Bounded distance equivalence of Pisot substitution tilings

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Fractal Geometry and Stochastics 6

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Bad Herrenalb

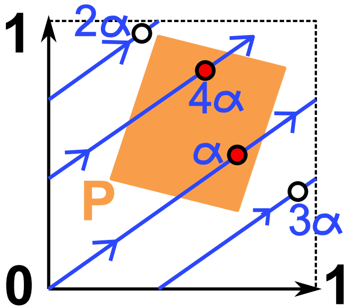
- Bounded remainder sets
- Bounded distance equivalence
- Cut-and-project sets
- Pisot substitution tilings
- ...and how they are connected

Consider a subset $P \subset [0,1]^d$.
Fix a very irrational* $\alpha \in \mathbb{R}^d$ and count how often

$$\alpha \bmod 1, 2\alpha \bmod 1, \ldots, n\alpha \bmod 1$$

hits P. Call these numbers $h(n)$.

(*: $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \notin \mathbb{Q}$, $\alpha_i/\alpha_j \notin \mathbb{Q}$ for $i \neq j$)



Then in many cases (e.g. P is a polygon)

$$\frac{h(n)}{n} \to \mathrm{vol}(P)$$

(In other words: $|h(n) - n \cdot \mathrm{vol}(P)| \in o(n)$)

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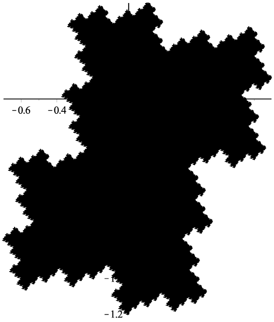
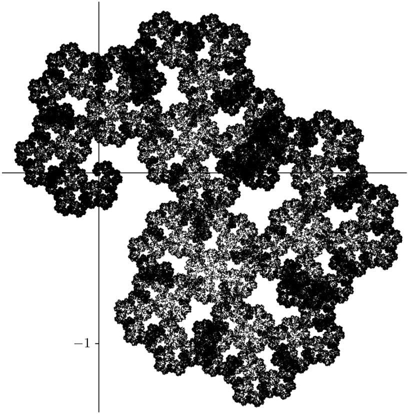
(There are some technicalities regarding the choice of the starting point, including a "for almost all", but for this talk this is irrelevant)

Question: Is this one a BRS?



Or this one?

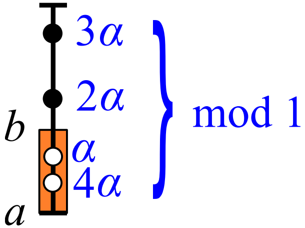


In Dimension 1:

Theorem (Kesten 1966)

Let $\alpha \in [0,1]$, $0 \le a < b \le 1$. Then $[a,b]$ is a BRS wrt α if and only if $b - a \in \mathbb{Z} + \alpha\mathbb{Z}$.

(if-part: Hecke 1921, Ostrowski 1927)



There is an analogue of Kesten's theorem in higher dimensions:

Theorem (Grepstad-Lev 2015)

Let $\alpha \in \mathbb{R}^d$ be very irrational.

1. Any parallelepiped spanned by vectors $v_1, \ldots, v_k$ belonging to $\mathbb{Z}^d + \alpha\mathbb{Z}$ is a BRS.

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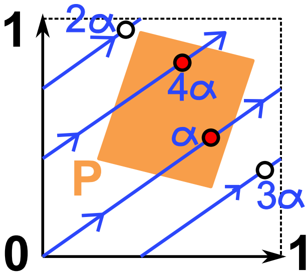
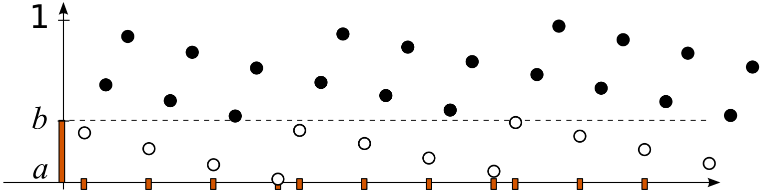
2. A Riemann measurable set $S \in \mathbb{R}^d$ is a BRS wrt α if and only if S is $(\mathbb{Z}^d + \alpha\mathbb{Z})$-equidecomposable to some parallelepiped spanned by vectors in $\mathbb{Z}^d + \alpha\mathbb{Z}$.



Back to dimension 1: Let us take a different viewpoint:



The image shows $\{(k, k\alpha \bmod 1) \mid k = 0, 1, 2, \ldots\}$.
Let $\Lambda_b = \{k \mid 0 \le k\alpha \bmod 1 \le b,\ k \in \mathbb{Z}\}$.

