Tilings with dense tile orientations

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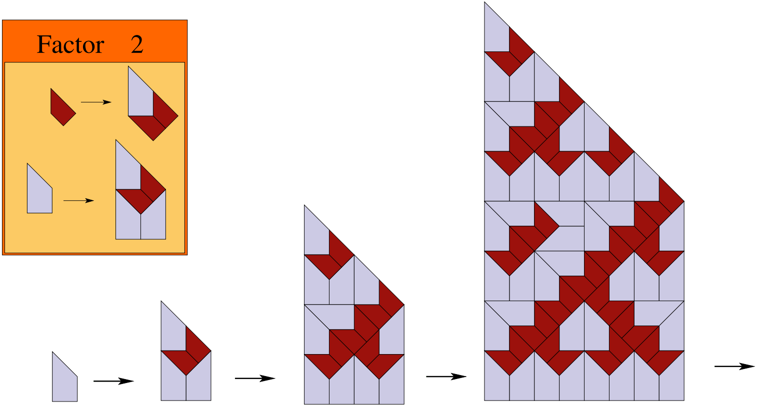
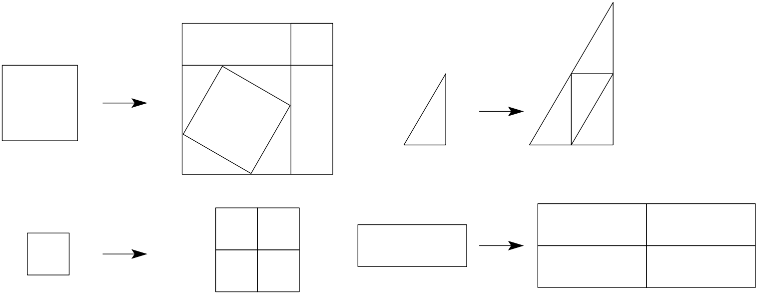
1. Substitution tilings with tiles in infinitely many orientations and dense tile orientations (DTO)
2. Tile shapes forbidding DTO
3. Tilings with DTO and rotational symmetry

Substitution tilings:



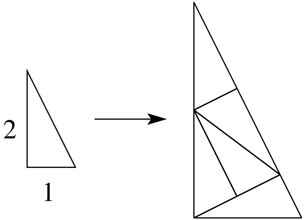
Usually, tiles occur in finitely many different orientations only.

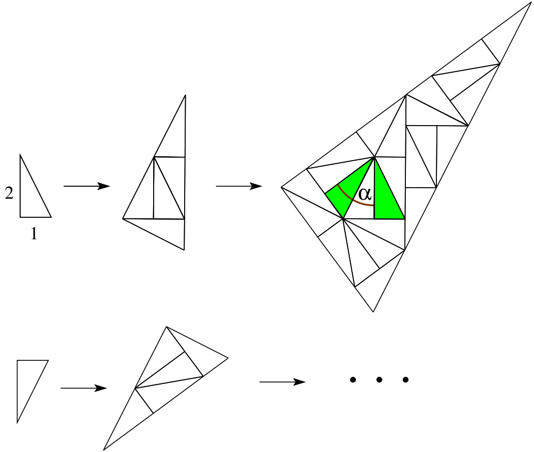
Not always. Cesi's example (1990):



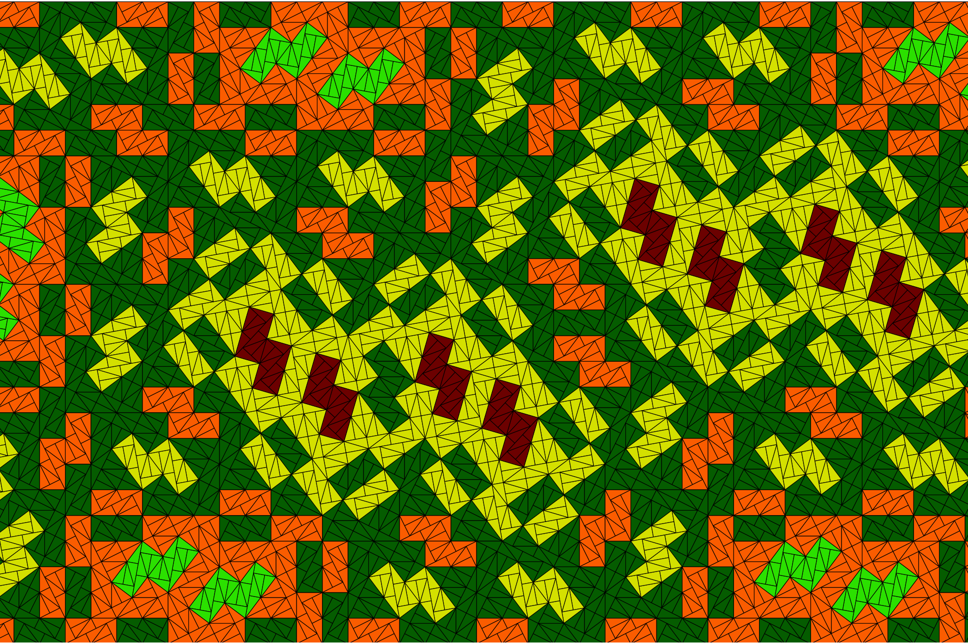
A substitution σ is *primitive*, if for any tile T there is $k \geq 1$ such that $\sigma^k(T)$ contains all tile types.

