Highly symmetric fundamental cells for lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$

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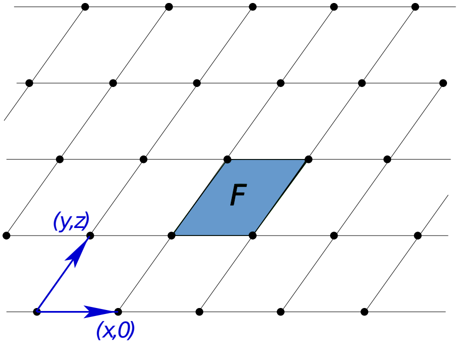
Point lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

Fundamental cell of Γ: $\overline{\mathbb{R}^d/\Gamma}$.



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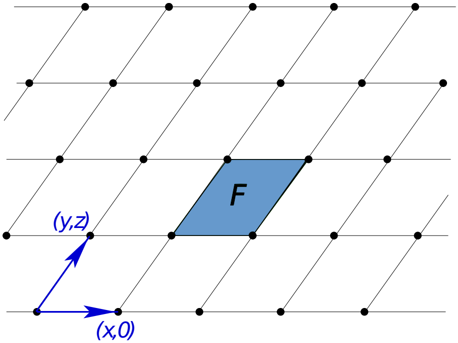
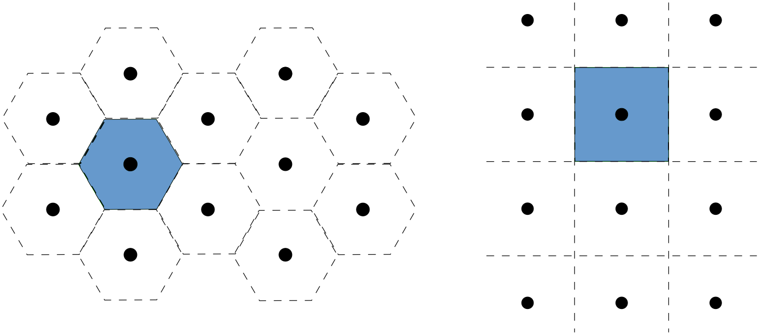


Point group $P(\Gamma)$ of Γ: All isometries g with $g\Gamma = \Gamma$.

Trivial: any lattice Γ has a fundamental cell whose symmetry group is $P(\Gamma)$.

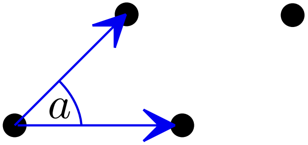
For instance, take the *Voronoi cell* of a lattice point x. (That is the set of points closer to x than to each other lattice point.)



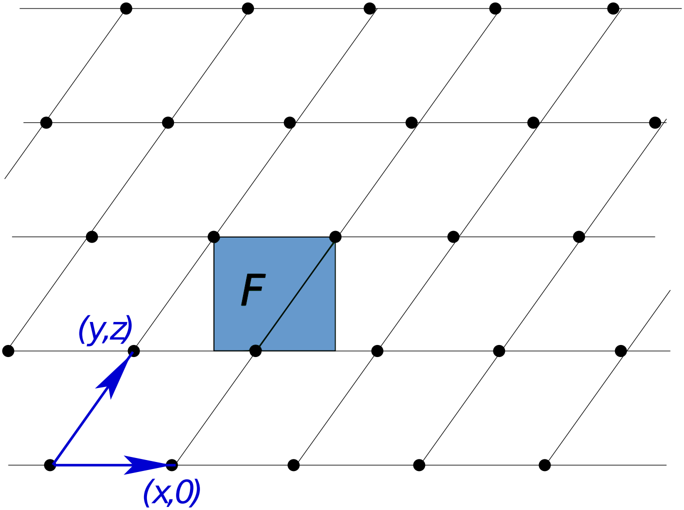
Theorem (Elser, F 2013)

Let $\Gamma \subset \mathbb{R}^2$ be a lattice, but not a rhombic lattice. Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $\quad [S(F) : P(\Gamma)] = 2$.

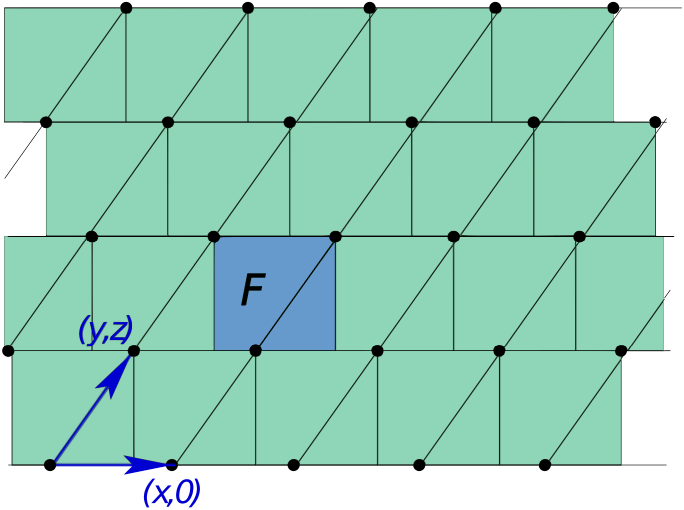
'Rhombic lattice' means: one with basis vectors of equal length, but neither a square lattice nor a hexagonal lattice.



