Substitution tilings with dense tile orientations

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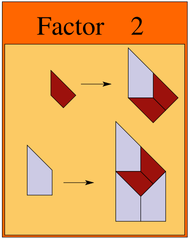
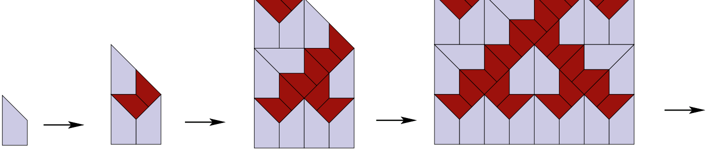
1. Substitution tilings with tiles in infinitely many orientations
2. Dense tile orientations (DTO)
3. Tilings with rotational symmetry and DTO

Substitution tiling with *substitution factor* 2:





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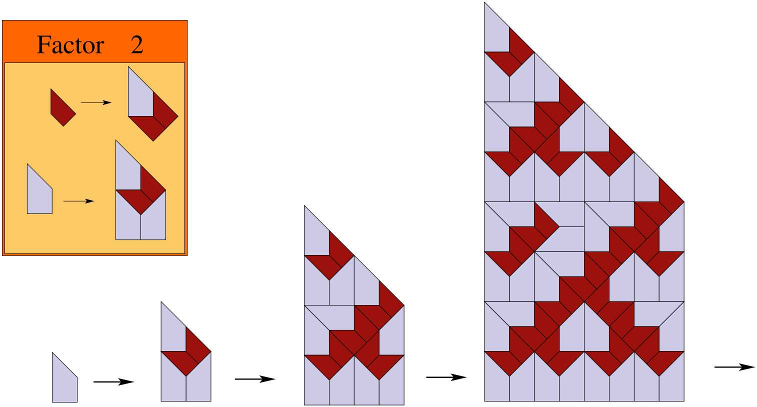
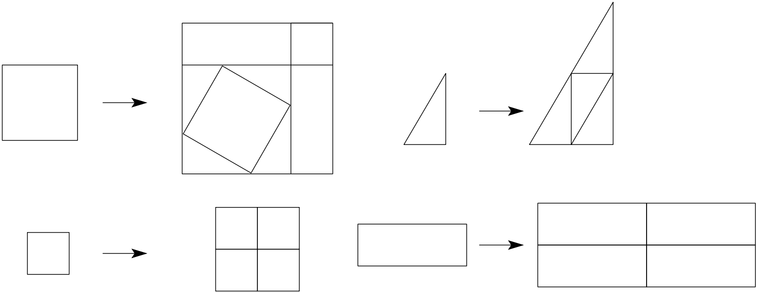


Substitution matrix here $M = \left(\begin{smallmatrix} 2 & 2 \\ 1 & 3 \end{smallmatrix}\right)$.

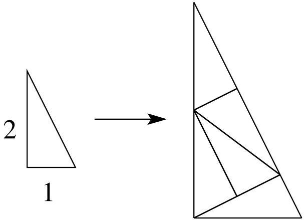
Fact: if λ is the substitution factor, then λ^2 is the largest eigenvalue of the substitution matrix.

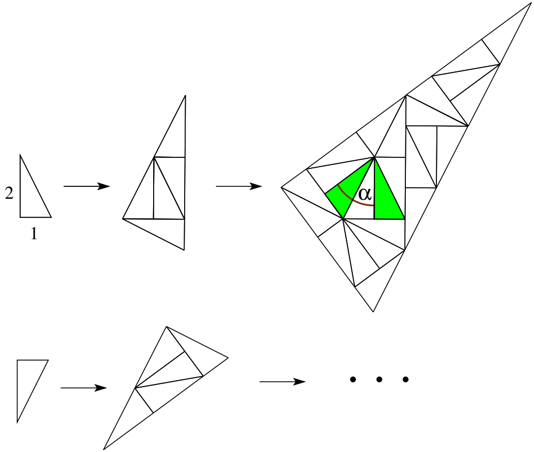
Usually, tiles occur in finitely many different orientations only. Not always. Cesi's example (1990):