Conway's Pinwheel substitution (1991):

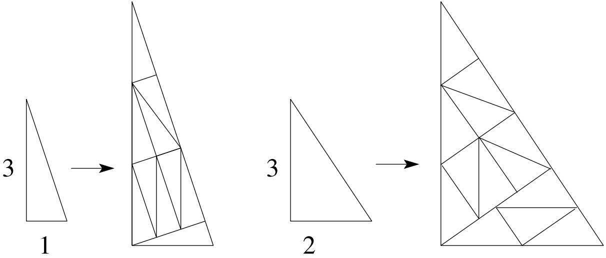




The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



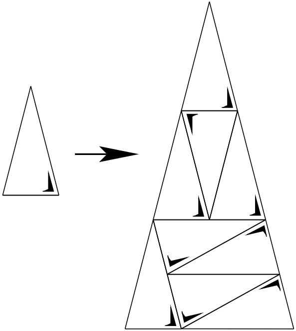
Obvious generalizations: Pinwheel (n, k)

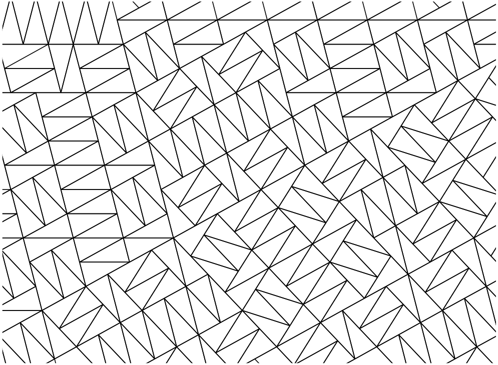


$n = 3, k = 1$ $n = 3, k = 2$

etc.

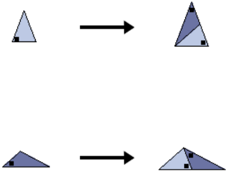
Unknown (< 1996, communicated to me by Danzer):

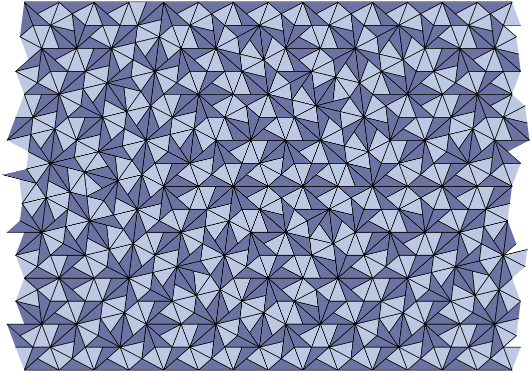




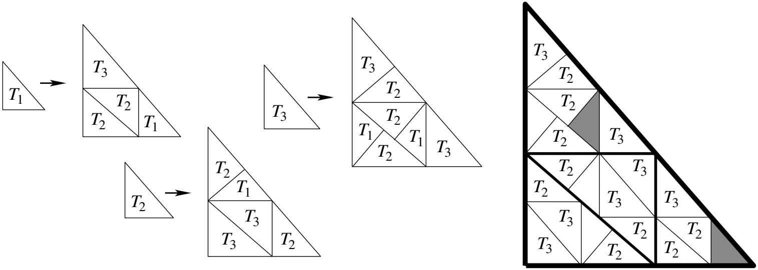
(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):





Pythia (m, j), here: $m = 3$, $j = 1$.