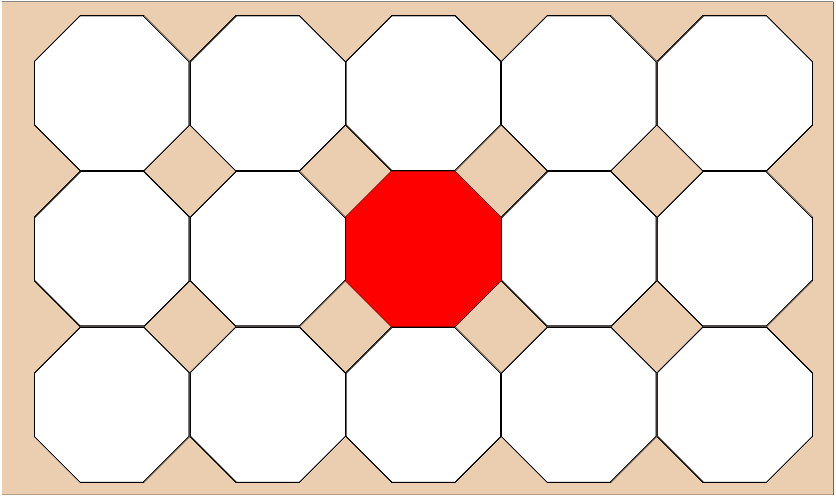
Proof: Case 1: Oblique lattice:

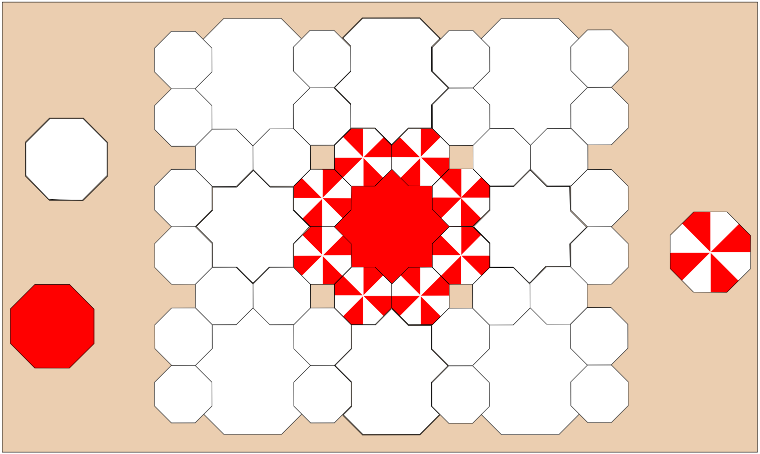


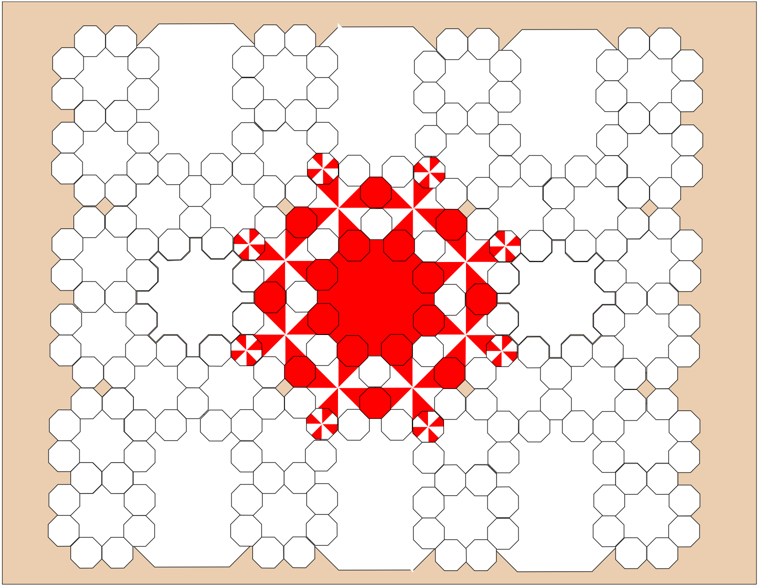
Oblique lattice:

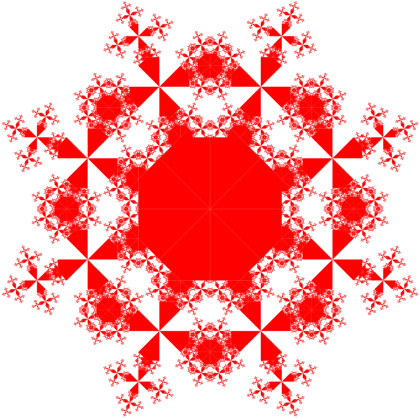


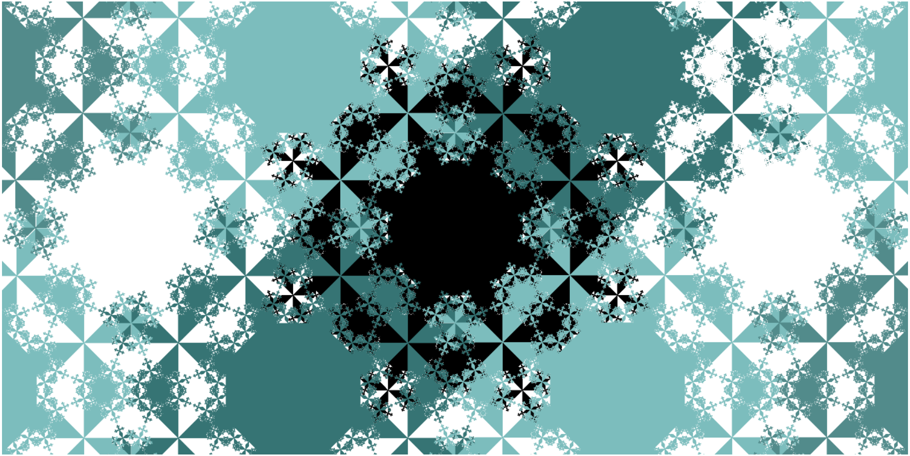
Case 2: Square lattice (V. Elser)