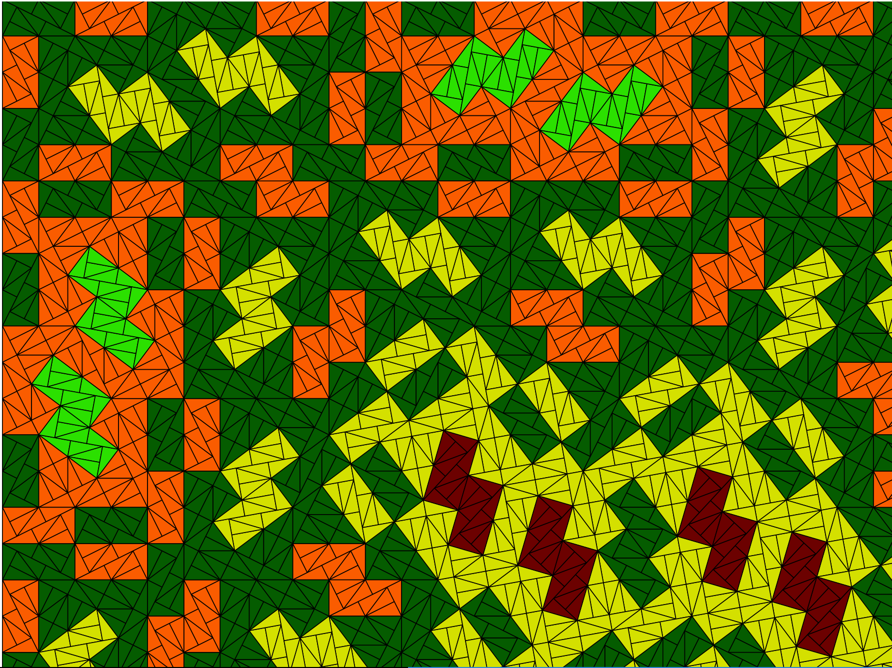
A substitution σ is *primitive*, if for any tile T there is $k \geq 1$ such that $\sigma^k(T)$ contains all tile types.

Conway's Pinwheel substitution (1991):

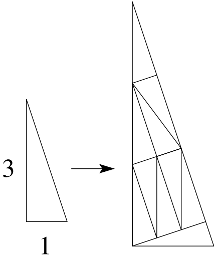




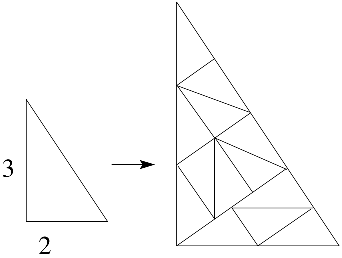
The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



Obvious generalizations: Pinwheel (n, k)



