On colour symmetries of hyperbolic regular tilings and cyclotomic integers

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$S(X)$: full *symmetry group* of some pattern X (including reflections).

$R(X)$: proper *symmetry group* of some pattern X (without reflections).

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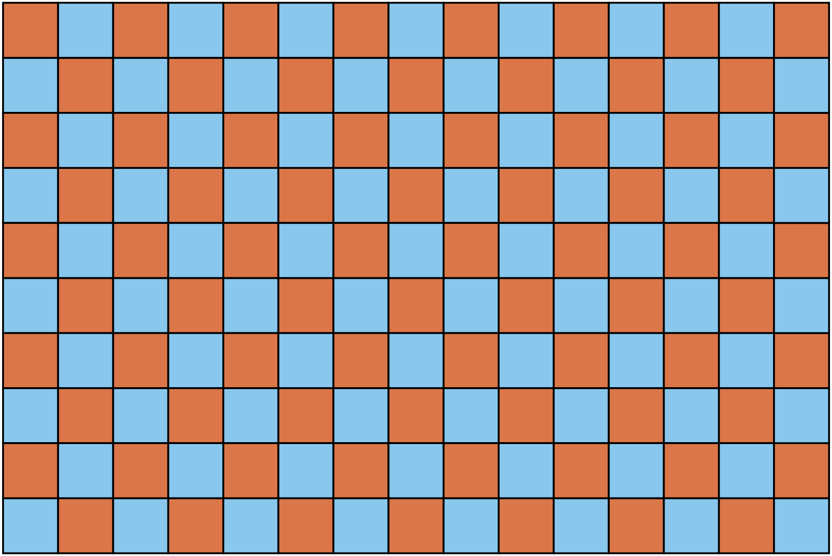
$R(X)$: proper *symmetry group* of some pattern X (without reflections).

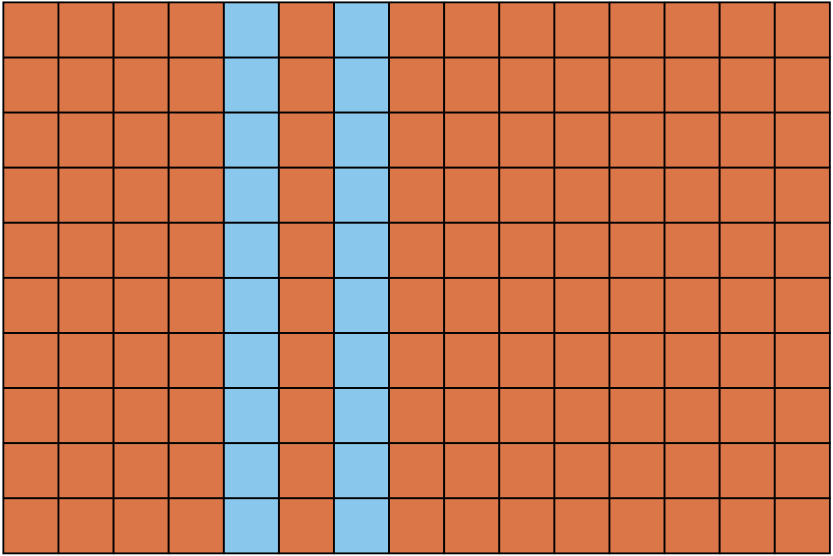
Perfect colouring: Colouring of some pattern X, where each $f \in S(X)$ acts as a global permutation of colours.

Chirally perfect: dito for $R(X)$.