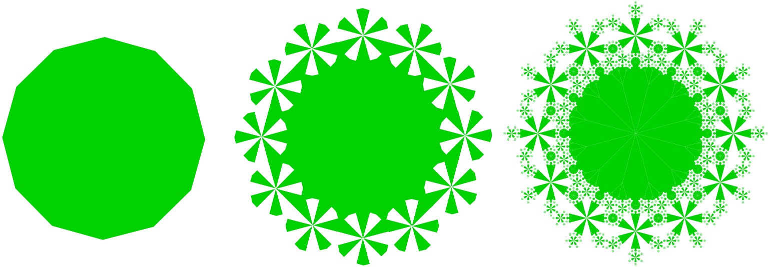




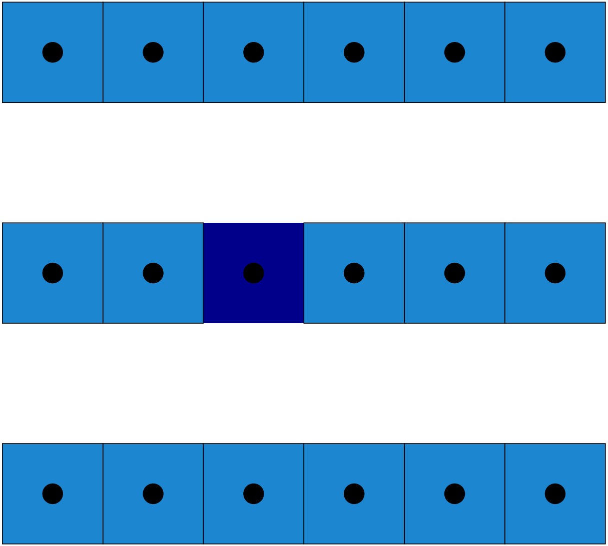


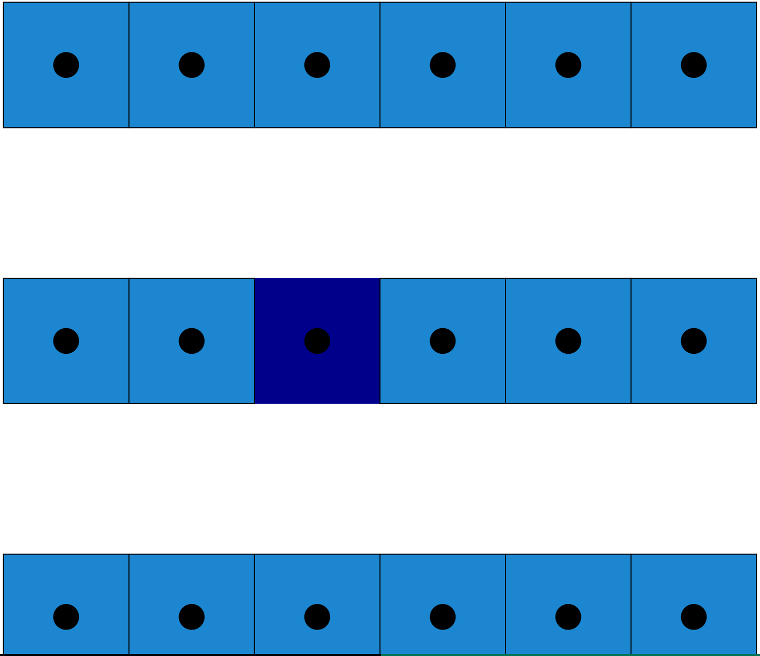
Case 3: Hexagonal lattice

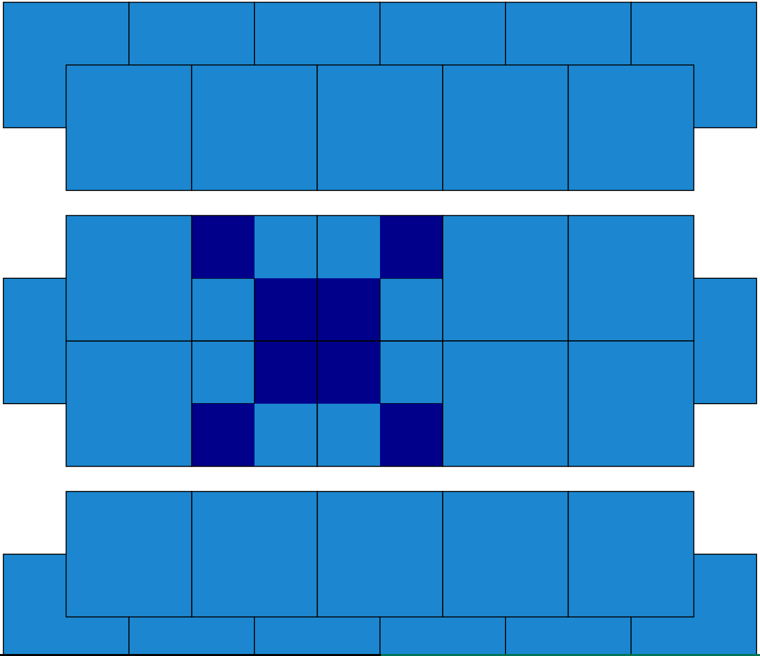
(Elser-Cockayne, Baake-Klitzing-Schlottmann):

