About substitution tilings with statistical circular symmetry

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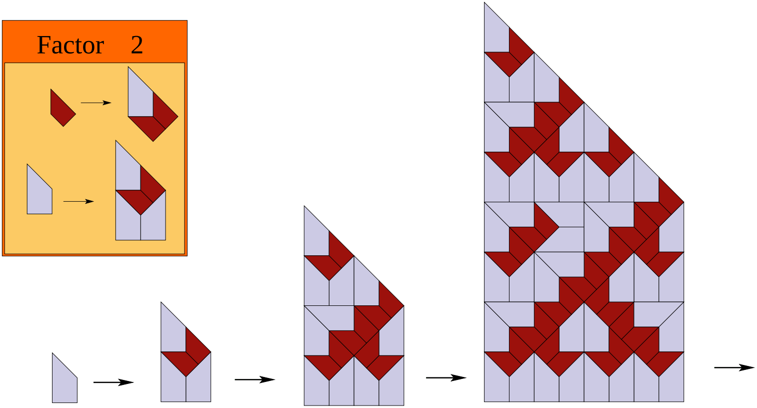
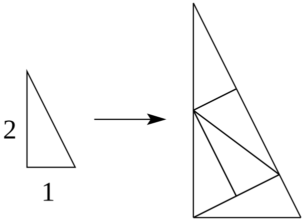
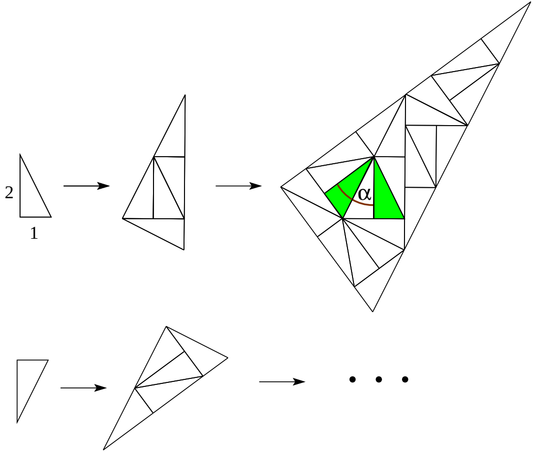
Substitution tilings:



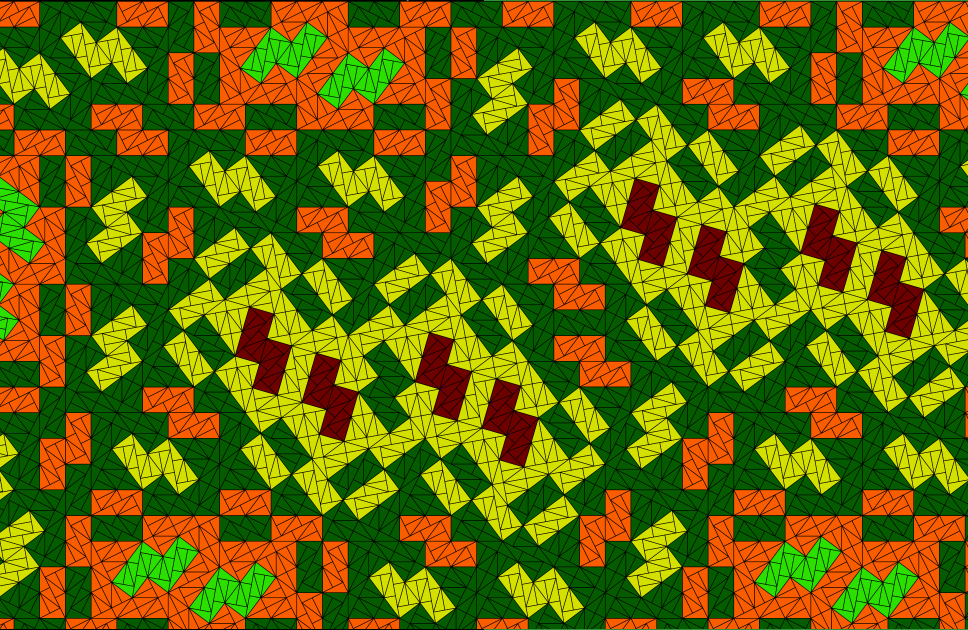
Usually, tiles occur in finitely many different orientations.