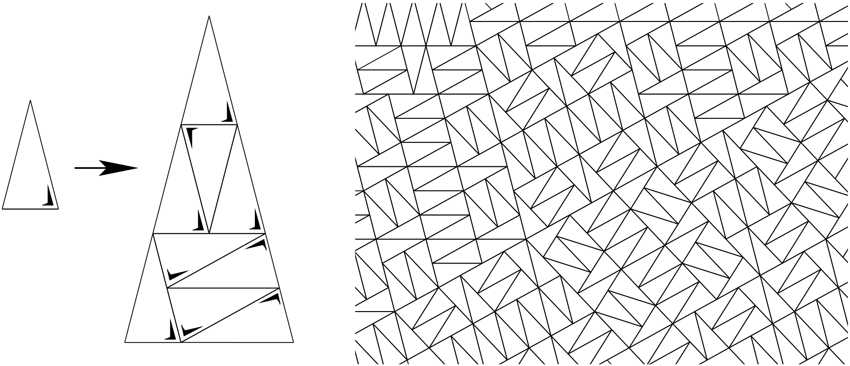
$n = 3, k = 1$



$n = 3, k = 2$

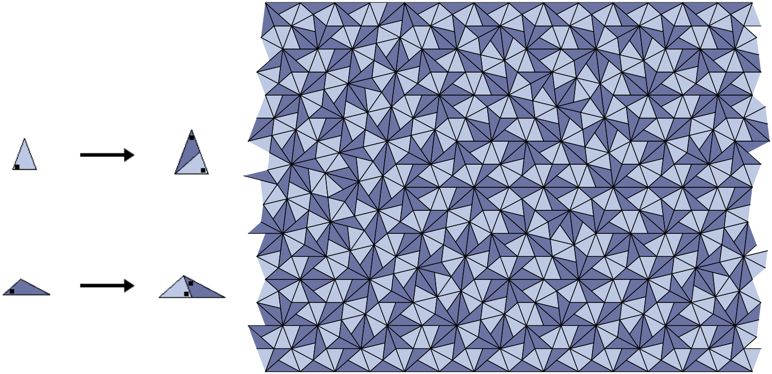
etc.

Unknown (< 1996, communicated to me by Danzer):

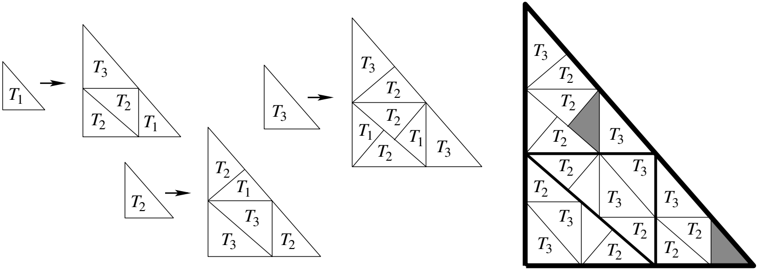


(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):



Pythia (m, j), here: $m = 3$, $j = 1$.



Dense Tile Orientations (DTO)

For all examples: the orientations are dense in $[0, 2\pi[$.

Even more: The orientations are equidistributed in $[0, 2\pi[$.

Theorem (F. '08)

In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in $[0, 2\pi[$.

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In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in $[0, 2\pi[$.

Recall: $(\alpha_j)_j$ is *equidistributed* in $[0, 1[$, if for all $0 \le a < b < 1$ holds:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[a,b]}(\alpha_j) = b - a$$

Here: in a tiling $\mathcal{T} = T_1, T_2, \ldots$ the orientations of the tiles are equidistributed, if for all $0 \le a < b < 2\pi$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[a,b]}(\alpha(T_j)) = \frac{b-a}{2\pi}$$

where $\alpha(T_j)$ is the angle of tile T_j (wrt some fixed copy of T_j).

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Because the sum is not absolutely convergent, the order matters!

Here it is OK to order the tiles wrt distance from 0.

Proof needs:
Weyl's criterion: (a_n) equidistributed mod 1 iff

$$\forall\, \ell \in \mathbb{Z} \setminus \{0\} : \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i \ell a_j} = 0.$$

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Perron's Theorem: $M \in \mathbb{R}^{n \times n} \geq 0$ (i.e., non-negative entries only) and $M^k > 0$ for some k, then

- There is a biggest eigenvalue $\mu \in \mathbb{R}$ with $\mu > 0$
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- There is a biggest eigenvalue $\mu \in \mathbb{R}$ with $\mu > 0$
- μ has a positive eigenvector v
- $\lim\limits_{n\to\infty} \frac{1}{\mu^n} M^n$ exists, the columns are multiples of v
- If $0 \leq A \leq M$, $A \neq M$, then the biggest eigenvalue of A is less than μ.