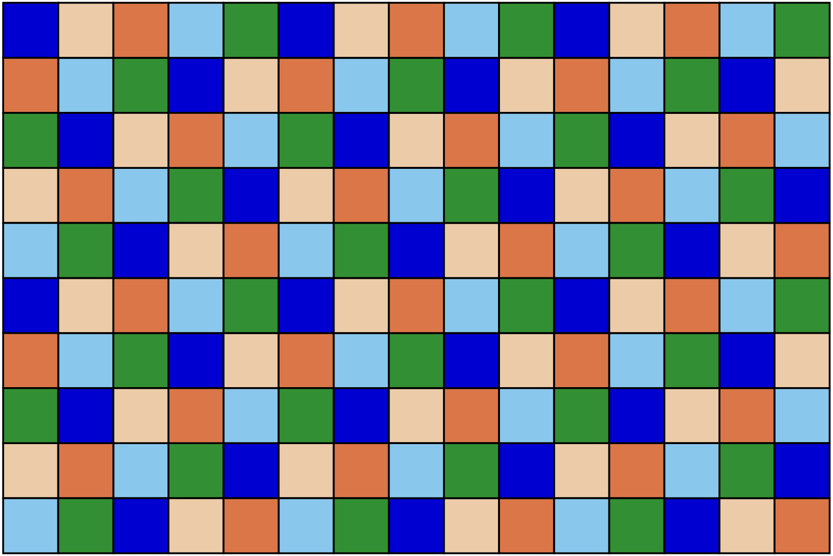
Perfect colouring of (4^4) with two colours:



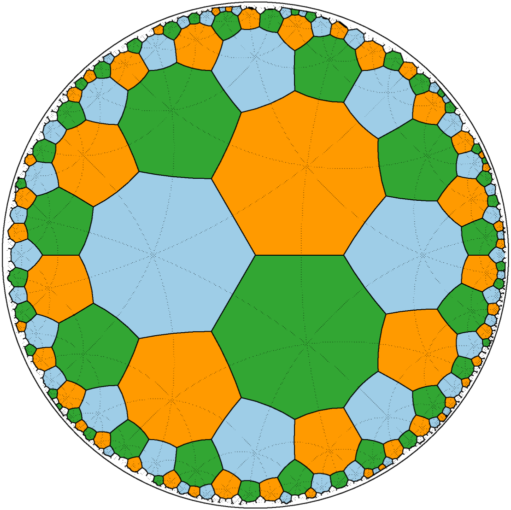
Not a perfect colouring of (4^4):



Chirally perfect colouring of (4^4) with five colours:



Perfect colouring of (8^3) with three colours:



Questions: Given some nice pattern X,

1. for which number of colours does there exist a perfect colouring?
2. how many for a certain number of colours?
3. what is the algebraic structure of the colour symmetry group?

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Well known for lattices, regular tilings etc. in $\mathbb{R}^2, \mathbb{R}^3, \ldots$

(Belov & Shubnikov, ... van der Waerden, Schwarzenberger, ... Grünbaum & Shephard, Conway)

Few is known for regular tilings in $\mathbb{H}^2$

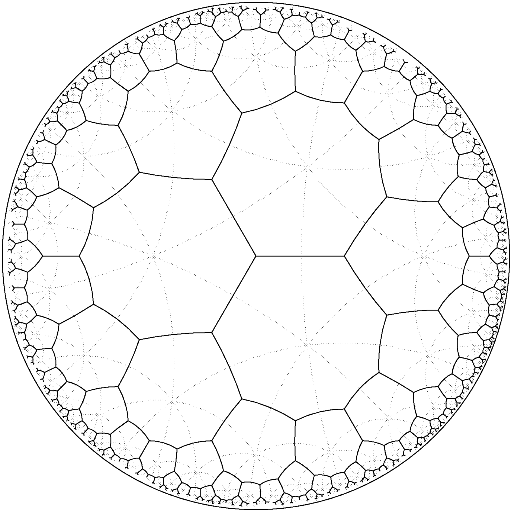
Regular tiling (p^q): edge-to-edge tiling by regular p-gons, where q tiles meet at each vertex.

In $\mathbb{R}^2$: three regular tilings: (4^4), (3^6), (6^3).

In $\mathbb{S}^2$: five regular tilings: (3^3), (4^3), (3^4), (5^3), (3^5).

In $\mathbb{H}^2$: Infinitely many regular tilings: (p^q), where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.