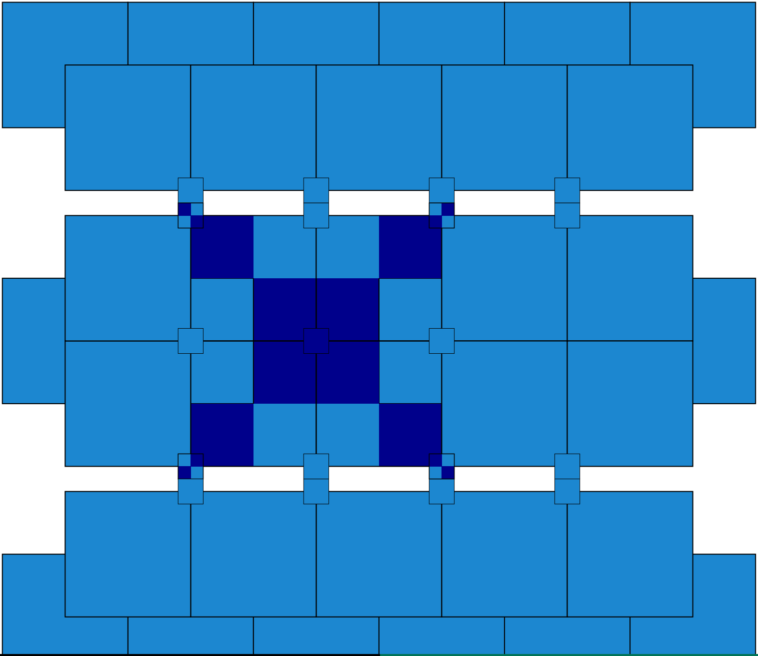


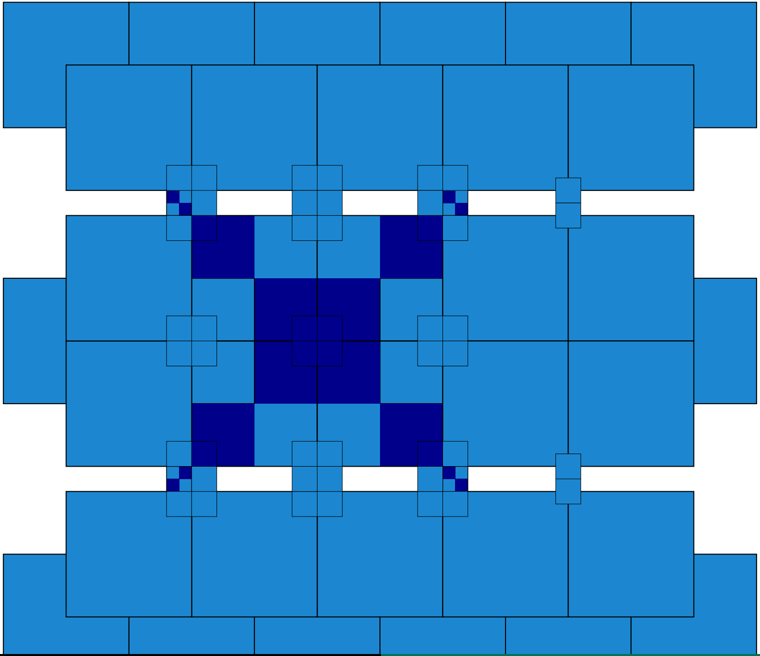
Case 4: Rectangular lattice

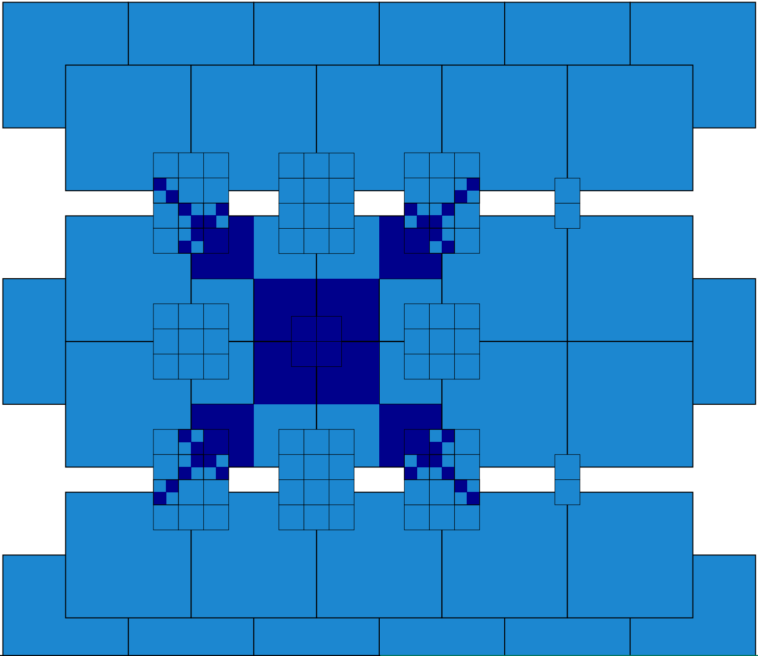


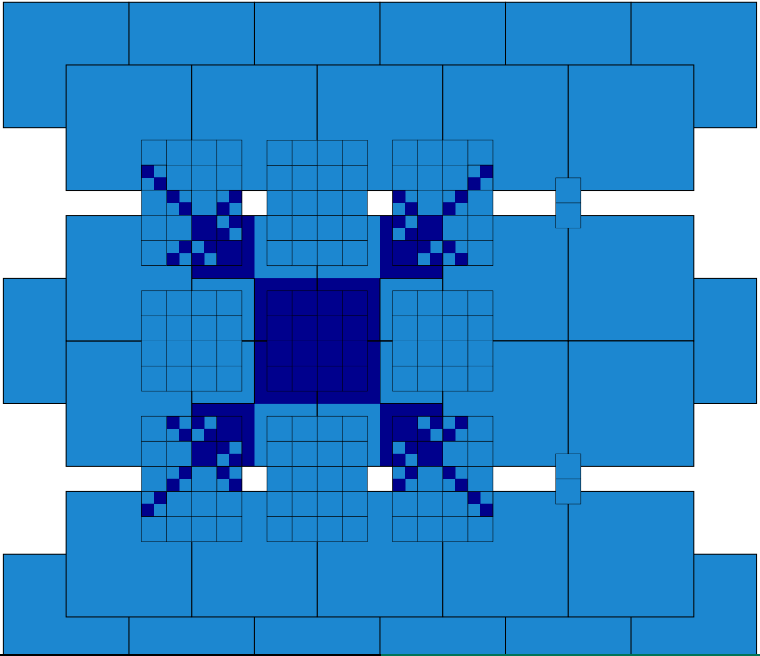


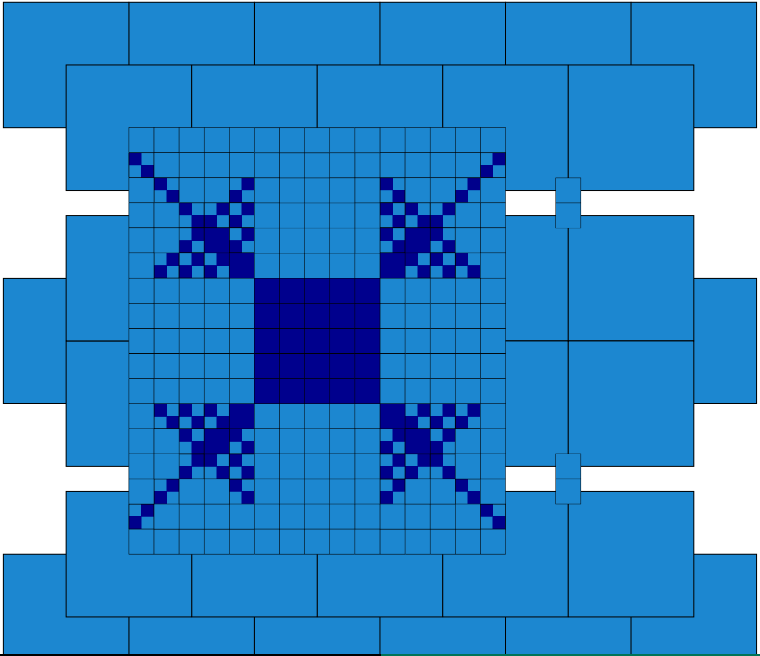


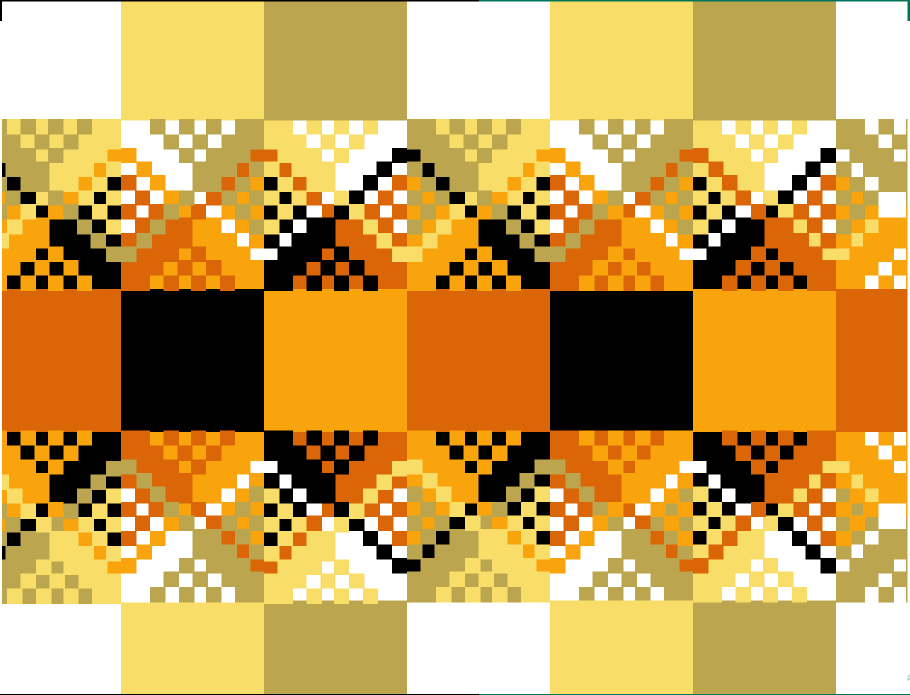




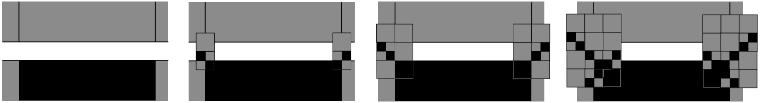








Euclidean algorithm at work:



Edge length of the rectangular gap: a, b with $a > b$.

