Dynamical properties of tiling spaces with statistical circular symmetry (Dynamische Eigenschaften von Parketträumen mit statistischer Kreissymmetrie)

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1. Tiling spaces (Parketträume)
2. Statistical circular symmetry
3. Examples of tilings with statistical circular symmetry
4. Dynamics of tilings with statistical circular symmetry

1. Tiling spaces

Let $\mathcal{T}, \mathcal{T}'$ be tilings of the plane $\mathbb{R}^2$.

tiling metric: $\mathcal{T}$ and $\mathcal{T}'$ are ε-close:
$\mathcal{T} + x$ and $\mathcal{T}' + y$ agree on $B_{1/\varepsilon}(0)$ for $|x|, |y| < \varepsilon/2$.

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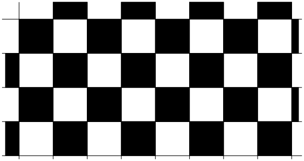
$$d(\mathcal{T}, \mathcal{T}') = \min\{2^{-1/2}, \text{supremum of these } \varepsilon\}$$

This defines a metric, which yields a topology.

The closure of $\{\mathcal{T} + x \mid x \in \mathbb{R}^2\}$ is the *tiling space* $X_{\mathcal{T}}$

First case: Periodic tilings



$X(\mathcal{T})$ is a 2D-torus.

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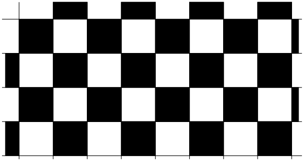


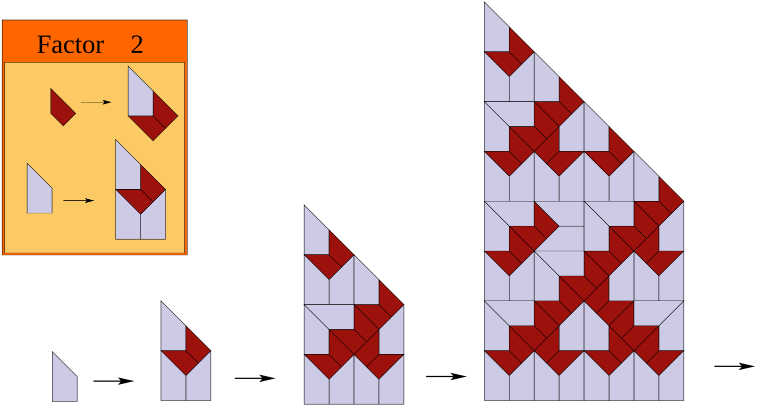
$X(\mathcal{T})$ is a 2D-torus.

Next case: Tiling of the line with white tiles and one black tile:

$X(\mathcal{T})$ is a dyadic solenoid.

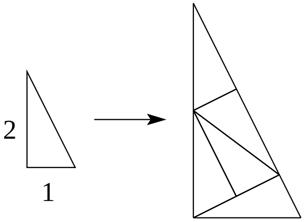
Interesting case: Nonperiodic substitution tilings.

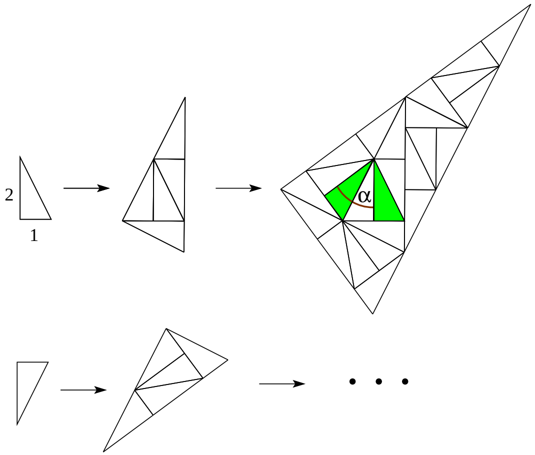
Substitution tilings:



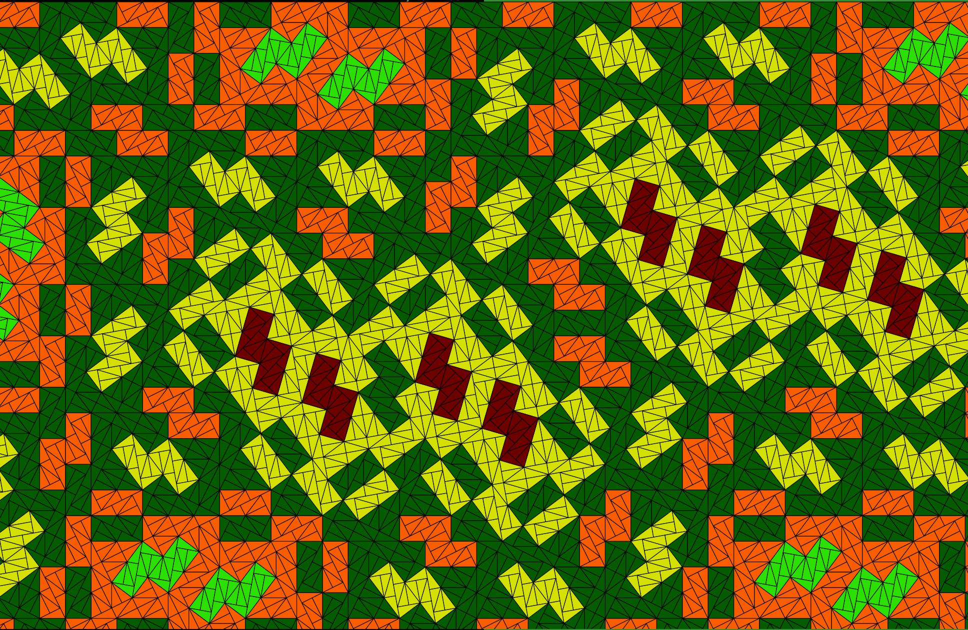
Many examples show tiles in finitely many orientations only.

But not all: Conway's Pinwheel substitution (1991):





The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



2. Statistical circular symmetry

Definition

A tiling has TIMOR *(***T***iles in* **I***nfinitely* **M***any* **OR***ientations), if some tile type occurs in infinitely many orientations.*

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True for the pinwheel. Even more is known:

Theorem (Radin '95, see also Moody-Postnikoff-Strungaru '06)

The pinwheel tiling is of statistical circular symmetry, i.e. (roughly spoken) the orientations are equidistributed on the circle.

Recall: $(\alpha_j)_j$ is *equidistributed* in $[0, 2\pi[$, if for all $0 \le x < y < 2\pi$ holds:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[x,y]}(\alpha_j) = \frac{x - y}{2\pi}$$

Because the sum is not absolutely convergent, the order matters!

Recall: $(\alpha_j)_j$ is *equidistributed* in $[0, 2\pi[$, if for all $0 \le x < y < 2\pi$ holds:

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Because the sum is not absolutely convergent, the order matters!

Definition

A substitution tiling $\mathcal{T} = \{T_1, T_2, \ldots\}$ is of statistical circular symmetry, (SCS) if

- *for each n exists $\ell \ge n$ such that $\{T_1, \ldots, T_\ell\}$ is congruent to some supertile $\sigma^k(T_i)$, and*
- *for all $0 \le x < y < 2\pi$ holds:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[x,y]}(\angle(T_j)) = \frac{x-y}{2\pi}$$

Probably, this Def can be made simpler for primitive substitution tilings (order tiles wrt distance to 0).

Theorem (F. '08)

Each primitive substitution tiling with TIMOR is of statistical circular symmetry.