Dense Tile Orientations (DTO)

For all examples: the orientations are **dense** in $[0, 2\pi[$.

Even more: The orientations are **equidistributed** in $[0, 2\pi[$.

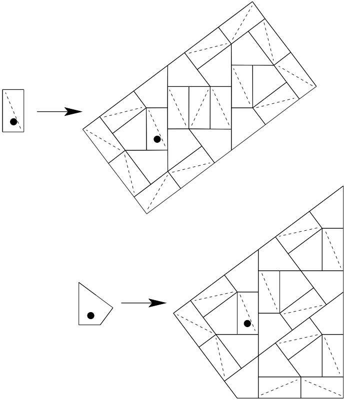
Theorem (F. '08)

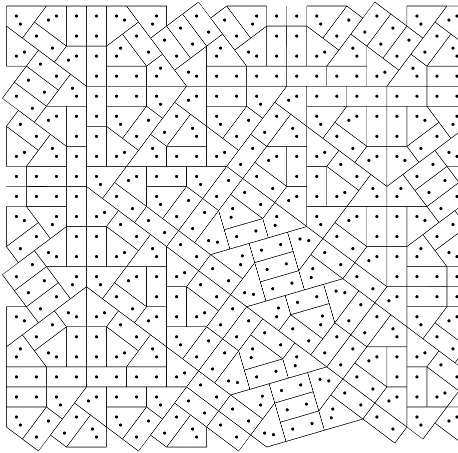
In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in $[0, 2\pi[$.

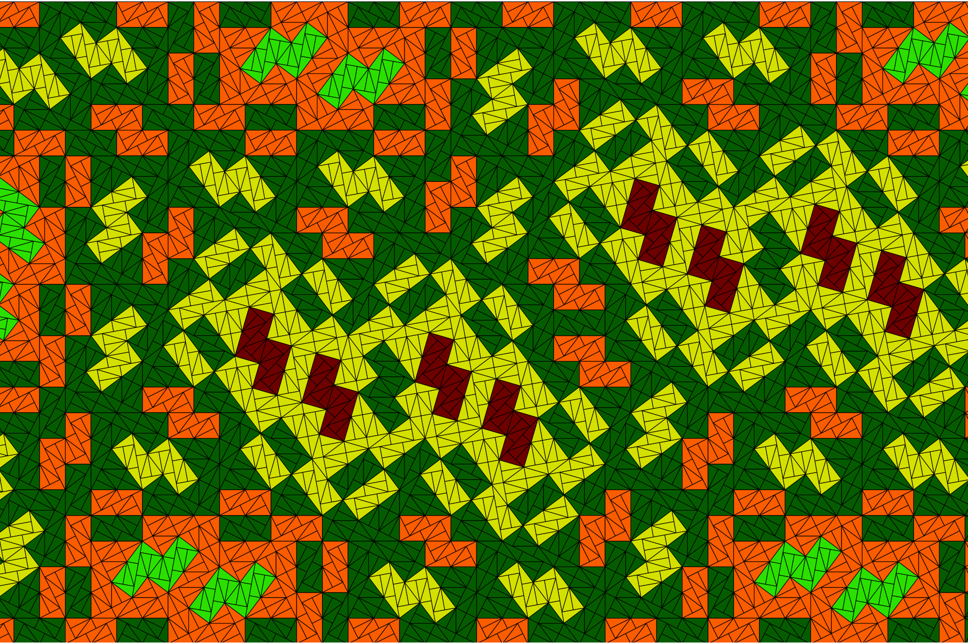
D.F.: Substitution tilings with statistical circular symmetry, *European J. Combin.* 29 (2008) 1881-1893
`arXiv:0704.2521`

So far: tiles are always triangles. One exception:

Kite Domino (equivalent with Pinwheel):







Tile shapes forbidding DTO

Question: Can we find examples with rhombic tiles for instance?

Answer: No.