Sketch of proof: Let M be the substitution matrix, with biggest eigenvalue μ.

$$\text{Let } A(\ell) = \Big(\sum_{j=1}^{M_{km}} e^{i\alpha(T_j)\ell} \Big)_{km} \qquad (\ell \in \mathbb{Z})$$

be the matrix containing the orientations $\alpha(T_j)$ times ℓ.
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By irrationality of the angles

$$|A(\ell)|^n \leq M^n \text{ and } |A(\ell)|^n \neq M^n \qquad \text{(from some } n \text{ on)}$$

We need to show:

$$\lim_{n\to\infty} \frac{(A(\ell)^n)_{km}}{(M^n)_{km}} = 0$$

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Corollary

In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are dense in $[0, 2\pi[$.

So far: tiles are always triangles. No wonder:

Theorem (F.-Harriss, 2013)

Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many prototiles (i.e., finitely many different tile shapes). Let all prototiles be centrally symmetric convex polygons. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

Some people (e.g. Lorenzo Sadun, UT Austin) compute cohomologies of tiling spaces (...which means: consider the set of all tilings to a given substitution. Define when two tilings are “close”. This yields a topological object which has cohomologies)

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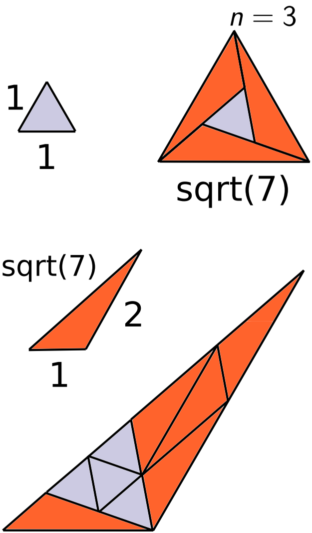
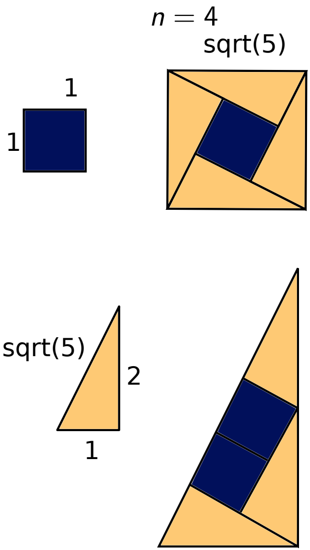
Question: Are there tilings with DTO and n-fold rotational symmetry for $n \geq 3$?

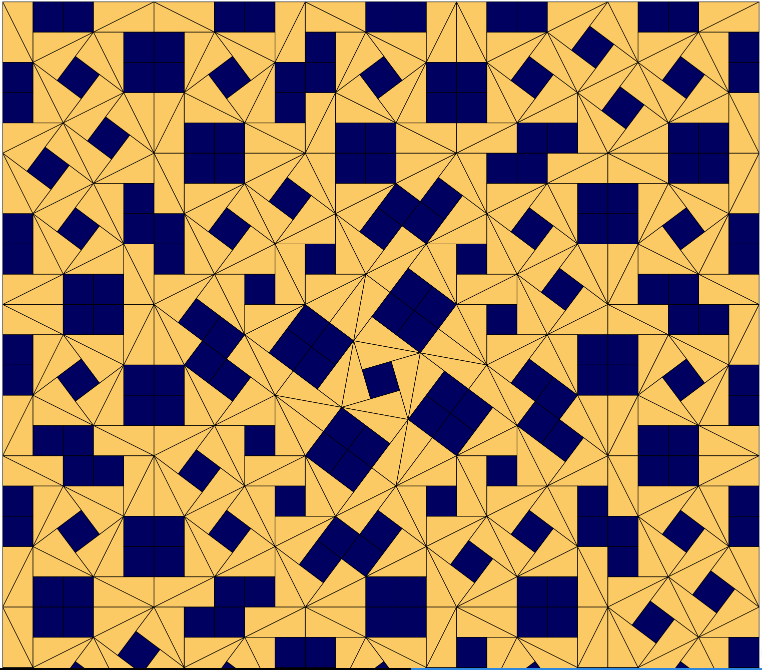
Answer: Yes. At least for $n \in \{3, 4, 5, 6, 7, 8\}$.

