Bounded distance equivalence classes of cut-and-project sets on the line

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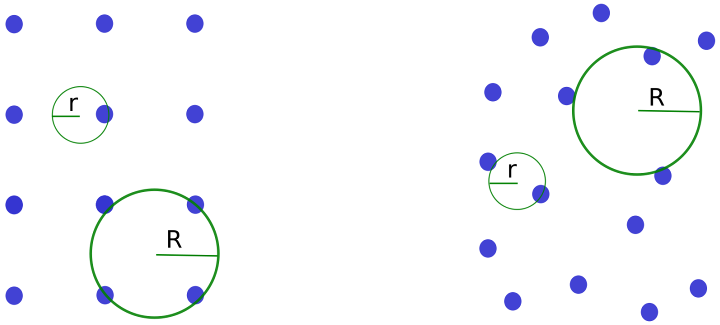
Essentially all of this talk is contained in:

D.F., Alexey Garber:
Pisot substitution sequences, one dimensional cut-and-project sets and bounded remainder sets with fractal boundary,
Indagationes Mathematicae 29 (2018) 1114-1130.
`arXiv:1711.01498`

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each ball of radius R contains at least one point of Λ (*relatively dense*)

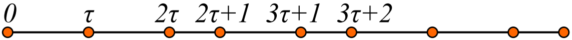


crystallographic disordered

Aperiodic Delone sets are in between.

In this talk all Delone sets are one-dimensional.

E.g. Fibonacci sequence, silver mean sequence, all *a*s in the period doubling sequence ($a \to ab$, $b \to aa$)....



$$\cdots abaaabababaaabaaabaaabababaaabababaaabababaaabaaab \cdots$$

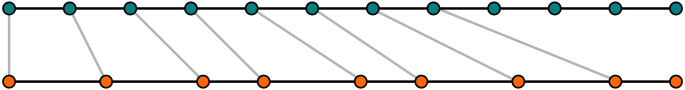
The *hull* $\mathbb{X}_\Lambda$ of a Delone set Λ is the closure of the orbit of Λ under translations in the local topology:

$$\mathbb{X}_\Lambda = \overline{\{t + \Lambda \mid t \in \mathbb{R}\}}$$

Definition

Let Λ, Λ' be Delone sets. We say that Λ and Λ' are bounded distance equivalent ($\Lambda \stackrel{\mathrm{bd}}{\sim} \Lambda'$) if there is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$



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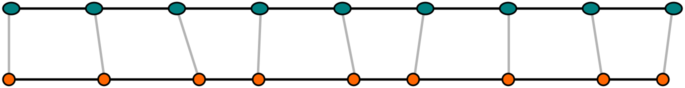


Lemma

Bounded distance equivalence is an equivalence relation.

Questions: Given a cut-and-project set Λ,

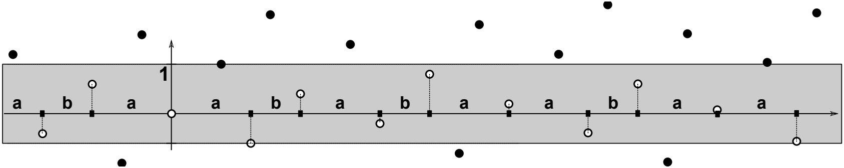
- is $\Lambda \stackrel{\text{bd}}{\sim} b\mathbb{Z}$ for some $b > 0$?
- how many equivalence classes (wrt b.d.e.) does the hull $\mathbb{X}_\Lambda$ contain?

Cut-and-Project Sets (CPS)

$$\begin{array}{ccccc} G & \overset{\pi_1}{\longleftarrow} & G \times H & \overset{\pi_2}{\longrightarrow} & H \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

Here always: $G = \mathbb{R}$, $H = \mathbb{R}^e$

- Γ a *lattice* in $G \times H$
- π_1, π_2 *projections*
  - $\pi_1|_\Gamma$ injective
  - $\pi_2(\Gamma)$ dense

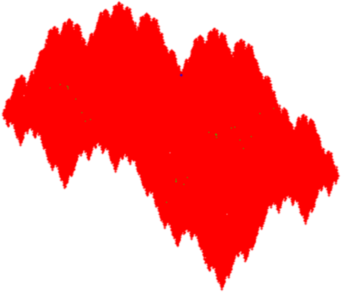


Fibonacci: $e = 1$

An example with $e = 2$ ($G = \mathbb{R}^1$, $H = \mathbb{R}^2$).:

$$\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML$$

...uses a window W that looks like a fractal:



Regarding the 2nd question: Let Λ be a CPS.
How many equivalence classes does the hull $\mathbb{X}_\Lambda$ contain?

