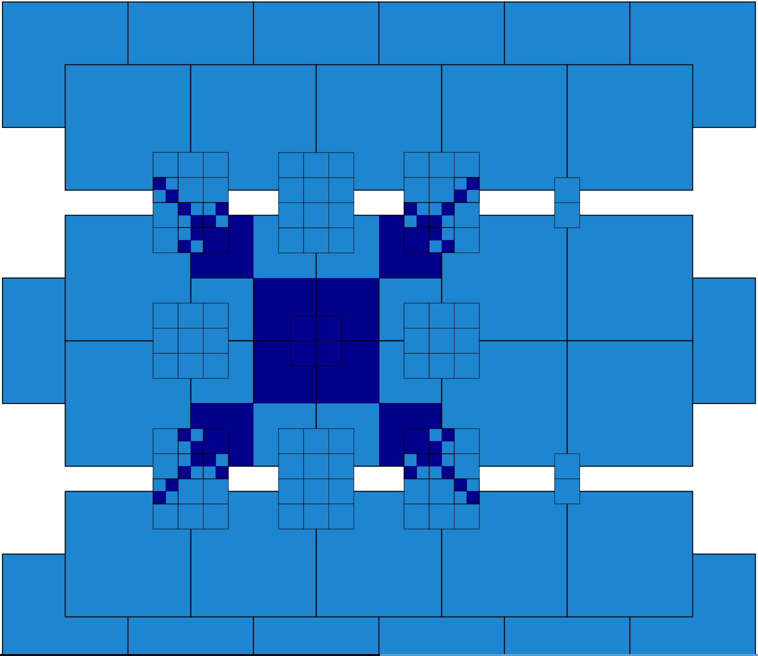
Case 4: Rectangular lattice

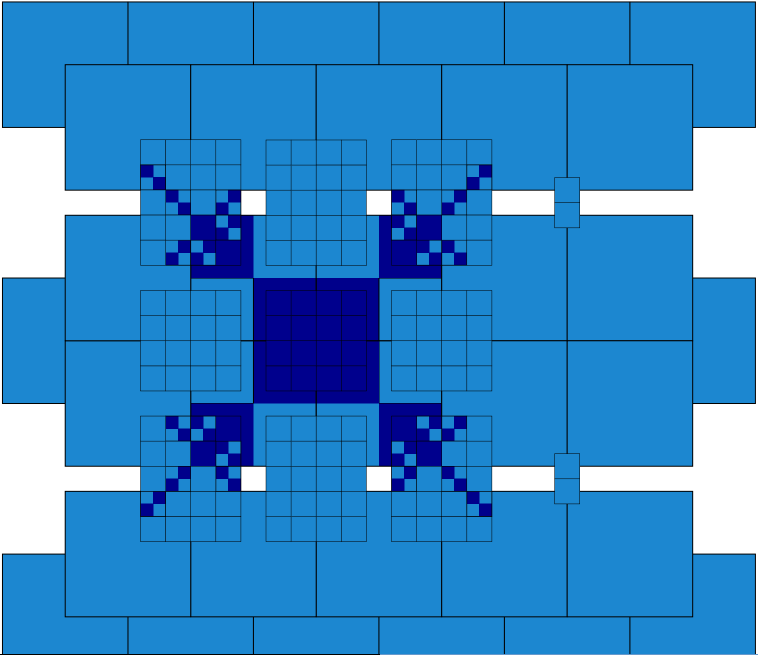


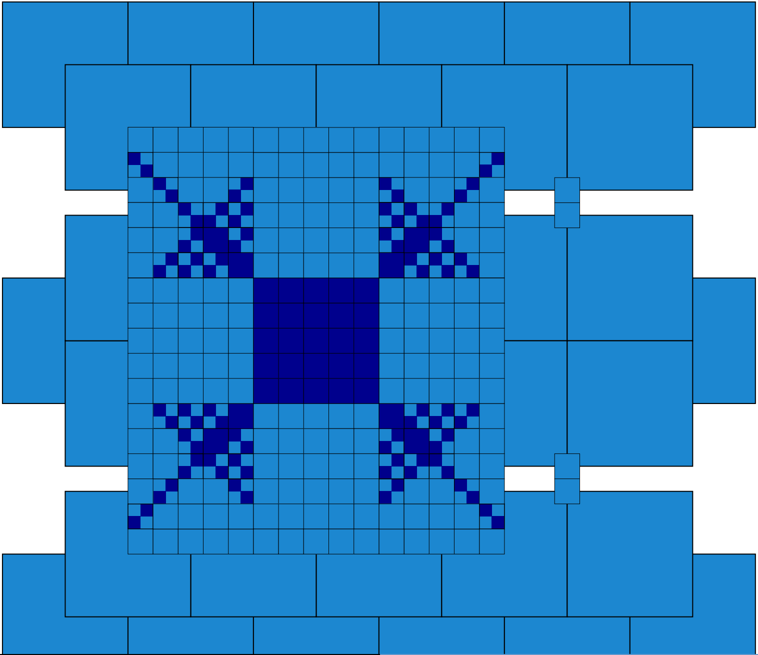


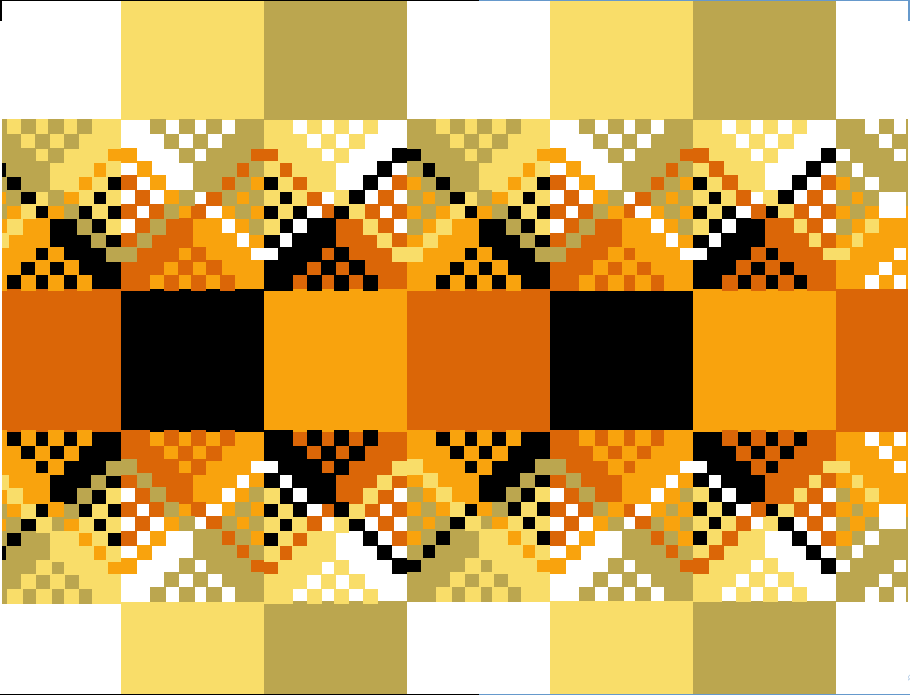




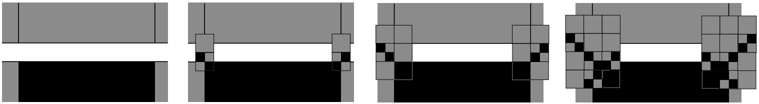








Euclidean algorithm at work:



Edge length of the rectangular gap: a, b with $a > b$.

$$a,\ a-b,\ a-2b,\ a-3b, \ldots, a - \left\lfloor \frac{a}{b} \right\rfloor b$$

Leaves a gap with edge length b, $c := a - \left\lfloor \frac{a}{b} \right\rfloor b$.

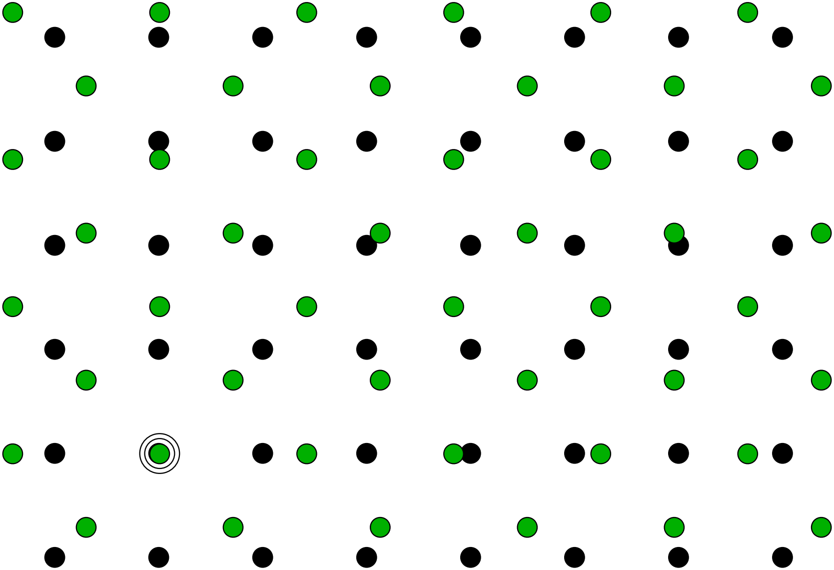
Continue.