Proof uses just Perron's theorem, Weyl's Lemma and a technical result:

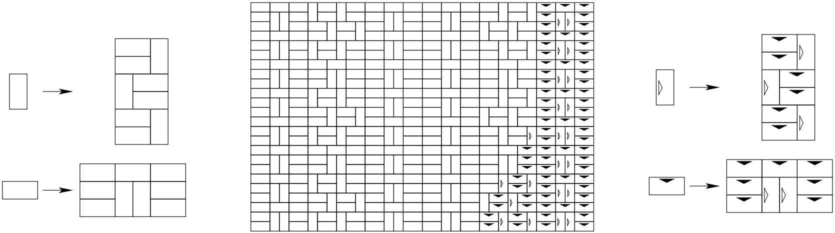
$$\text{“Bad angles”} \Leftrightarrow \text{TIMOR}$$

(Clear: Bad angles $\Rightarrow$ TIMOR)

Btw:

Theorem (F. '08)

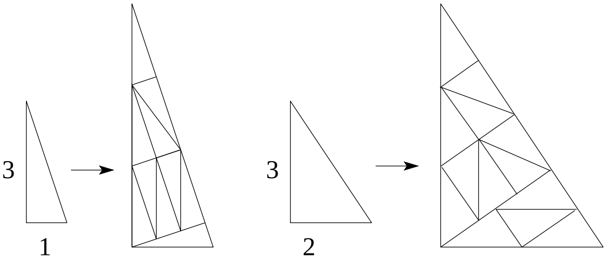
In each primitive substitution tiling, each prototile occurs with the same frequency in each of its orientations.



2. Examples of substitution tilings with SCS

Just seen: Conway's Pinwheel tilings.

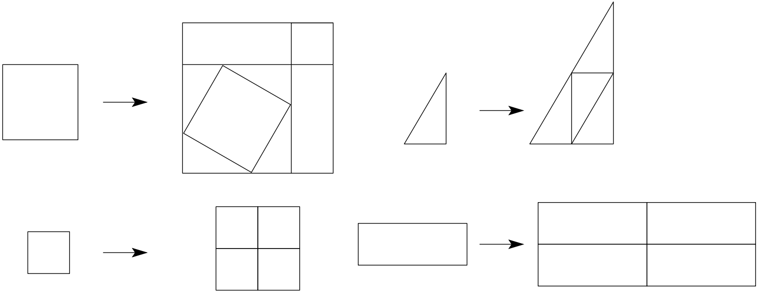
Obvious generalizations: Pinwheel (n, k)



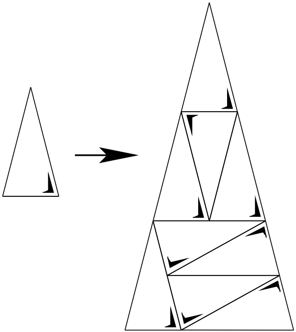
$n = 3, k = 1$ $\qquad$ $n = 3, k = 2$

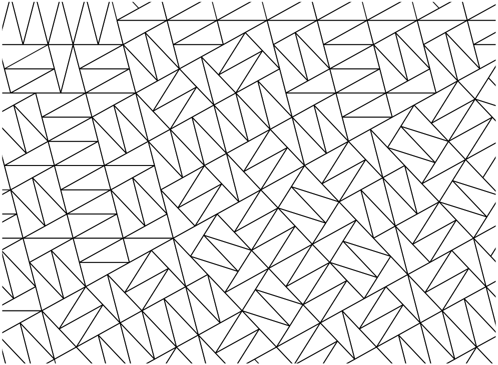
etc.

Cesi's example (1990):



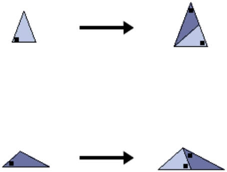
Penrose (< 1995)