Regular hyperbolic tiling (8^3):



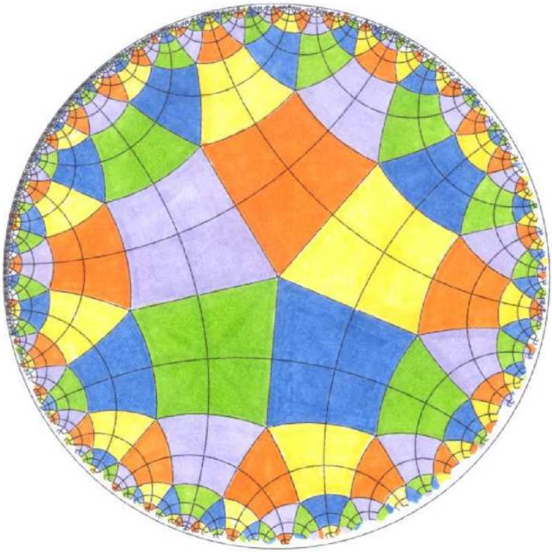
Perfect colourings (F. 2008):

(4^4)	$1, 2, 4, 8, 9, 16, 18, 25, 32, 36, \dots$
(3^6)	$1, 2, 4, 6, 8, 16, 18, 24, 25, 32, \dots$
(6^3)	$1, 3, 4, 9, 12, 16, 25, 27, 36, \dots$
(7^3)	$1, 8, 15, 22, 24, 30, 36^2, 44, 50^5, \dots$
(3^7)	$1, 22, 28^5, 37, 42^4, 44, 49^7, 50^3, \dots$
(8^3)	$1, 3, 6, 12, 17, 21^4, 24, 25^5, 27^3, 29^4, 31^4, 33^6, 37^6, 39^8, \dots$
(3^8)	$1, 2, 4, 8, 10^2, 12, 14, 16^2, 18, 20^4, 24^3, 25^5, 26, 28^{12}, 29, 30^2, \dots$
(5^4)	$1, 2, 6, 11, 12, 16^2, 21^3, 22^5, 24, 26^9, 28, \dots$
(4^5)	$1, 5^2, 10^4, 11, 15^7, 16, 20^9, 21^3, 22, 25^{27}, 26, 27^3, 30^{38}, \dots$
(6^4)	$1, 2, 4, 6, 8, 10^2, 12^7, 13^4, 14, 15^2, 16^{13}, 18^{13}, 19^{10}, 20^{23}, 21^{10} \dots$
(4^6)	$1, 2, 3, 5, 6^3, 9^4, 10^1, 11^2, 12^7, 13^5, 14^2, 15^{16}, 16^2, 17^9, 18^{26}, \dots$

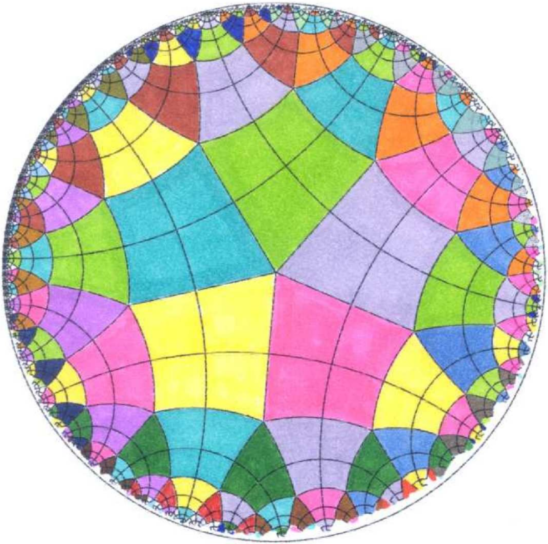
Chirally perfect colourings (F. 2008):