Theorem (F.-Harriss, 2013)

Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many prototiles (i.e., finitely many different tile shapes). Let all prototiles be centrally symmetric convex polygons (i.e., $P = -P$). Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

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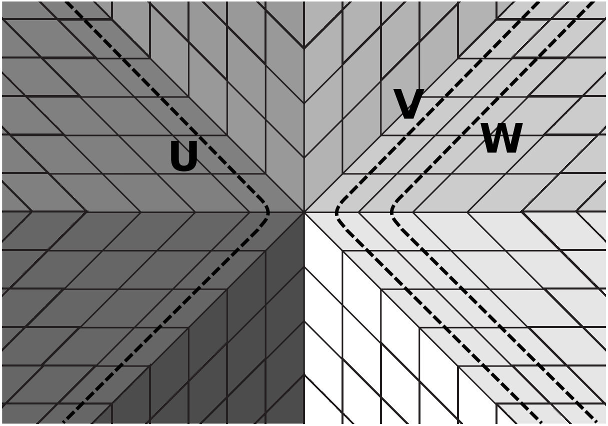
Theorem (F.-Harriss, 2013)

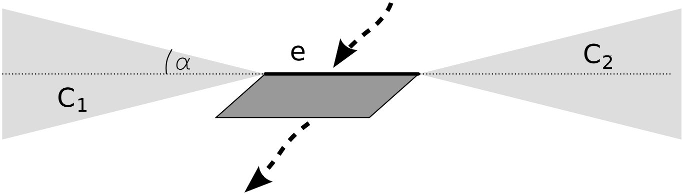
Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many parallelograms as prototiles. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

Assume all tiles are vertex-to-vertex.



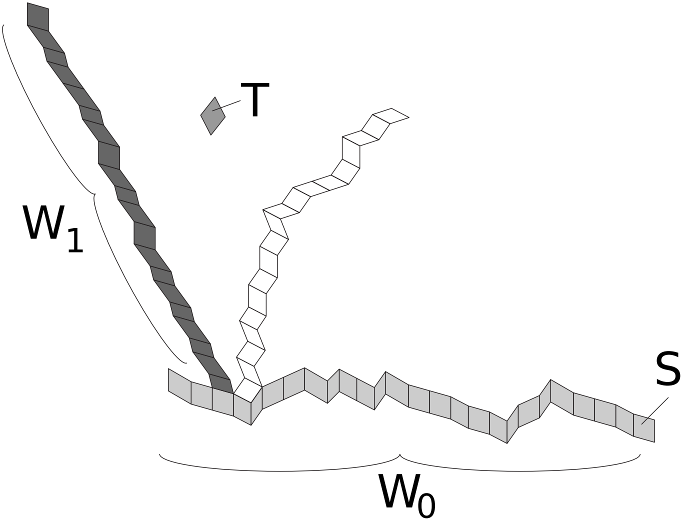
A *worm* is a sequence of tiles $\ldots, T_{-1}, T_0, T_1, T_2, \ldots$ where T_k and T_{k+1} share a common edge, and all shared edges are parallel.

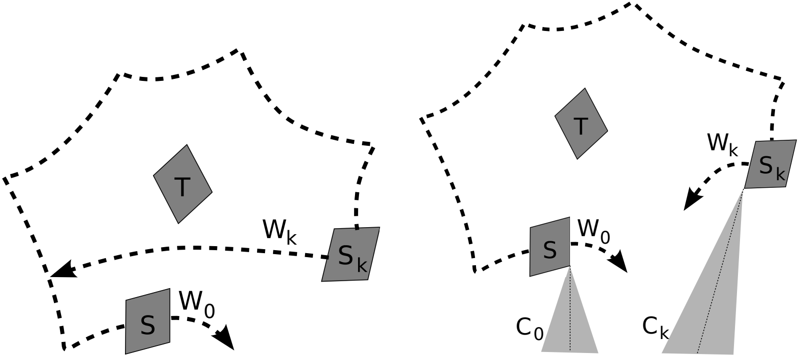
Cone Lemma: A worm defined by edge e cannot enter C_1 or C_2.
(α the minimal interior angle in the prototiles)



Loop Lemma: A worm has no loop.

Travel Lemma: Any two tiles can be connected by a finite sequence of finite worm pieces. (At most $k = \lceil \frac{2\pi}{\alpha} \rceil$ many.)