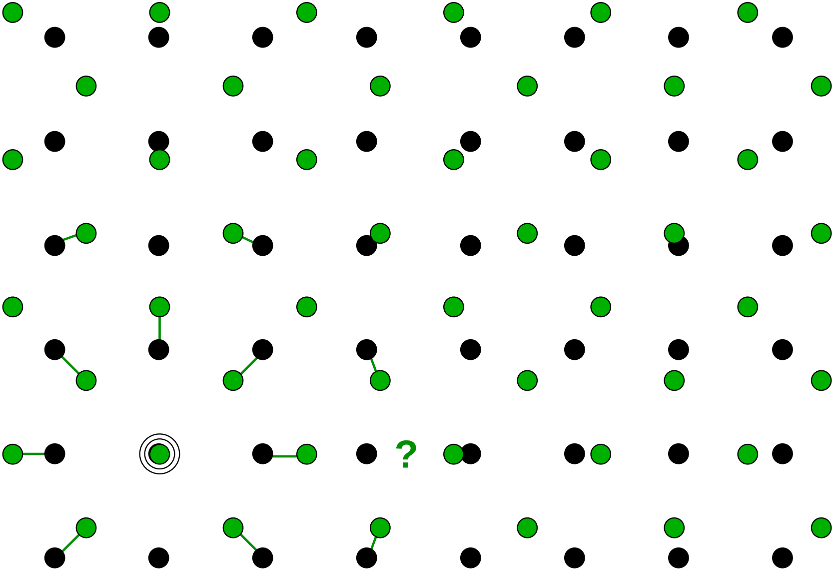
Application: Short perfect matchings

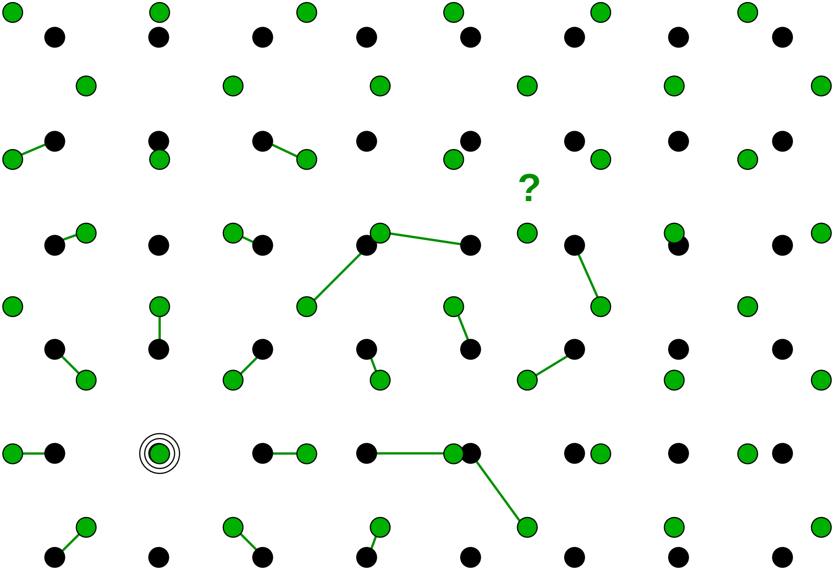
Consider the square lattice $\mathbb{Z}^2$, and $R_{45}\mathbb{Z}^2$, the square lattice rotated by 45°.

Problem: Find a perfect matching between $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$ with maximal distance not larger than $C > 0$. How small can C be?

(It is easy to see that $C \geq \frac{\sqrt{2}}{2} = 0.7071...$.)







Naively: difficult.

Using the 8-fold fundamental cell F yields a matching with $C = 0.92387...$.

How?

Naively: difficult.

Using the 8-fold fundamental cell F yields a matching with $C = 0.92387\ldots$.