$$a,\ a-b,\ a-2b,\ a-3b, \ldots, a - \left\lfloor \frac{a}{b} \right\rfloor b$$

Leaves a gap with edge length b, $c := a - \left\lfloor \frac{a}{b} \right\rfloor b$.

Continue.

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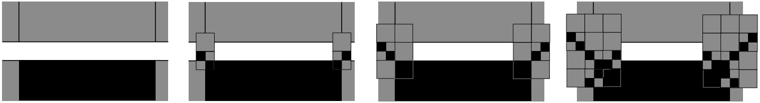

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Continue.

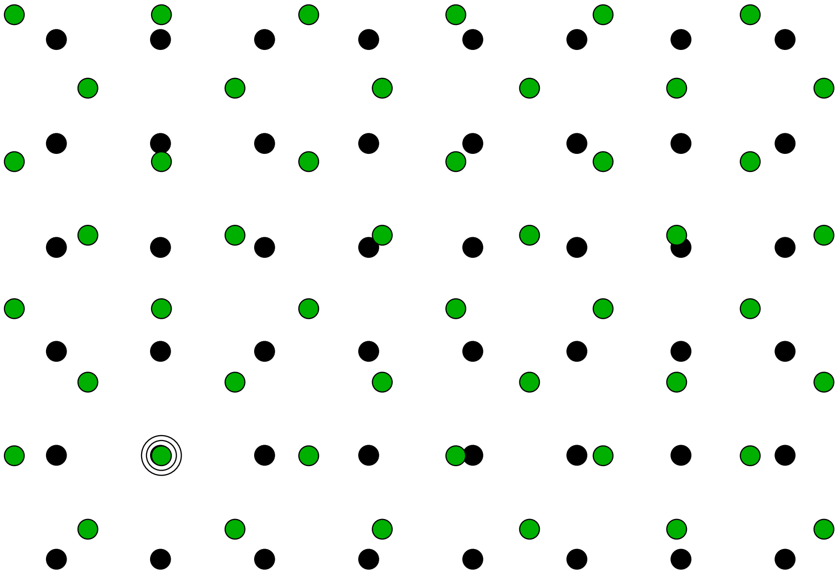
Case 5: rhombic lattices: still unsolved.