(4^4)	$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25^2, 26, 29, 32, \ldots$
(3^6)	$1, 2, 4, 6, 7, 8, 13, 14, 16, 18, 19, 24, 25, 26, 28, 31, \ldots$
(6^3)	$1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, \ldots$
(7^3)	$1, 8, 9, 15^2, 22^7, 24, \ldots$
(3^7)	$1, 7, 8, 14^6, 21^2, 22^7, \ldots$
(8^3)	$1, 3, 6, 9, 10, 12, 13^2, 15, 17^5, 18^5, 19^5, \ldots$
(3^8)	$1, 2, 4, 8^4, 10^3, 12, 13^2, 14^2, 16^{12}, 17^5, 18, 19^5, \ldots$
(5^4)	$1, 2, 6^2, 11^3, 12^6, 16^{12}, 17^4, \ldots$
(4^5)	$1, 5^2, 6, 10^6, 11^3, 15^{15}, 16^2, 17^4, \ldots$
(6^4)	$1, 2, 4^2, 6, 7^2, 8^3, 9^2, 10^6, 12^{11}, \ldots$
(4^6)	$1, 2, 3, 5, 6^4, 7^2, 8, 9^8, 10^3, 11^5, 12^{15}, \ldots$

Perfect colouring of (4^5) with five colours (R. Lück, Stuttgart):



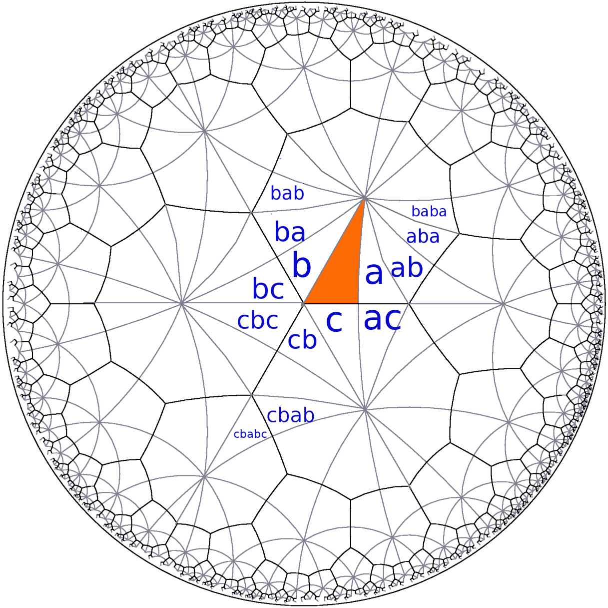
Perfect colouring of (4^5) with 25 colours (R. Lück, Stuttgart):



How to obtain these values?

The (full) symmetry group of a regular tiling (p^q) is a Coxeter group:

$$G_{p,q} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (ac)^2 = (bc)^q = \mathrm{id} \rangle$$


bab
baba
ba
aba
b
a
ab
bc
c
ac
cbc
cb
cbab
cbabc

Left coset colouring of (p^q):

Let F be the fundamental triangle.

- Choose a subgroup S of $G_{p,q}$ such that $a, b \in S$
- Assign colour 1 to each $f\,F$ $(f \in S)$
- Analoguosly, assign colour i to the i-th coset S_i of S

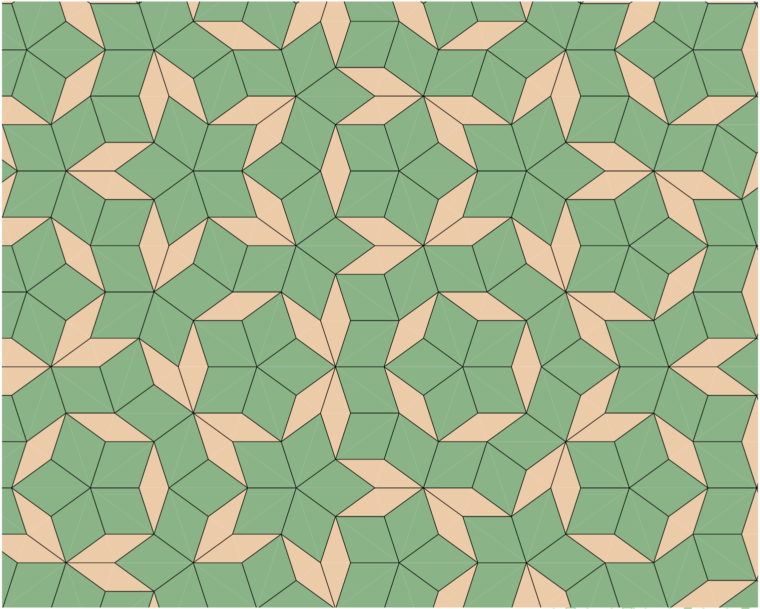
Yields a colouring with $[G_{p,q} : S]$ colours.