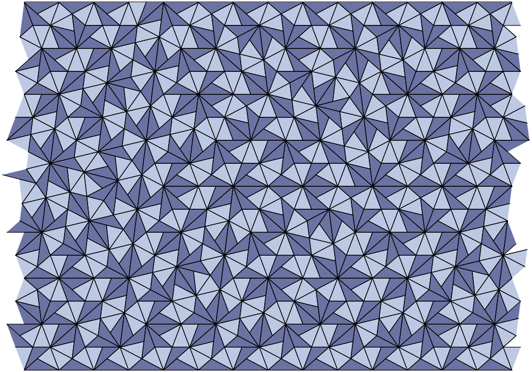




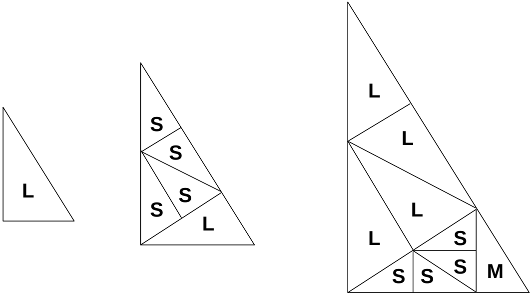
(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):



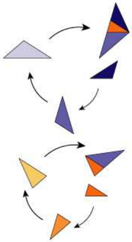


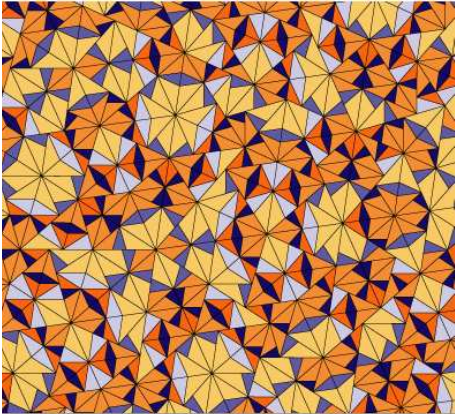
Sadun's generalized Pinwheels (1998):



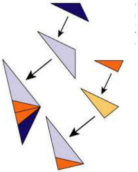
Yields infinitely many proper tile-substitutions.

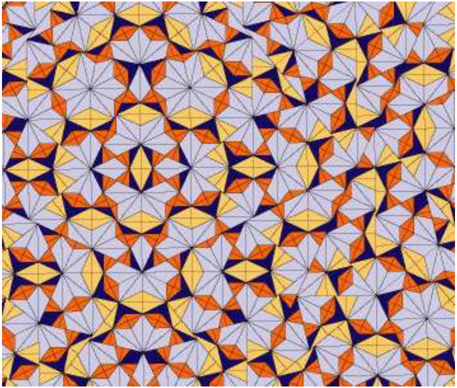
Harriss' Cubic Pinwheel (2004 ± 1):



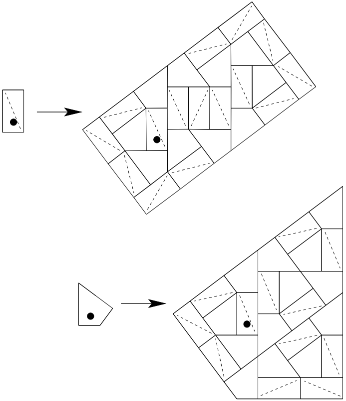


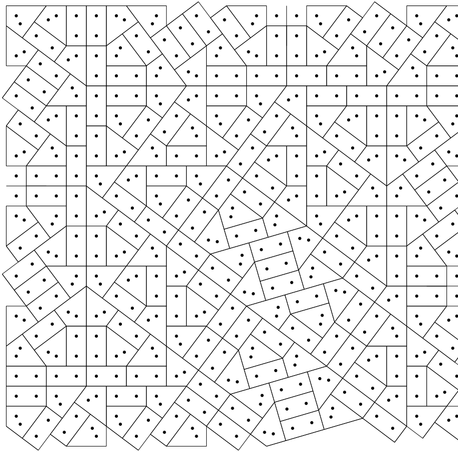
Harriss' Quartic Pinwheel (2004 ± 1):



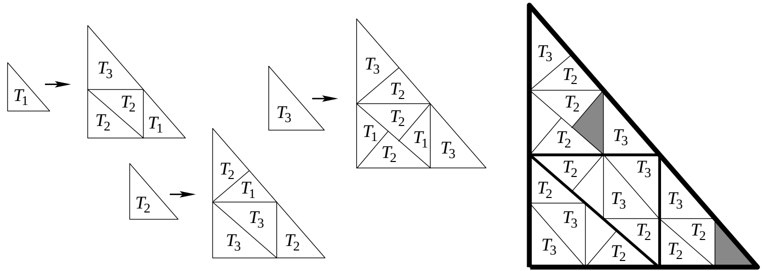


Kite domino (equivalent with pinwheel):





Pythia (m,j), here: $m = 3, j = 1$.