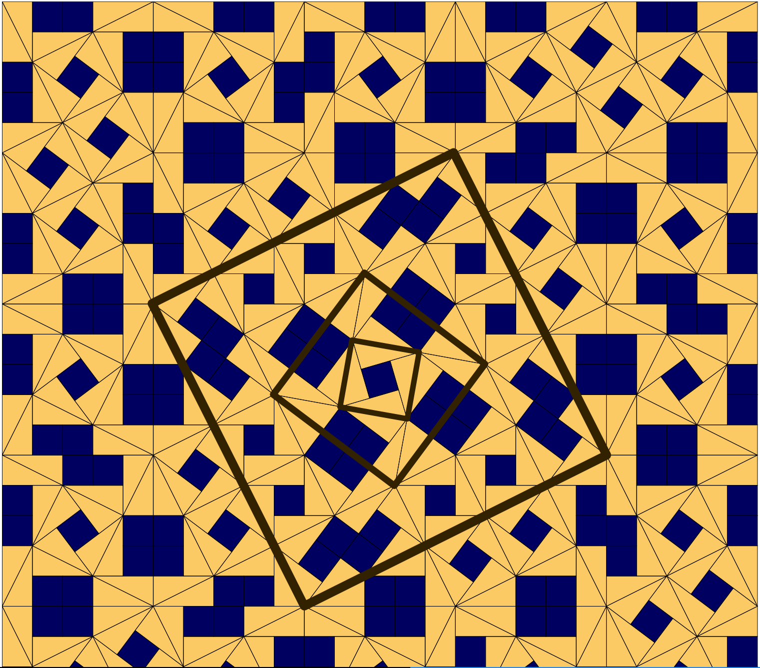


n = 3
1
1
sqrt(7)
sqrt(7)
2
1

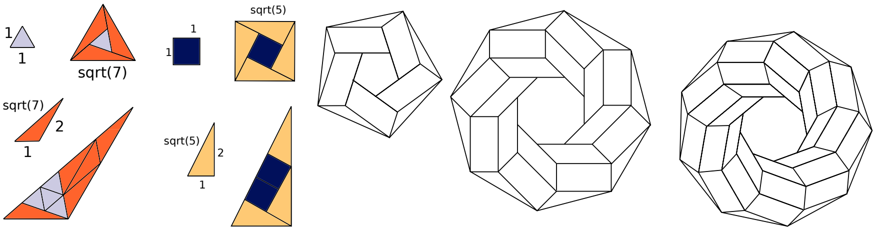


n = 4
sqrt(5)
1
1
sqrt(5)
2
1

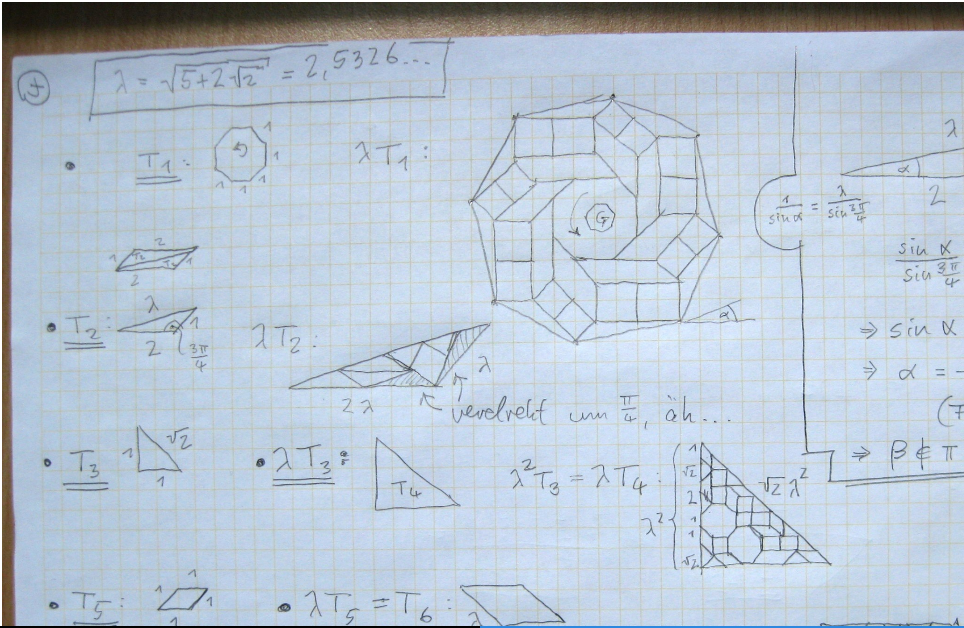




Considering the analogues for larger n



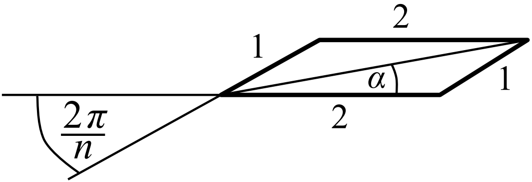
E.g.


λ = √(5+2√2) = 2,5326...
T1 :
λT1 :
T2 :
λT2 :
verdreht um π/4, äh...
T3
λT3 :
λ²T3 = λT4 :
T5 :
λT5 = T6 :
sin α
sin 3π/4
⇒ sin α
⇒ α =
⇒ β ∉ π

...we found (rediscovered?):

Theorem (F.-Say-Awen-de las Peñas 2017)

In a parallelogram with edge lengths 1 and 2, and interior angle β:
If $\beta = \frac{2\pi}{n}$ $(n \geq 4)$ then $\alpha \notin \pi\mathbb{Q}$.