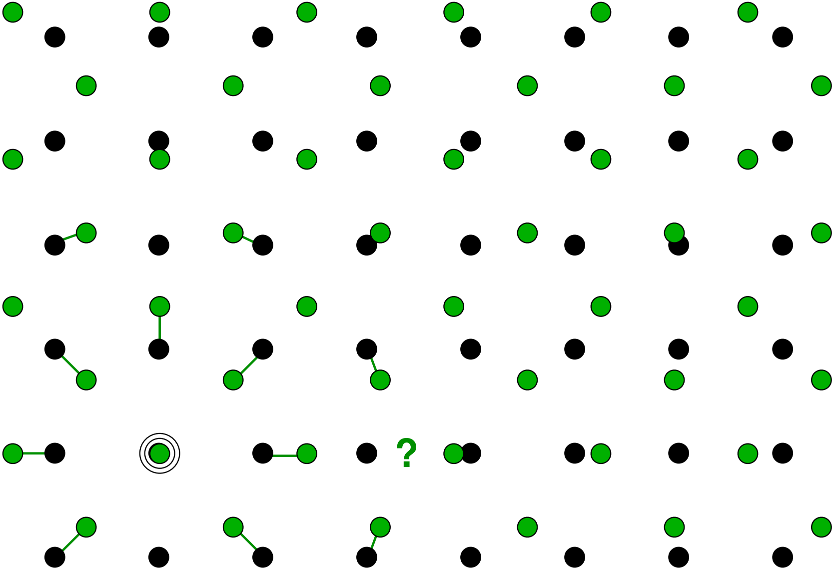
Application: Short perfect matchings

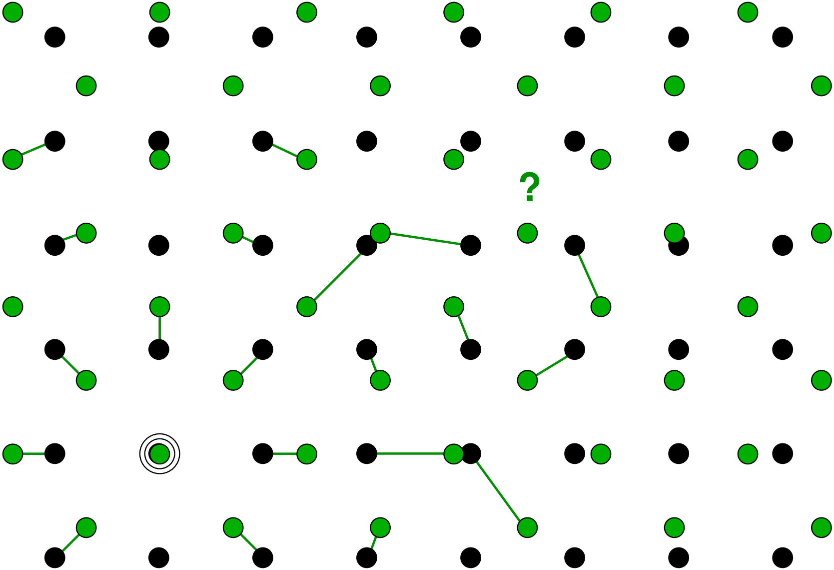
Consider the square lattice $\mathbb{Z}^2$, and $R_{45}\mathbb{Z}^2$, the square lattice rotated by 45°.

Problem: Find a perfect matching between $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$ with maximal distance not larger than $C > 0$. How small can C be?

(It is easy to see that $C \geq \frac{\sqrt{2}}{2} = 0.7071\ldots$.)







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Using the 8-fold fundamental cell F yields a matching with $C = 0.92387\ldots$.

How?

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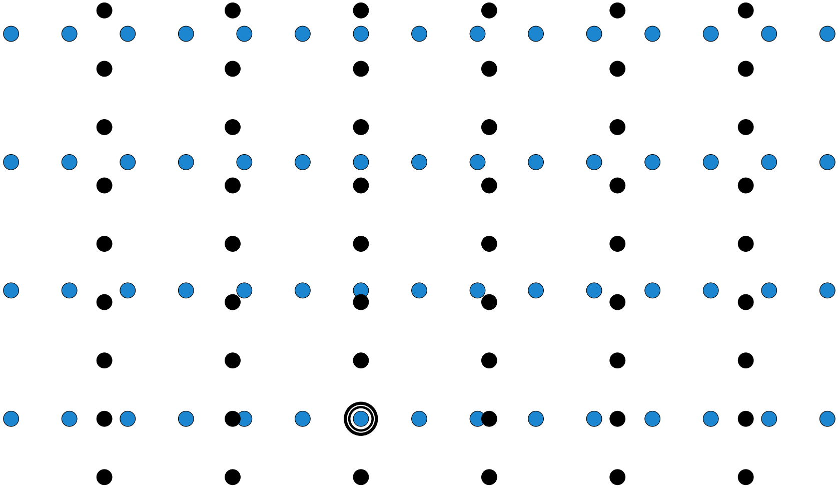
How?