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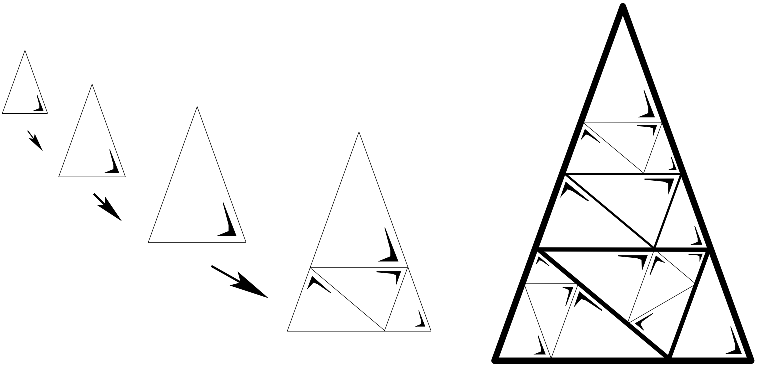
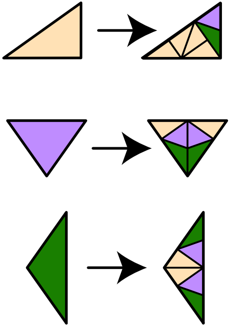
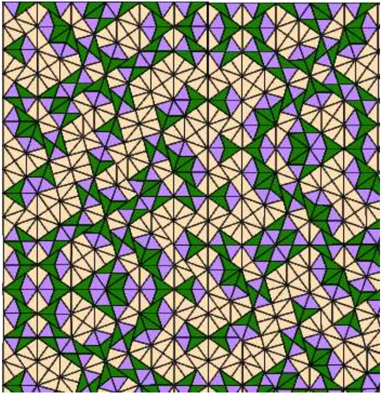


Dale Walton: several single examples





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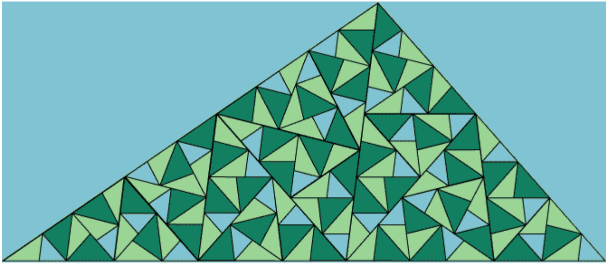
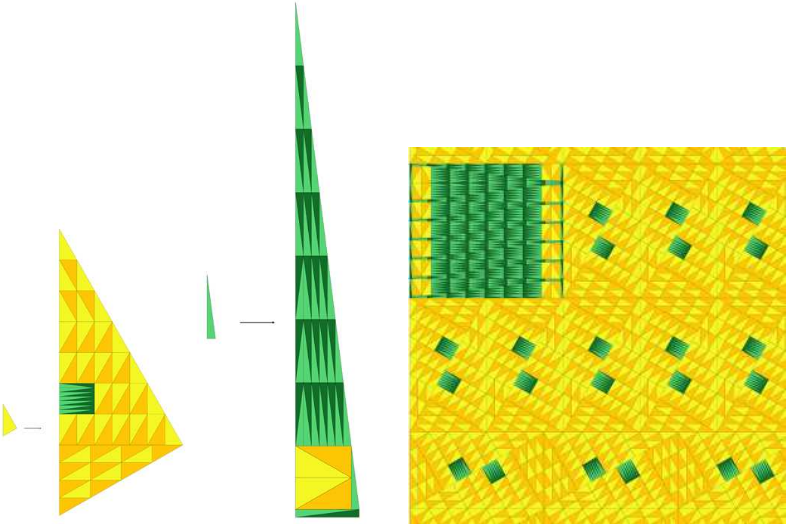