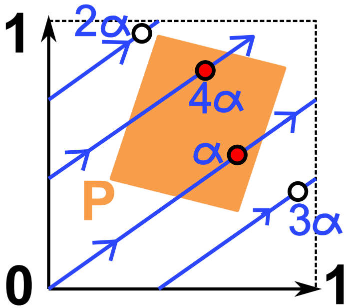
There is a nice connection between windows of CPS and *bounded remainder sets* (BRS).

Consider a subset $P \subset [0,1]^d$.
Fix a very irrational* $\alpha \in \mathbb{R}^d$ and count how often

$$\alpha \bmod 1, 2\alpha \bmod 1, \ldots, n\alpha \bmod 1$$

hits P. Call these numbers $h(n)$.

(*: $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\quad \alpha_i \notin \mathbb{Q}$, $\quad \alpha_i/\alpha_j \notin \mathbb{Q}$ for $i \neq j$)



Then in many cases (e.g. P is a polygon)

$$\frac{h(n)}{n} \to \mathrm{vol}(P)$$

(In other words: $|h(n) - n \cdot \mathrm{vol}(P)| \in o(n)$)

Sometimes even better:

$$\exists C > 0 : \quad |h(n) - n \cdot \mathrm{vol}(P)| < C$$

(In other words: $|h(n) - n \cdot \mathrm{vol}(P)| \in O(1)$)

In the latter case P is called a *bounded remainder set* (BRS).

Fact: Let $\Lambda \subset \mathbb{R}$ be a CPS with window W.

$\Lambda \overset{\mathrm{bd}}{\sim} b\mathbb{Z}$ for some $b > 0 \Leftrightarrow W$ is a BRS (wrt a certain α).

Theorem (Kesten 1966)

Let $\alpha \in [0,1]$, $0 \le a < b \le 1$. Then $[a,b]$ is a BRS wrt α if and only if $b - a \in \mathbb{Z} + \alpha\mathbb{Z}$.

This means for 1×1-CPS Λ where the window W is an interval (for a certain α that can be read off from the CPS) :

$$|W| \in \alpha\mathbb{Z} + \mathbb{Z} \;\Leftrightarrow\; \Lambda \overset{\text{bd}}{\sim} b\mathbb{Z} \text{ for all } \Lambda \in \mathbb{X}_\Lambda$$

E.g. Fibonacci sequences:

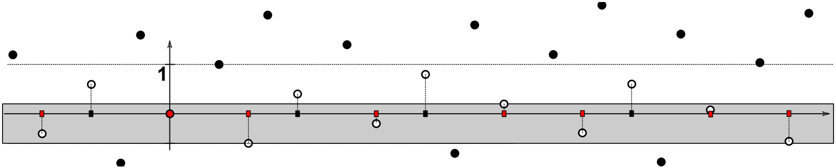
- $\alpha = \frac{\sqrt{5}+1}{2}$
- $|W| = 0 + \alpha \in \mathbb{Z} + \alpha\mathbb{Z}$.

Hence all Fibonacci sequences are b.d.e. $b\mathbb{Z}$ (for $b = \frac{2+\alpha}{1+\alpha}$).

In other words: the Fibonacci hull $\mathbb{X}_{Fib}$ consists of just one equivalence class wrt b.d.e.

Kesten's Theorem allows also the opposite:

"Half-Fibonacci sequence": the CPS with half the window.



- $\alpha = \frac{\sqrt{5}+1}{2}$
- $|W| = \frac{1}{2}\alpha \notin \mathbb{Z} + \alpha\mathbb{Z}$.

Hence all Half-Fibonacci sequences are *not* b.d.e. to $b\mathbb{Z}$ (for *any* $b > 0$).