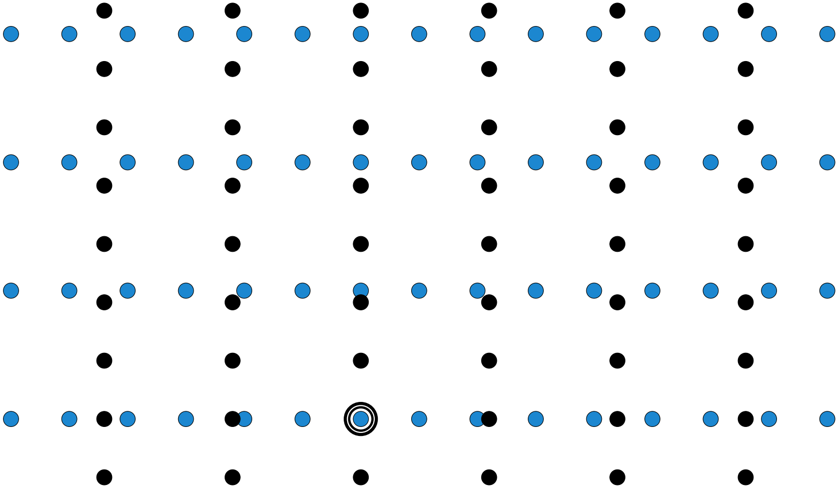
How?

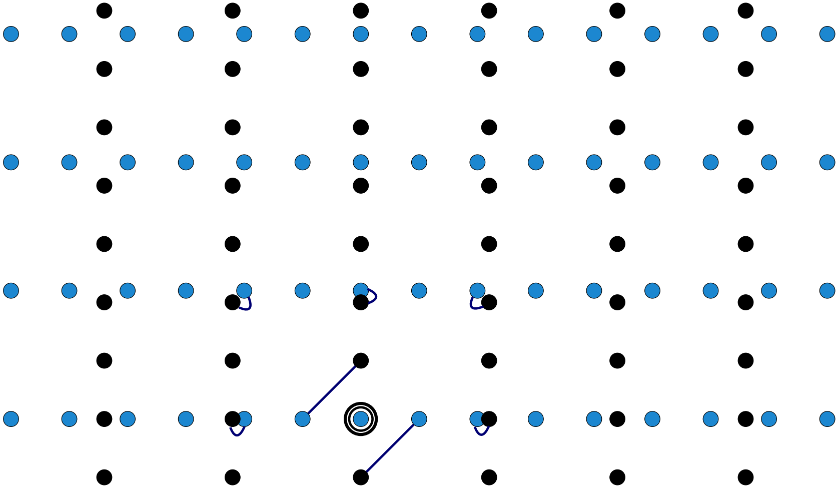
- Consider $\mathbb{Z}^2 + F$. Each $x + F$ ($x \in \mathbb{Z}^2$) contains exactly one point of $\mathbb{Z}^2$ in its centre.
- F is also fundamental cell for $R_{45}\mathbb{Z}^2$. Thus each $x + F$ ($x \in \mathbb{Z}^2$) contains exactly one point $x' \in R_{45}\mathbb{Z}^2$.
- Match x and x'.

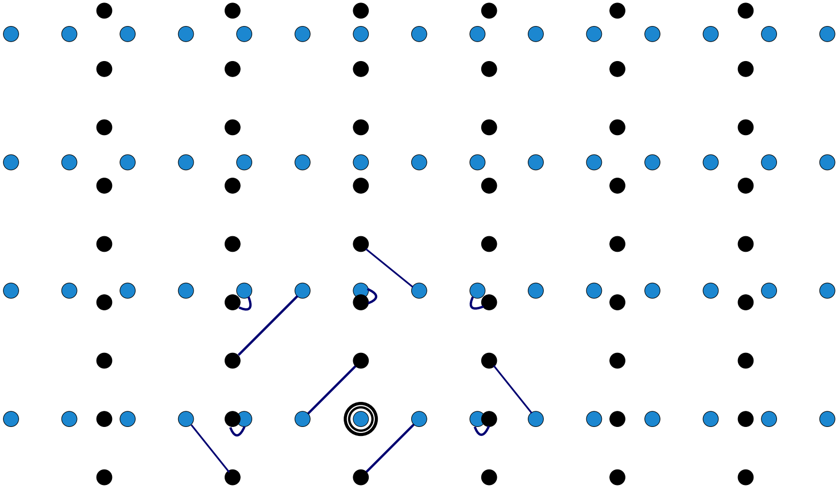
This (and its analogues) yields good matchings for