The Uberpinwheel: orientation indexed by *two* parameters



4. Dynamics

The tiling space $\mathbb{X}_{\mathcal{T}}$ of a tiling $\mathcal{T}$:

- wrt *translations*: the closure of $G\mathcal{T}$, where G is the group of translations in $\mathbb{R}^2$
- wrt *Euclidean motions*: the closure of $G\mathcal{T}$, where G is the group of Euclidean motions in $\mathbb{R}^2$

'Closure' wrt an appropriate topology, e.g.

- tiling top
- wiggle top
- local rubber top

- *tiling top*: (as above) $\mathcal{T}$ and $\mathcal{T}'$ are ε-close: $\mathcal{T}+x$ and $\mathcal{T}'+y$ agree on $B_{1/\varepsilon}(0)$ for $|x|,|y|<\varepsilon/2$.
- *wiggle top*: $\mathcal{T}$ and $\mathcal{T}'$ are ε-close: $R_\alpha\mathcal{T}+x$ and $\mathcal{T}'+y$ agree on $B_{1/\varepsilon}(0)$ for $|x|,|y|<\varepsilon/2$, $|\alpha|<\varepsilon$.
- *local rubber top* (for discrete point sets): Λ and Λ' are ε-close: Λ and Λ' agree on $B_{1/\varepsilon}(0)$, after moving each point *individually* by an amount $<\varepsilon$.

For primitive substitution tilings without TIMOR: All three yield the same hull.

For those with TIMOR: tiling top not appropriate!
Then: $\mathbb{X}_{\mathcal{T}}$ not compact, $(\mathbb{X}_{\mathcal{T}}, G)$ not ergodic...

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For primitive substitution tilings of FLC: wiggle top and local rubber top yield the same hull.

Then, in nice cases:

- $\mathbb{X}_{\mathcal{T}}$ compact
- $(\mathbb{X}_{\mathcal{T}}, E(2))$ minimal
- $(\mathbb{X}_{\mathcal{T}}, E(2))$ uniquely ergodic

where $E(2)$ denotes the Euclidean motions in $\mathbb{R}^2$.

What does "nice" mean?

An *r-patch* is $\mathcal{T} \cap B_r(x)$ (all tiles in $\mathcal{T}$ contained in some ball of radius r)

- *finite local complexity* (FLC): For each $r > 0$ there are finitely many different r-patches

- *Repetitive*: For all $r > 0$, each r-patch occurs relatively dense
- *Linearly repetitive*: There is $c > 0$, s.t. for all r, each r-patch occurs in each ball of radius cr.

- *uniform patch frequency* (UPF): For all van Hove sequences $(F_n)_n$, and for all patches $P \subset \mathcal{T}$

$$\mathrm{freq}(P) := \lim_{r \to \infty} \frac{1}{\mathrm{vol}\, F_n} \#\{P' \subset \mathcal{T} \cap F_n \,|\, P' \equiv P \text{ for some } t \in \mathbb{R}^d\}$$

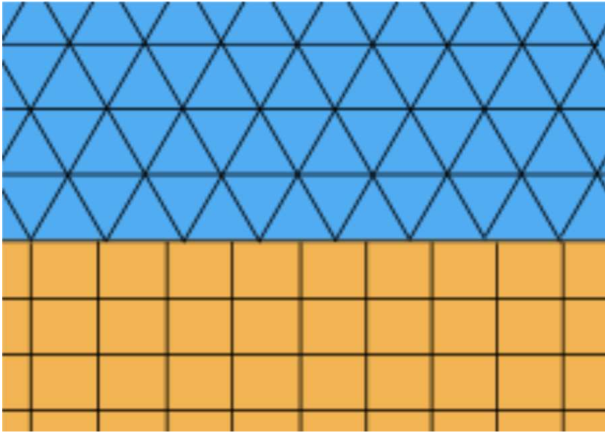
exists, and is independent of the choice of $(F_n)_n$.

van Hove sequence: Sequence of sets $F_n \subset \mathbb{R}^d$, such that

$$\lim_{n\to\infty} \frac{\operatorname{vol} \partial^{+\delta} F_n}{\operatorname{vol} F_n} = 0 \quad \text{for all } \delta > 0,$$

where $\partial^{+\delta} F_n := \{x \in \mathbb{R}^d \mid d(x, \partial F) < \delta\}$.