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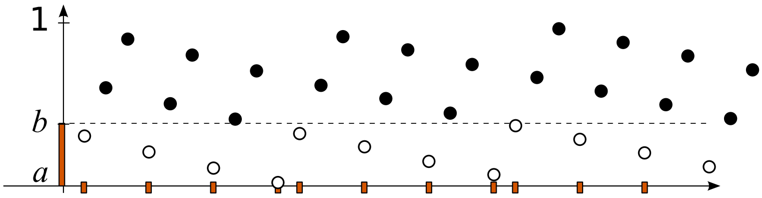

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This image (plus some arguments) yields:

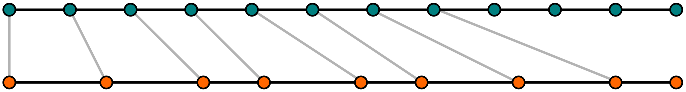
The interval $[0, b]$ is a BRS $\Leftrightarrow$ Λ_b is in bounded distance to $\frac{1}{b}\mathbb{Z}$.

Let us consider discrete point sets Λ on the line (*Delone sets*).

Definition

Let Λ, Λ' be Delone sets. We say that Λ and Λ' are bounded distance equivalent ($\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$) if there is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$



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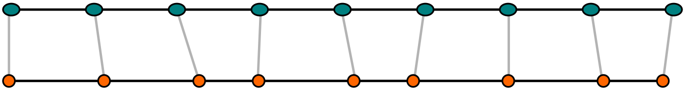

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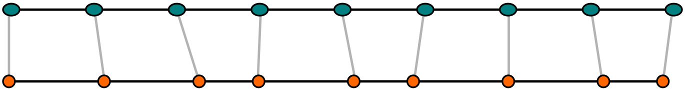

Lemma

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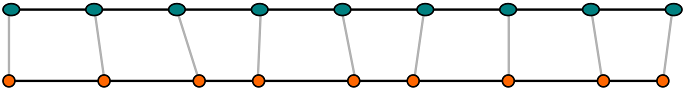


Lemma

Bounded distance equivalence is an equivalence relation.

Where are we: Given a Delone set as above,

$$\text{The interval } [0, b] \text{ is a BRS} \iff \Lambda \overset{\mathrm{bd}}{\sim} \frac{1}{b}\mathbb{Z}.$$

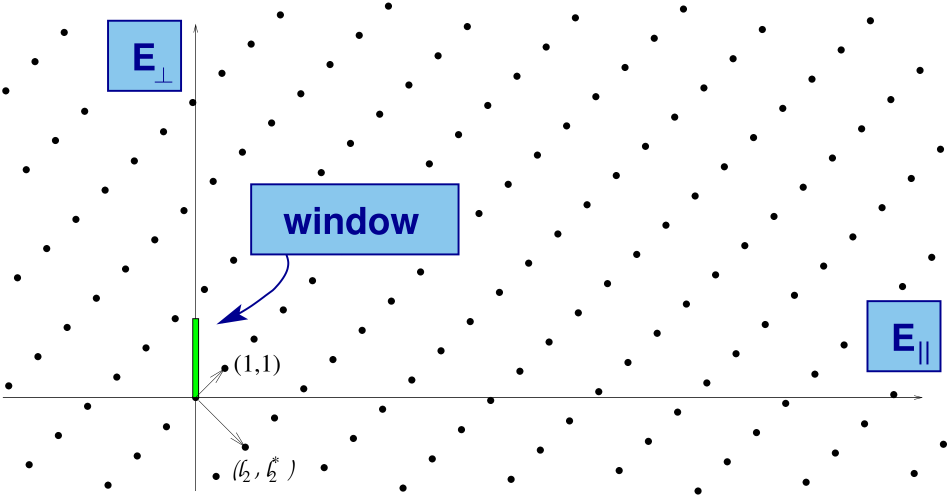
Particularly nice Delone sets: *Cut-and-Project Sets*

$$\begin{array}{ccccc} E_{||} = \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^e & \xrightarrow{\pi_2} & \mathbb{R}^e = E_{\perp} \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

- Γ a *lattice* in $\mathbb{R}^d \times \mathbb{R}^e$
- π_1, π_2 *projections*
  - $\pi_1|_\Gamma$ injective
  - $\pi_2(\Gamma)$ dense
- W *compact* ("window", somehow nice, e.g. ∂W has zero measure)

Then $\Lambda = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set* (CPS).