How to count perfect colourings now?

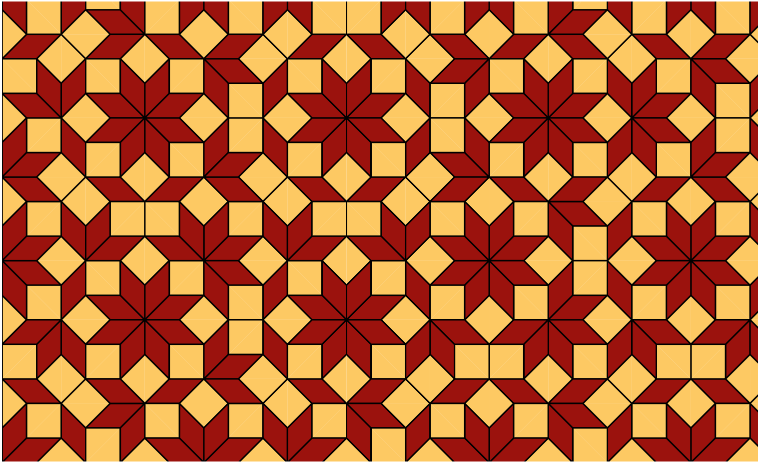
- Show that each of these colourings is perfect (simple)
- Show that each perfect colouring is obtained in his way
- Count subgroups of index k in $G_{p,q}$ (hard)

Using `GAP` yields the tables above.

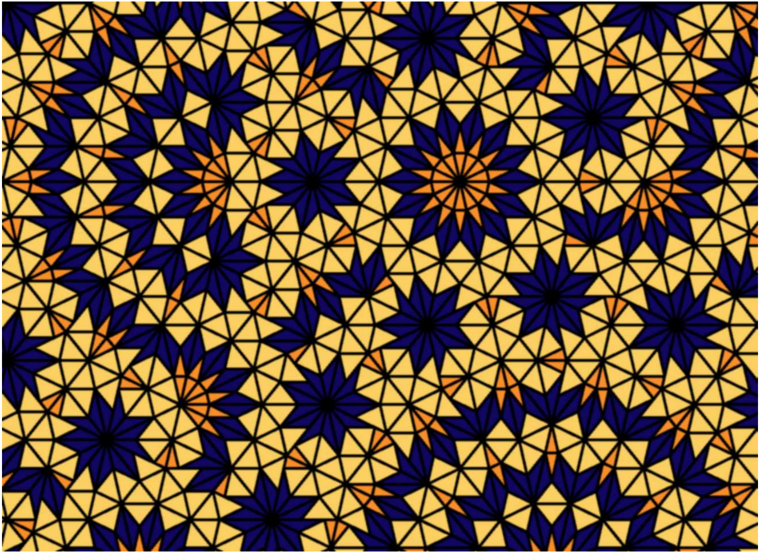
Nonperiodic tilings: Penrose



Nonperiodic tilings: Ammann-Beenker



Nonperiodic tilings: Danzer's $k\pi/7$



What are colour symmetries/perfect colourings here?

- Fourier space approach (Mermin, Lifshitz)
- Cyclotomic integers (Moody, Baake)

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Consider $\mathbb{Z}[\xi_n]$, $\xi_n = e^{2\pi i/n}$, as a point set in the plane.