Example:



By no means nice.... neither FLC, nor repetitive, nor UPF.

"Classical" results: ($\mathcal{T}$ has not TIMOR)

- $\mathcal{T}$ FLC then $\mathbb{X}_{\mathcal{T}}$ compact (Radin-Wolff? Solomyak 97)
- $\mathcal{T}$ repetitive iff $(\mathbb{X}_{\mathcal{T}}, \mathbb{R}^2)$ minimal (Radin-Wolff? Solomyak 97)
- $\mathcal{T}$ UPF then $(\mathbb{X}_{\mathcal{T}}, \mathbb{R}^2)$ uniquely ergodic (Solomyak 97, Schlottmann 98)

Lagarias-Pleasants: linear rep implies UPF

"New" results:

- $\mathcal{T}$ E(2)-FLC then $\mathbb{X}_{\mathcal{T}}$ compact (analoguous)
- $\mathcal{T}$ E(2)-rep (or wiggle-rep) iff $(\mathbb{X}(\mathcal{T}), E(2))$ minimal (Yokonuma 05, F 08)
- For $\mathcal{T}$ FLC: $\mathcal{T}$ E(2)-UPF iff $(\mathbb{X}(\mathcal{T}), E(2))$ uniquely ergodic (Müller-Richard)

Aim: Show for FLC primitive substitution tilings $\mathcal{T}$ with TIMOR

$$(\mathbb{X}(\mathcal{T}), E(2)) \text{ is uniquely ergodic.}$$

Done: Just use a classical result on UPF for prim subst tilings and Müller-Richard

Aim: Show for FLC primitive substitution tilings $\mathcal{T}$ with TIMOR

$$(\mathbb{X}(\mathcal{T}), \mathbb{R}^2) \text{ is uniquely ergodic. } (\mathbb{X}(\mathcal{T}) \text{ wrt wiggle-top})$$

Plan A:

- SCS and $E(2)$-UPF imply linear wiggle-repetitivity
- Along Lagarias-Pleasants: lin wiggle-rep implies wiggle-UPF
- Apply Müller-Richard to show that $(\mathbb{X}(\mathcal{T}), \mathbb{R}^2)$ is uniquely ergodic

Aim: Show for FLC primitive substitution tilings $\mathcal{T}$ with TIMOR

$$(\mathbb{X}(\mathcal{T}), \mathbb{R}^2) \text{ is uniquely ergodic. } (\mathbb{X}(\mathcal{T}) \text{ wrt wiggle-top})$$

Plan B:

- SCS and $E(2)$-UPF imply wiggle-UPF
- Apply Müller-Richard to show that $(\mathbb{X}(\mathcal{T}), \mathbb{R}^2)$ is uniquely ergodic

Aim: Show for FLC primitive substitution tilings $\mathcal{T}$ with TIMOR

$$(\mathbb{X}(\mathcal{T}), \mathbb{R}^2) \text{ is uniquely ergodic. } (\mathbb{X}(\mathcal{T}) \text{ wrt wiggle-top})$$

Plan B:

- SCS and $E(2)$-UPF imply wiggle-UPF
- Apply Müller-Richard to show that $(\mathbb{X}(\mathcal{T}), \mathbb{R}^2)$ is uniquely ergodic

Plan C:

- SCS and $E(2)$-UPF imply wiggle-UPF
- Show that this implies $(\mathbb{X}(\mathcal{T}), \mathbb{R}^2)$ is uniquely ergodic

Further plan: generalise Lagarias-Pleasants: linear wiggle-rep implies wiggle-UPF