Cut-and-Project Sets



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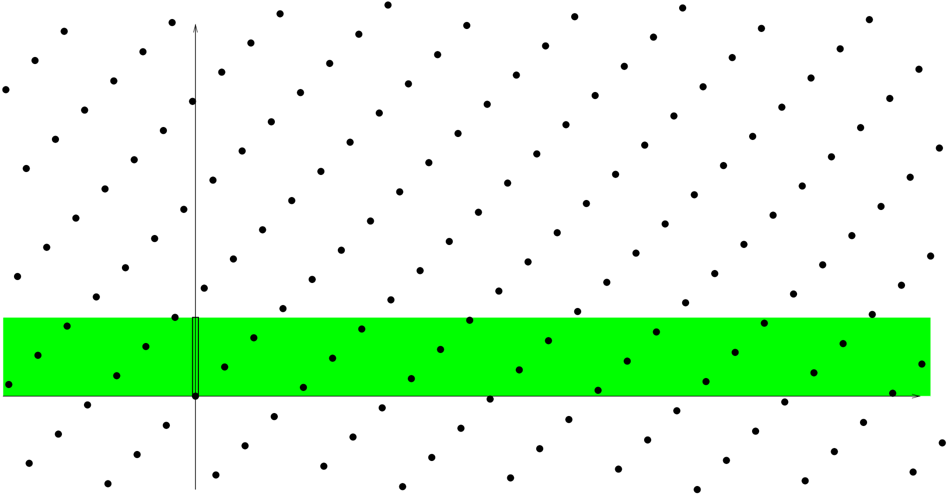

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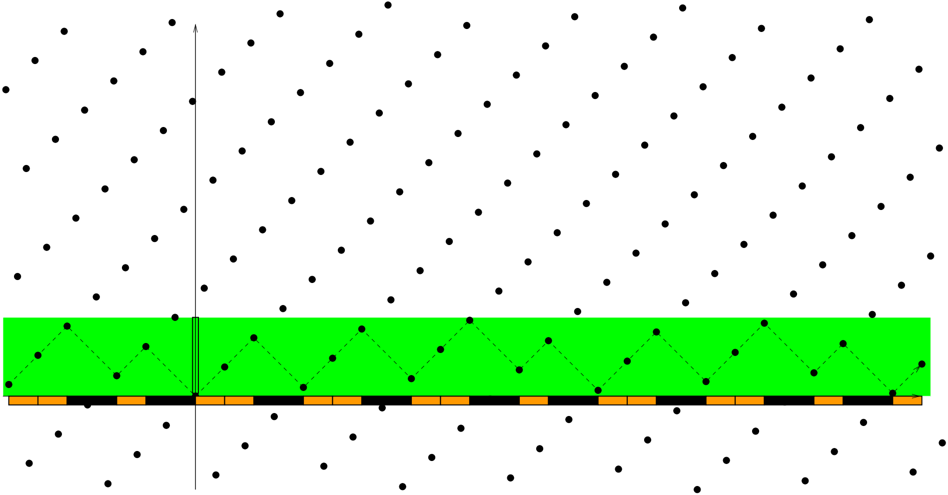

The last one uses $d = e = 1$ $(\mathbb{R}^1 \times \mathbb{R}^1)$.

An example with $d = 1$, $e = 2$ $(\mathbb{R}^1 \times \mathbb{R}^2)$.:

$$\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML$$

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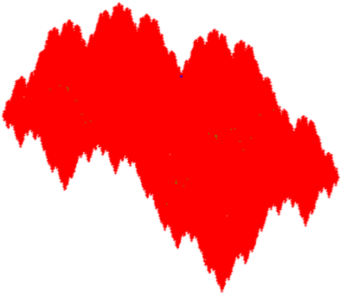
...uses a window W that looks like a fractal:



The result above holds in any dimension:

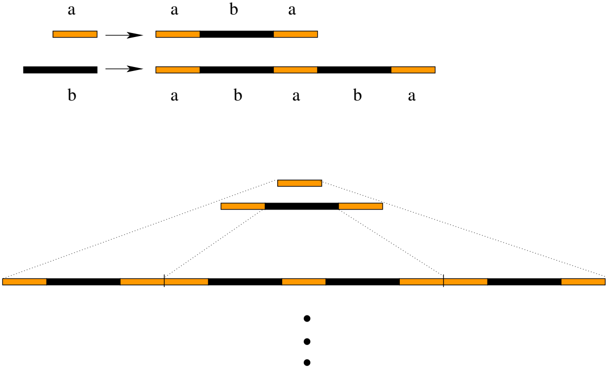
Theorem (1)

Let Λ be a cut-and-project set in $\mathbb{R}$ with window $W \subset \mathbb{R}^d$. Then $\Lambda \overset{\mathrm{bd}}{\sim} c\mathbb{Z}$ if and only if W is a BRS (where $c = \frac{1}{\mathrm{dens}(\Lambda)}$).