- $n = 4$: square lattice
- $n = 3, n = 6$: hexagonal lattice
- $n = 5, n \geq 7$: dense point sets

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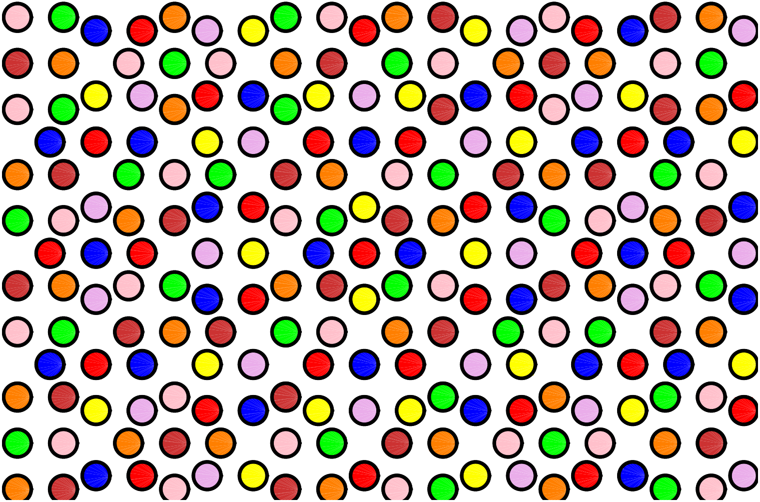
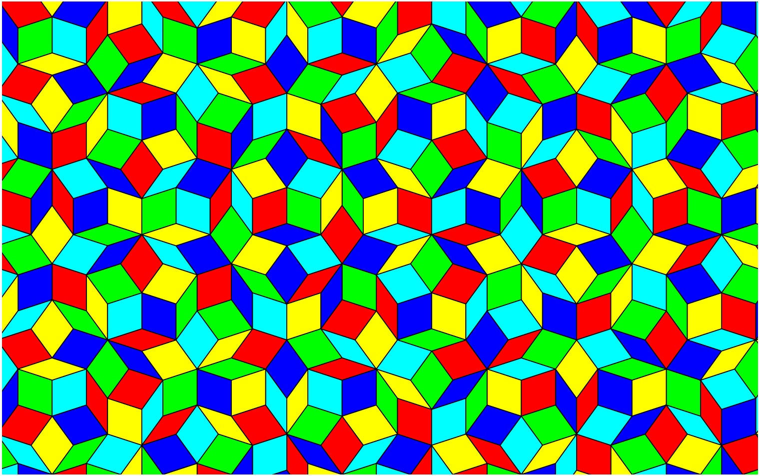
Consider colourings of $\mathbb{Z}[\xi_n]$, the tilings inherit the colours from those.





Ideals in the ring $\mathbb{Z}[\xi_n]$ generate colourings.

If $\mathbb{Z}[\xi_n]$ has class number one, each ideal is a principal ideal, generated by some $q \in \mathbb{Z}[\xi_n]$.

Thm (Bugarin, de las Peñas, F 2009) Let $\mathbb{Z}[\xi_n]$ have class number one, C a colouring of $\mathbb{Z}[\xi_n]$.

- C is generated by an ideal, iff C is chirally perfect.
- C generated by an ideal (q) is perfect, iff q is *balanced*.

q *balanced*: in the unique factorization

$$q = \varepsilon \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i} \prod_{p_j \in \mathcal{C}} \omega_{p_j}^{\beta_j} \overline{\omega_{p_j}}^{\gamma_j} \prod_{p_k \in \mathcal{R}} p_k^{\delta_k}$$

holds: $\beta_j = \gamma_j$.

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holds: $\beta_j = \gamma_j$.

E.g.: $n = 4$ (square lattice):

- 4 colours: $q = 2 = (1+i)(1-i)$ balanced
- 5 colours: $q = (2+i)$ not balanced