T
W_k
S_k
S
W_0
T
W_k
S_k
S
W_0
C_0
C_k

Proof of theorem (parallelogram version): Fix some tile S. Every tile T can be connected to S by at most $\lceil \frac{2\pi}{\alpha} \rceil$ worm pieces. That is, with at most $\lceil \frac{2\pi}{\alpha} \rceil$ turns. □

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Proof of theorem (general) Any centrally symmetric convex polygon can be dissected into parallelograms.
(see e.g. Kannan-Soroker 1992)

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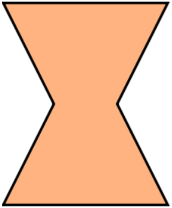
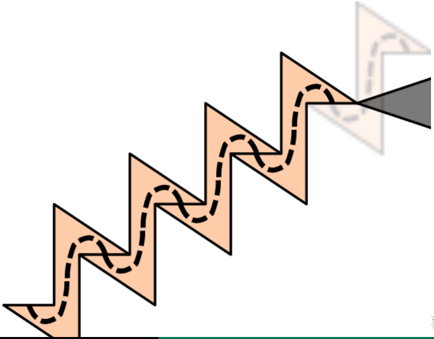
Can we drop "convex"? Hmm...

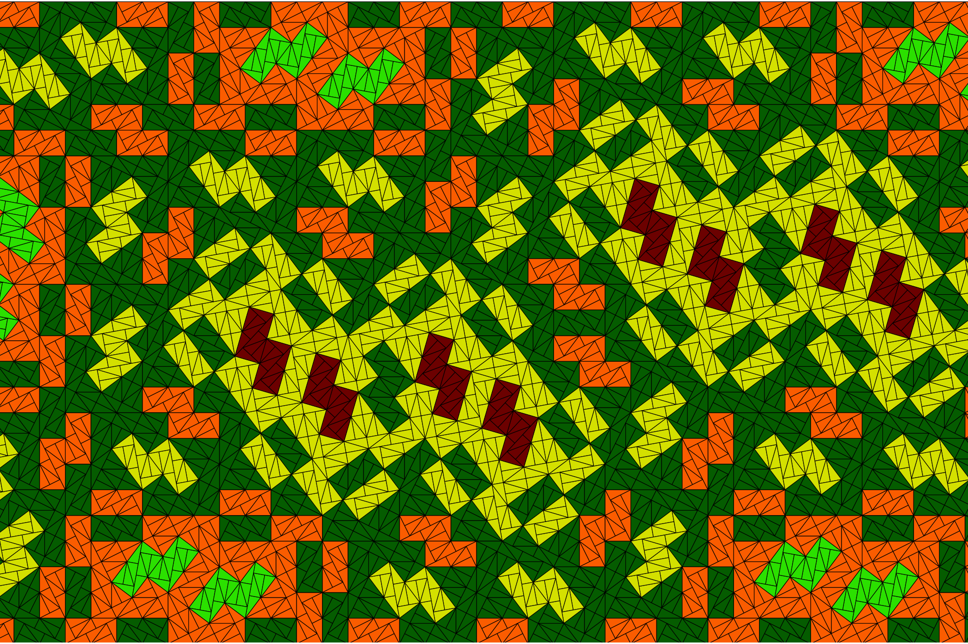




Tilings with rotational symmetry and DTO

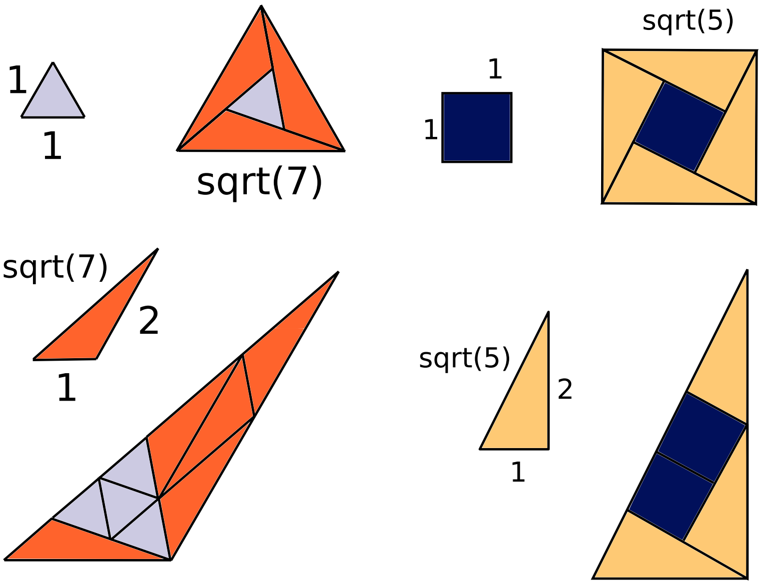
Some tilings with DTO show rotational symmetry.