(Implicitly in Duneau-Oguey 1990, explicitly in Haynes 2014, F-Garber 2018, elsewhere?)

Pisot substitutions

A one-dimensional *tile substitution* producing tilings of the line by intervals. The endpoints form some Delone set.



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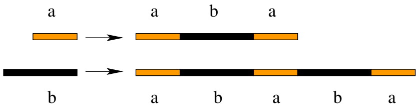

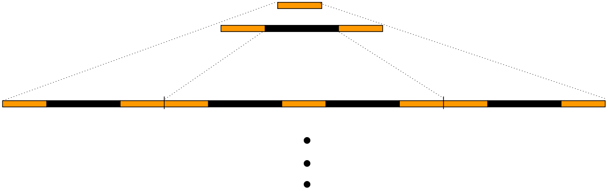

- $M_\sigma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$
- Inflation factor $\lambda = 2 + \sqrt{3}$
- $\text{length}(a) = 1$, $\text{length}(b) = \sqrt{3}$

A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution* if all eigenvalues of M_σ other than λ are less than one in modulus.

E.g. the examples above (with S,M,L, resp. a,b) are Pisot substitutions.

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Theorem (2)(F-Garber 2018)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to $c\mathbb{Z}$ (for some $c > 0$).

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Unfortunately:

Theorem (2)(Holton-Zamboni 1998, Dumont 1990)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to $c\mathbb{Z}$ (for some $c > 0$).

However, let's go on. Here is the Pisot conjecture:

Conjecture

All Pisot substitution tilings are cut-and-project sets.

(True for two tiles resp. letters, and for several small 3-letter cases)

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Finally we can assemble the results above:

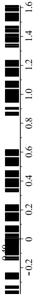
Theorem

Assuming the Pisot conjecture, the window of any Pisot tiling is a BRS (wrt to a certain α coming from the cut-and-projet setup).

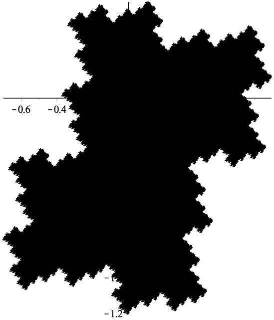
Let Λ be the vertex set of a Pisot substitution tiling.

- Theorem (2) implies $\Lambda \stackrel{\text{bd}}{\sim} c\mathbb{Z}$.
- Under the Pisot conjecture Λ is a cut-and-project set and has a window W
- By Theorem (1) the window W is a BRS

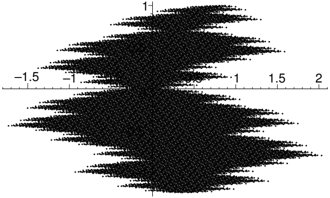
This result yields several non-trivial BRS
(beyond Kesten, and hard to decide by Grepstad-Lev)



$a \to aab$
$b \to ba$

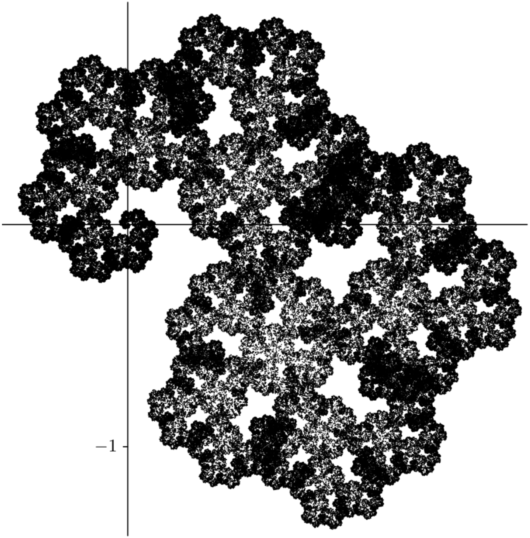


$a \to abc, b \to ab, c \to b$



$a \to abc, b \to ab, c \to a$

A particularly fuzzy BRS:



$$a \to b, \quad b \to c, \quad c \to ab$$

There is still much to explore. More here:

D.F., Alexey Garber:
Pisot substitution sequences, one dimensional cut-and-project sets and bounded remainder sets with fractal boundary,
Indagationes Mathematicae 29 (2018) 1114-1130
arXiv:1711.01498

and references therein.


Thank you.