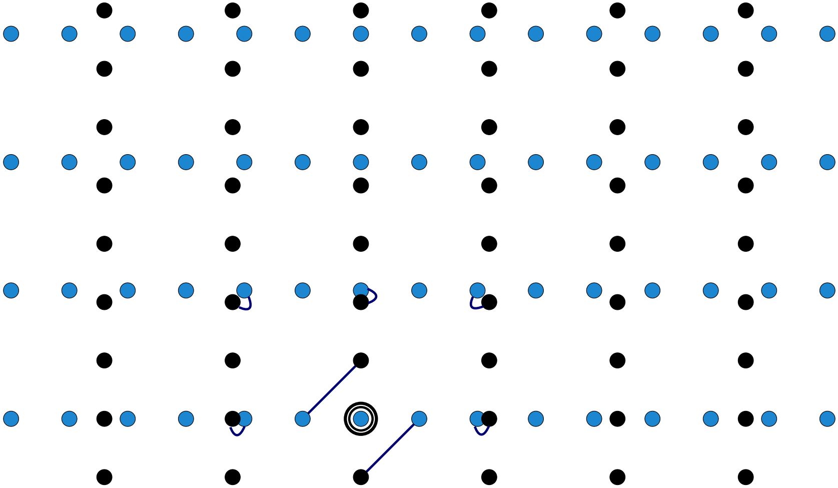
- Consider $\mathbb{Z}^2 + F$. Each $x + F$ ($x \in \mathbb{Z}^2$) contains exactly one point of $\mathbb{Z}^2$ in its centre.
- F is also fundamental cell for $R_{45}\mathbb{Z}^2$. Thus each $x + F$ ($x \in \mathbb{Z}^2$) contains exactly one point $x' \in R_{45}\mathbb{Z}^2$.
- Match x and x'.

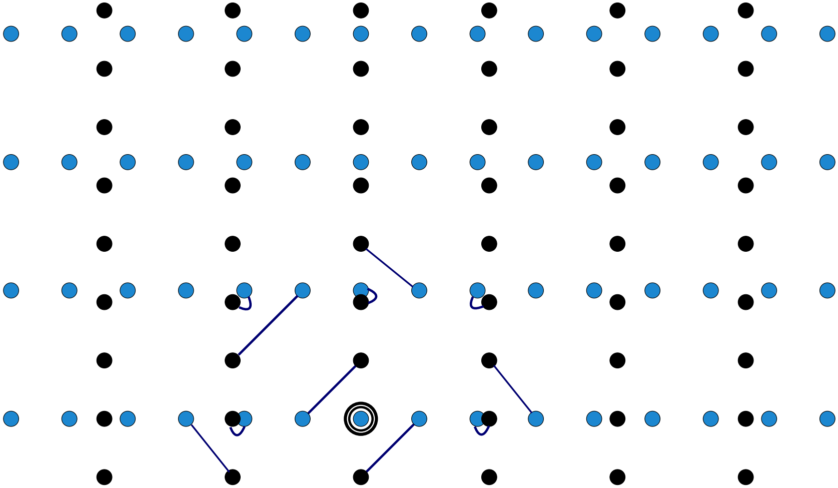
This (and its analogues) yields good matchings for

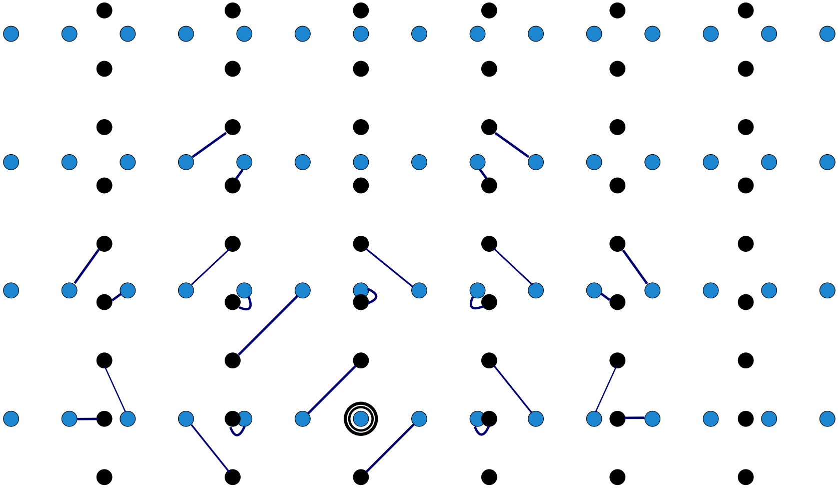
- $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$: $C = 0.92387\ldots.$
- The hexagonal lattice H and $R_{30}H$: $C = 0.78867\ldots$
- A rectangular lattice P and $R_{90}P$: $C \le \frac{1}{\sqrt{2}} \frac{\sqrt{5}+1}{2} b.$

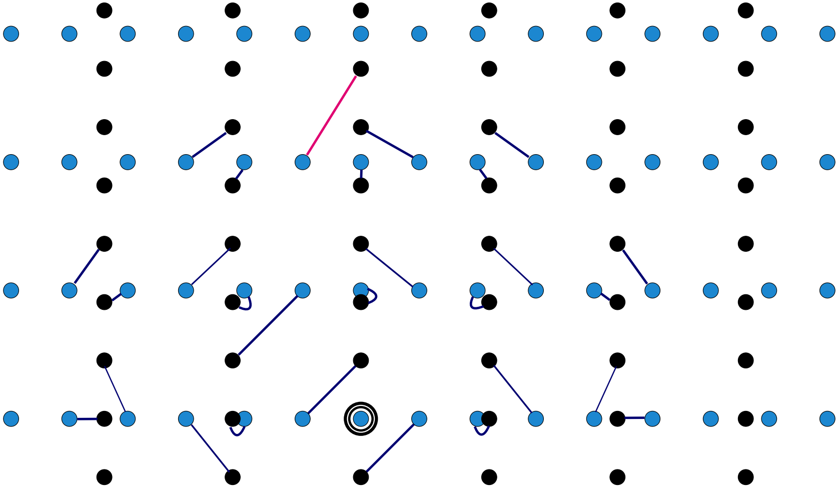
(b is the length of the longer lattice basis vector of P.)