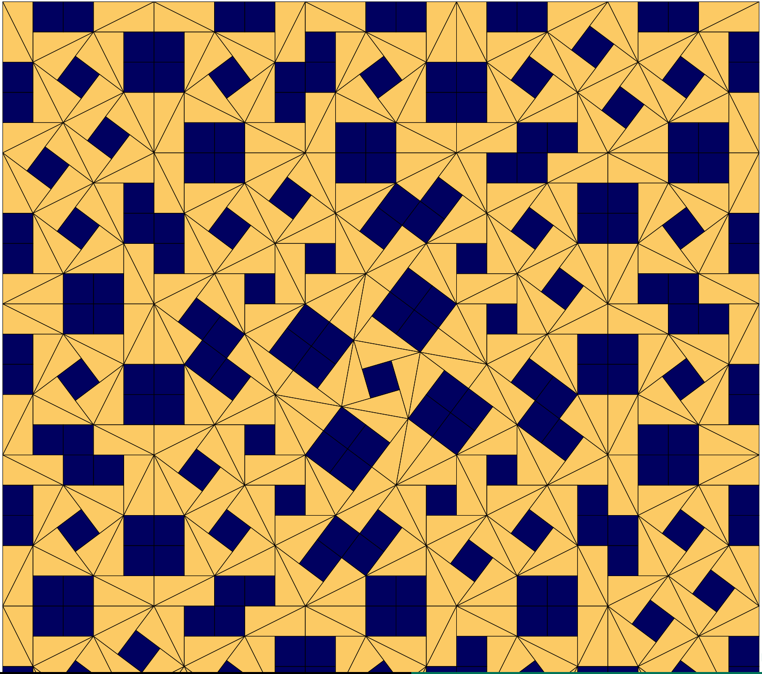
Rotational symmetry causes problems in computing cohomology of tiling spaces (...whatever this means; ask Paolo Bugarin).

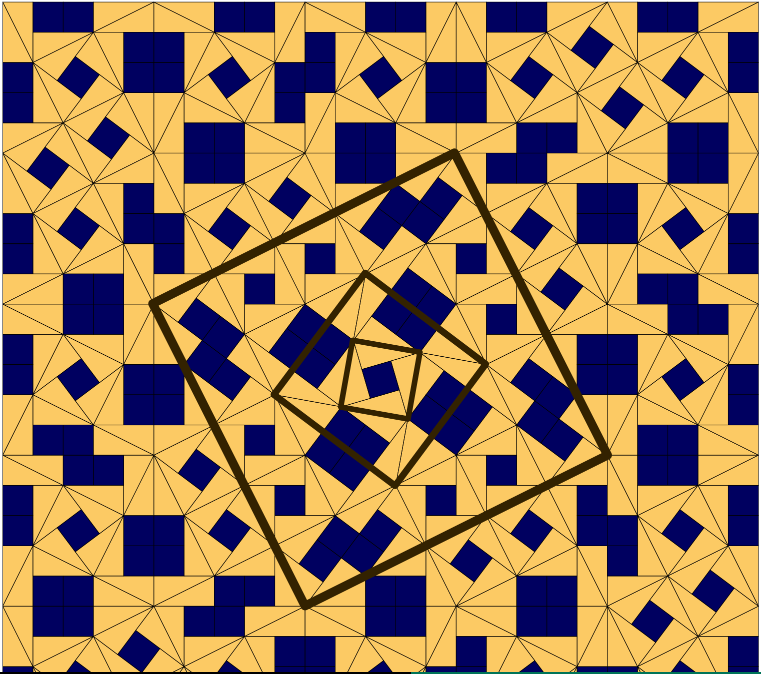


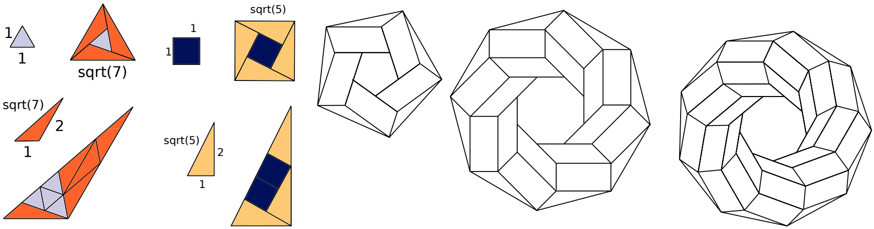
Question: Are there tilings with DTO and n-fold rotational symmetry for $n \geq 3$?