Embed the parallelogram in the complex plane:

- lower left vertex: 0
- upper left vertex: $\xi_n := e^{2\pi i/n}$
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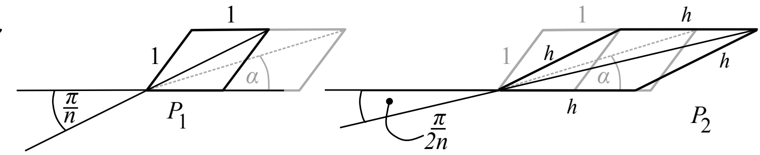
i.e., $\frac{z}{\overline{z}}$ is a complex mth root of unity.

Clearly, $\frac{z}{\overline{z}} \in \mathbb{Q}(\xi_n)$.

Theorem: All roots of unity in $\mathbb{Q}(\xi_n)$ are of the form $\pm\xi_n^k$.

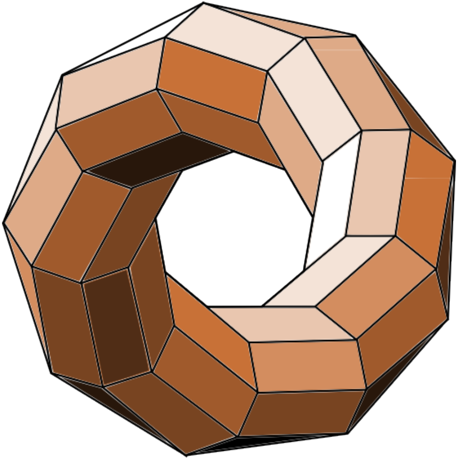
Hence $m = n$, or $m = 2n$ (if m is even and n is odd)



$m = n$: $\alpha < \frac{\pi}{n}$ (too small!),

$m = 2n$: $\frac{\pi}{2n} < \alpha < \frac{\pi}{n}$

Hence $\alpha \notin \pi\mathbb{Q}$.



Thank you!