Dimension 3

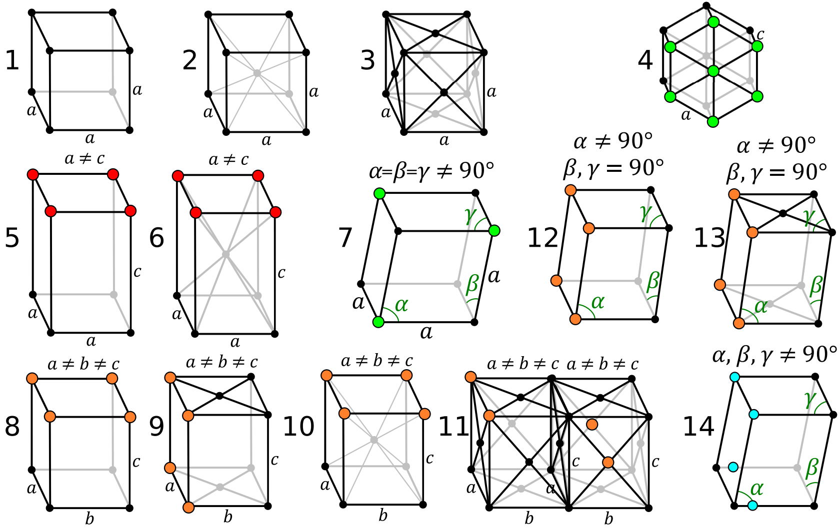
Theorem (F 2013)

Let $\Gamma \subset \mathbb{R}^3$ be a lattice, but not a cubic lattice. Then there is a fundamental cell F of Γ whose symmetry group $S(F)$ is strictly larger than $P(\Gamma)$: $\quad [S(F) : P(\Gamma)] = 2$.

"Cubic": One of $\mathbb{Z}^3$, $\mathbb{Z}^3 \cup \big(\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\big)$ ("*bcc*"), A_3 ("*fcc*").

Proof for $\mathbb{R}^3$: Consider the 14 cases:

Nr	Name	Point group	Order	2dim FC (# sym.)
1	$\mathbb{Z}^3$	$*432$	48	—
2	bcc	$*432$	48	—
3	fcc	$*432$	48	—
4	Hexagonal	$*622$	24	12fold (48)
5	Tetragonal prim.	$*422$	16	8fold (32)
6	Tetragonal body-c.	$*422$	16	8fold (32)
7	Rhombohedral	$2*3$	12	6fold (24) / 12fold(48)
8	Orthorhombic prim.	$*222$	8	4fold (16)
9	Orthorhombic base-c.	$*222$	8	4fold (16)
10	Orthorhombic body-c.	$*222$	8	4fold (16)
11	Orthorhombic face-c.	$*222$	8	4fold (16)
12	Monoclinic prim.	$2*$	4	2fold (8)/4fold(16)
13	Monoclinic base-c.	$2*$	4	2fold (8)/4fold(16)
14	Triclinic prim.	2	2	[monocl.(4)] / 2fold (8)


1
2
3
4
a ≠ c
a ≠ c
α=β=γ ≠ 90°
α ≠ 90°
β, γ = 90°
α ≠ 90°
β, γ = 90°
5
6
7
12
13
a ≠ b ≠ c
a ≠ b ≠ c
a ≠ b ≠ c
a ≠ b ≠ c a ≠ b ≠ c
α, β, γ ≠ 90°
8
9
10
11
14

Todo:

- Rhombic lattices
- Even more symmetry: $[S(F) : P(\Gamma)] > 2$
- Higher dimensions ($d \geq 4$)
- Hyperbolic spaces

- Fractal dimension of the boundaries
- Connectivity

- Better matchings
- ...


Thank you.

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- rotations and reflections ($d = 2$)
- rotations, reflections and rotoreflections ($d = 3$)

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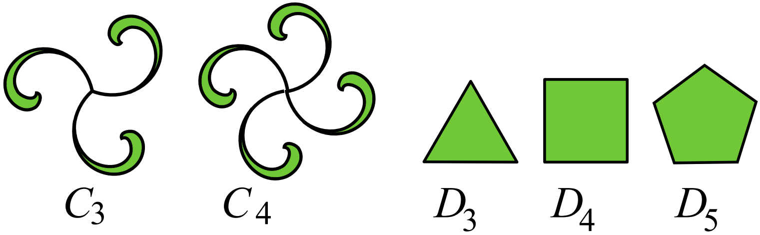
$\mathcal{C}_n$: cyclic group of order n, $\mathcal{D}_n$: dihedral group of order $2n$.



Crystallographic restriction: Rotational symmetries of 2-dim and 3-dim lattices are either 2-fold, 3-fold, 4-fold, or 6-fold.

The crystallographic restriction yields

$d = 2$: 10 candidates: $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_6$

$d = 3$: 32 candidates.

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Further considerations* yield: Only 4 lattice point groups in $\mathbb{R}^2$:

$$\mathcal{C}_2, \mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6 \qquad (2, *2, *4, *6 \text{ in orbifold notation})$$

(*: since, for instance, $x \mapsto -x$ is symmetry of any lattice)

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Only 7 lattice point groups in $\mathbb{R}^3$:

$$\mathcal{C}_2, \mathcal{D}_2, \mathcal{D}_2 \times \mathcal{C}_2, \mathcal{D}_3 \times \mathcal{C}_2, \mathcal{D}_4 \times \mathcal{C}_2, \mathcal{D}_6 \times \mathcal{C}_2, \text{cube group}$$

$$(2, *2, *222, 2*3, *422, *622, *432 \text{ in orbifold notation})$$