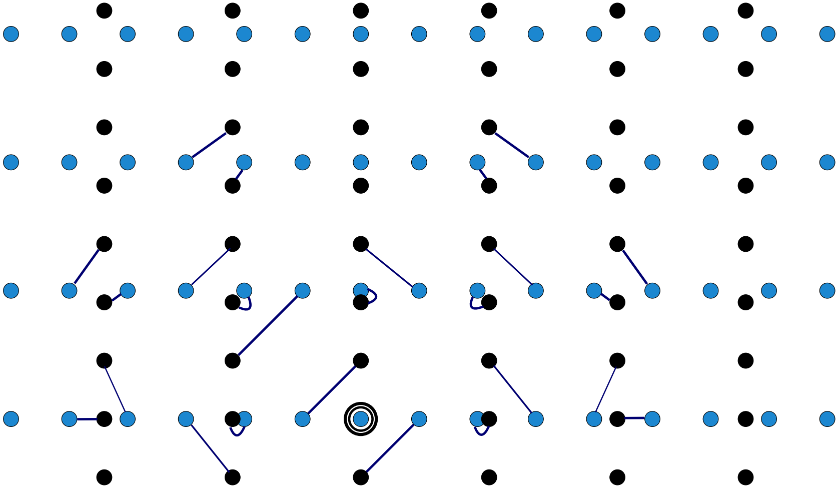
- $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$: $C = 0.92387\ldots$.
- The hexagonal lattice H and $R_{30}H$: $C = 0.78867\ldots$
- A rectangular lattice P and $R_{90}P$: $C \le \frac{1}{\sqrt{2}}\frac{\sqrt{5}+1}{2}b$.

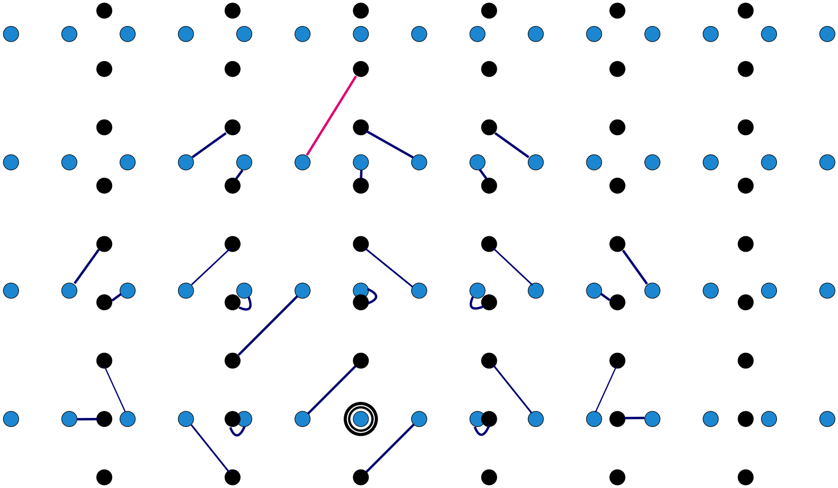
(b is the length of the longer lattice basis vector of P.)











Todo:

- Rhombic lattices
- Higher dimensions
- Hyperbolic spaces
- Dimension of the boundaries
- Connectivity
- Better matchings
- ...

New Results

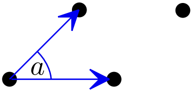
Theorem (F)

Let $\Gamma \subset \mathbb{R}^3$ be a lattice, but not a cubic lattice. Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $[S(F) : P(\Gamma)] = 2$.

"Cubic": One of $\mathbb{Z}^3$, $\mathbb{Z}^3 \cup \left(\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\right)$ ("*bcc*"), A_3 ("*fcc*").