But are all Half-Fibonacci sequences b.d.e to each other? No!

Theorem (F-Garber)

Let $\Lambda \overset{\mathrm{bd}}{\sim} b\mathbb{Z}$*, and* $\Lambda = \Lambda_1 \dot\cup \Lambda_2$ *such that* $\Lambda_1 \overset{\mathrm{bd}}{\sim} \Lambda_2$*.*

Then $\Lambda_1 \overset{\mathrm{bd}}{\sim} 2b\mathbb{Z}$ *and* $\Lambda_2 \overset{\mathrm{bd}}{\sim} 2b\mathbb{Z}$*.*

Let Λ_{Fib} be a Fibonacci CPS. We know $\Lambda_{Fib} \overset{\mathrm{bd}}{\sim} b\mathbb{Z}$.

Let Λ_1 be the Half-Fibonacci set using the upper half of the Fibonacci window, Λ_2 be the Half-Fibonacci set using the lower half.

Both are in the same hull: $\Lambda_1, \Lambda_2 \in \mathbb{X}_{HalfFib}$. Moreover,

$$\Lambda_{Fib} = \Lambda_1 \dot\cup \Lambda_2$$

If $\Lambda_1 \overset{\mathrm{bd}}{\sim} \Lambda_2$, the result above implies $\Lambda_1 \overset{\mathrm{bd}}{\sim} 2b\mathbb{Z}$; in contradiction to Kesten's Theorem.

Hence the number of equivalence classes in $\mathbb{X}_{HalfFib}$ is at least two.

Problems (hard? easy?)

- What is the number of equivalence classes in $\mathbb{X}_{HalfFib}$ wrt b.d.e.?
- Are the *a*s in the period doubling sequence b.d.e. to $b\mathbb{Z}$?
- Find a substitution tiling in $\mathbb{R}$ that is *not* b.d.e. to $b\mathbb{Z}$ (for any b).

Ad Question 1:

A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution* if for all other eigenvalues λ' of M_σ holds: $0 < \lambda' < 1$.

E.g. the examples above (Fibonacci, silver mean, S-M-L) are Pisot substitutions.

Theorem (Holton-Zamboni 1998, Dumont 1990, Rauzy?)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to $b\mathbb{Z}$ (for some $b > 0$).

Theorem (F-Garber 2018)

All one-dimensional substitutions with inflation factor $\lambda > 1$, where for all other eigenvalues λ' of M_σ holds: $0 \leq \lambda' < 1$, and M_σ is diagonisable over $\mathbb{C}$, are bounded distance equivalent to $b\mathbb{Z}$ (for some $b > 0$).

We can not yet use the Theorem above to answer the question "are all *a*s in the period doubling sequence bde to a lattice?"

But our version of the theorem can be used for:

$\sigma: \quad a \mapsto aabb, \quad b \mapsto ab.$

$\cdots aabbaabbababaabbab|aabbaabbababaabbaabbababaabbabaabbab \cdots$

$M_\sigma = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right)$, with eigenvalues 3 and 0. Hence for the geometric version holds

$$\forall \Lambda \in \mathbb{X}_\sigma: \quad \Lambda \overset{\mathrm{bd}}{\sim} \frac{3}{2}\mathbb{Z}.$$

(But here it is obvious anyway)

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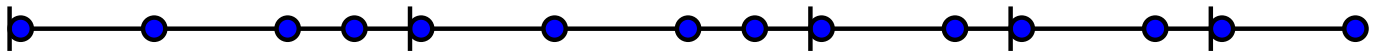

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Our theorem can also be used for

$\sigma : \quad a \mapsto ababc, \quad b \mapsto abc, \quad c \mapsto ab.$

$$\cdots ababcabcababcabcabababcabcabababcabcababcabcab \cdots$$

$M_\sigma = \left(\begin{smallmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$, with eigenvalues $\frac{3 \pm \sqrt{5}}{2}$ and 0. Hence for the geometric version holds

$$\forall \Lambda \in \mathbb{X}_\sigma : \quad \Lambda \stackrel{\text{bd}}{\sim} b\mathbb{Z}.$$

$* \quad *$

$*$

Thank you!