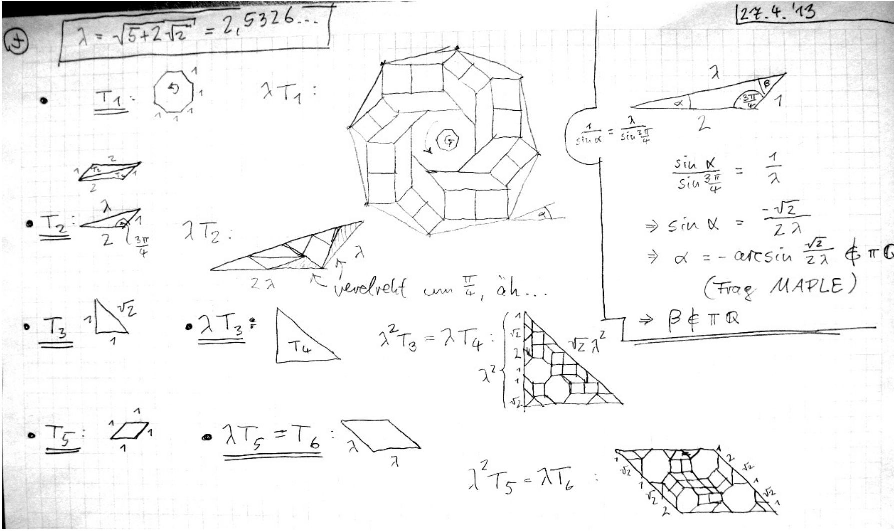
Answer: Yes. At least for $n \in \{3, 4, 5, 6, 8\}$.


1
1
sqrt(7)
1
1
sqrt(5)
sqrt(7)
2
1
sqrt(5)
2
1



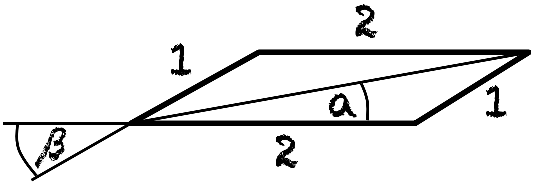



1
1
sqrt(7)
sqrt(7)
2
1
1
1
sqrt(5)
sqrt(5)
2
1


λ = √(5+2√2) = 2,5326...
27.4.'13
verdreht um π/4, äh...
(Frag MAPLE)

Conjecture or Fact (?):

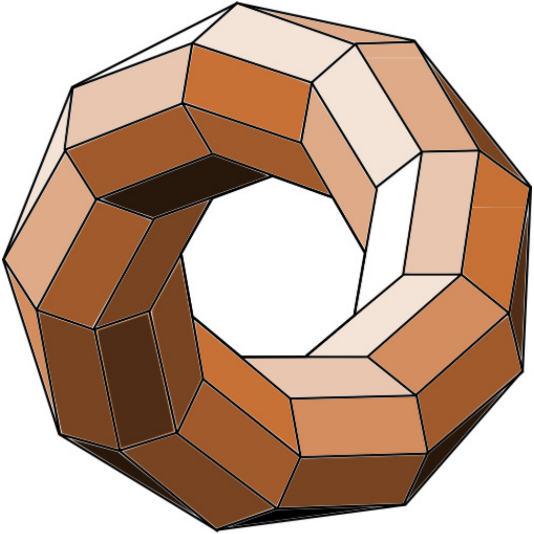
In a parallelogram with edge lengths 1 and 2, and interior angle β:
If $\beta = \frac{2\pi}{n}$ $(n \geq 4)$ then $\alpha \notin \pi\mathbb{Q}$.



True for $n \leq 100$ (Franz Gähler)

TODO:

- Find (deterministic) tilings with tile in infinitely many orientations, but without DTO
- Generalise F.-Harriss 2013
  (tiles centrally symm., convex $\Rightarrow$ finitely many orientations)
  - ...to higher dimensions: $d \geq 3$
  - ...to non-convex tiles
- Prove the conjecture/fact of the previous slide
- Construct examples with DTO and n-fold rotational symmetry for $n = 7$ and for $n \geq 9$



THANK YOU!