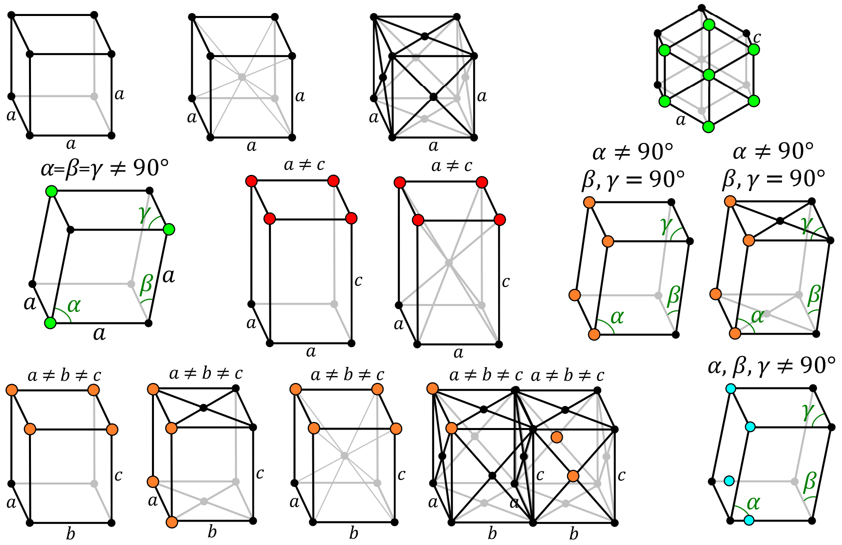
Theorem (hm, maybe)

Let $\Gamma \subset \mathbb{R}^2$ be a rhombic lattice, such that $\tan(\frac{\alpha}{2}) = \frac{p}{q} \in \mathbb{Q}$, and either p or q even (and furthermore ...).
Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $[S(F) : P(\Gamma)] = 2$.

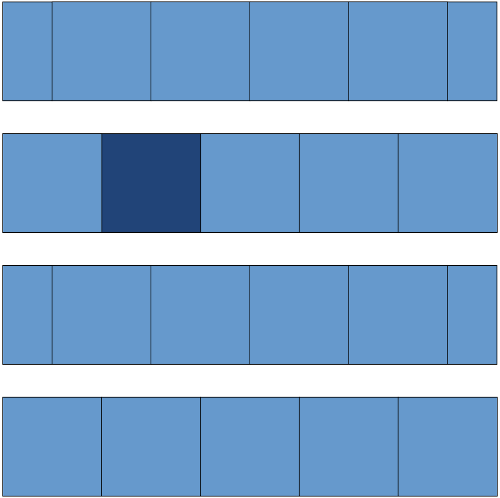


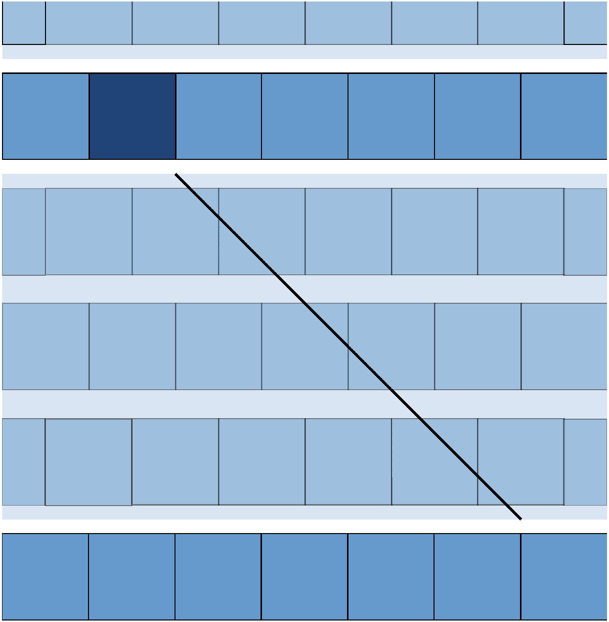
Proof for $\mathbb{R}^3$: Consider the 14 cases:

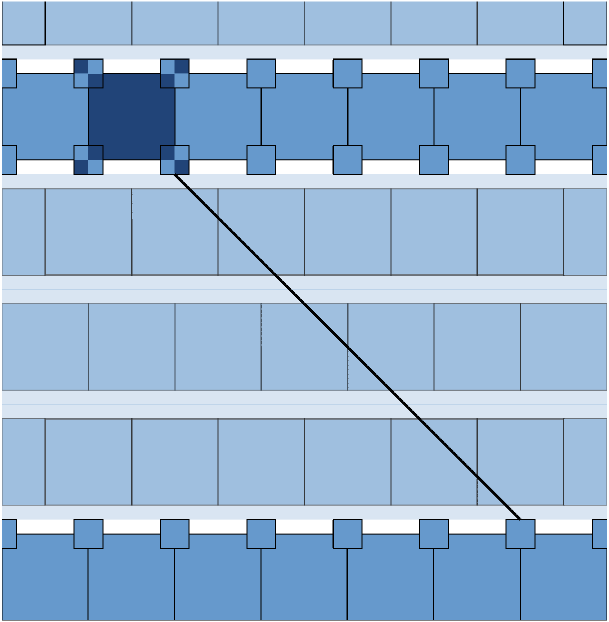
Nr	Name	Point group	Order	2dim FC (# sym.)
1	$\mathbb{Z}^3$	$*432$	48	—
2	bcc	$*432$	48	—
3	fcc	$*432$	48	—
4	Hexagonal	$*622$	24	12fold (48)
5	Tetragonal prim.	$*422$	16	8fold (32)
6	Tetragonal body-c.	$*422$	16	8fold (32)
7	Rhombohedral	$2*3$	12	6fold (24) / 12fold(48)
8	Orthorhombic prim.	$*222$	8	4fold (16)
9	Orthorhombic base-c.	$*222$	8	4fold (16)
10	Orthorhombic body-c.	$*222$	8	4fold (16)
11	Orthorhombic face-c.	$*222$	8	4fold (16)
12	Monoclinic prim.	$2*$	4	2fold (8)/4fold(16)
13	Monoclinic base-c.	$2*$	4	2fold (8)/4fold(16)
14	Triclinic prim.	2	2	[monocl.(4)] / 2fold (8)

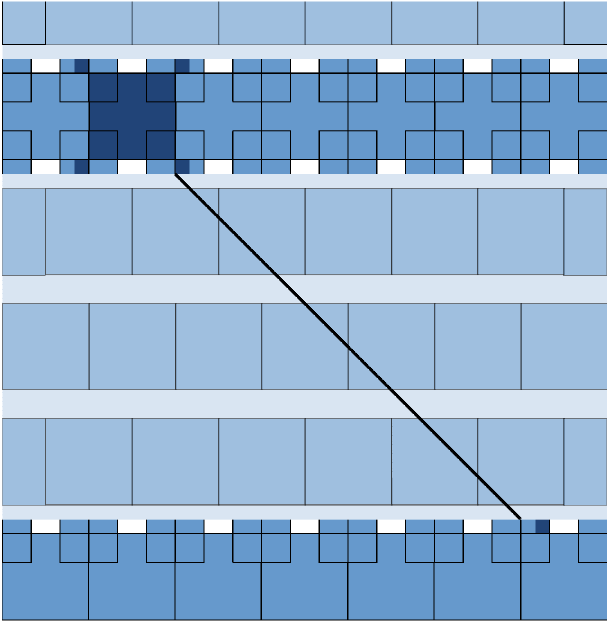

a
a
a
c
α=β=γ ≠ 90°
a ≠ c
a ≠ c
α ≠ 90°
β, γ = 90°
α ≠ 90°
β, γ = 90°
α
β
γ
a ≠ b ≠ c
a ≠ b ≠ c
a ≠ b ≠ c
a ≠ b ≠ c
a ≠ b ≠ c
α, β, γ ≠ 90°
b
c

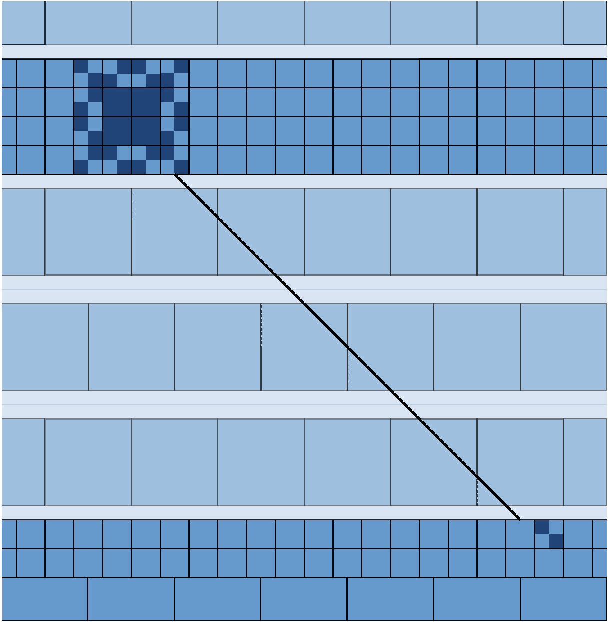
Idea for rhombic lattices in $\mathbb{R}^2$:












Thank you.