Not always. Conway's Pinwheel substitution (1991):

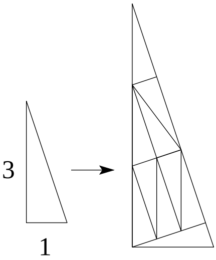




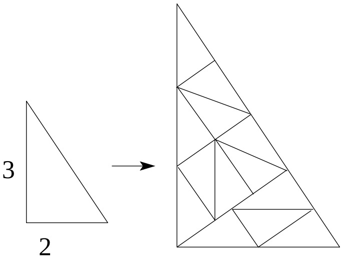
The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



Obvious generalizations: Pinwheel (n, k)



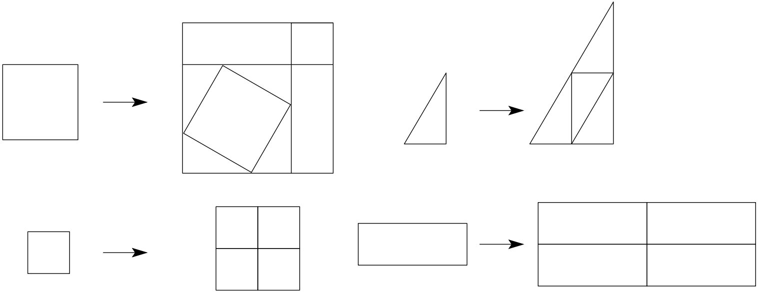
$n = 3, k = 1$



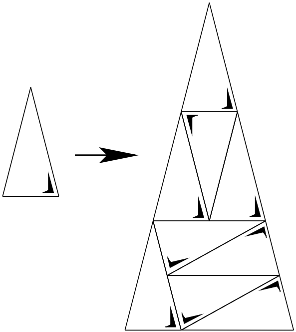
$n = 3, k = 2$

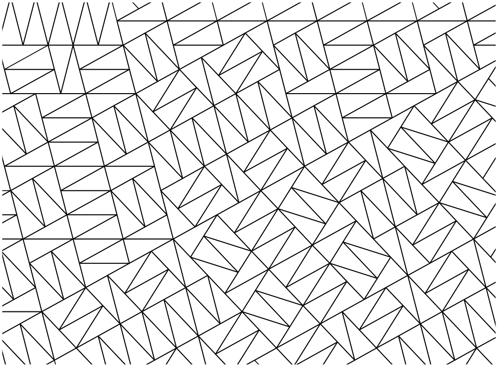
etc.

Cesi's example (1990):



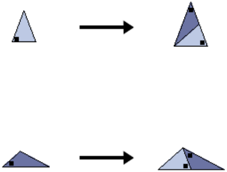
Unknown ($<$ 1996, communicated to me by Danzer):

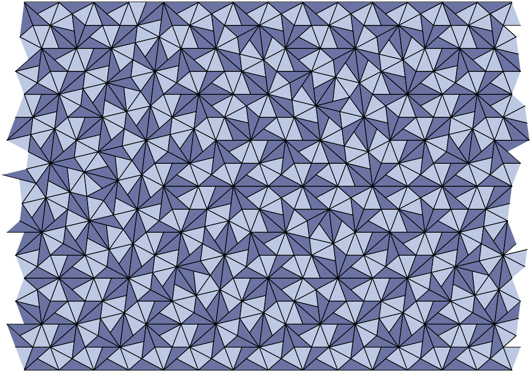




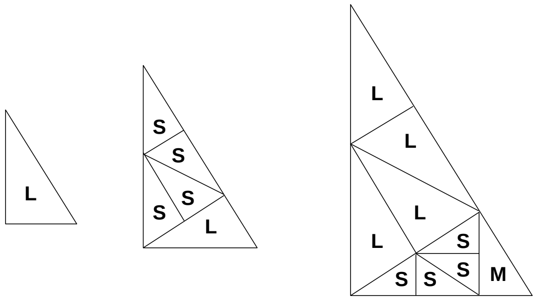
(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):



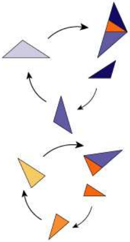


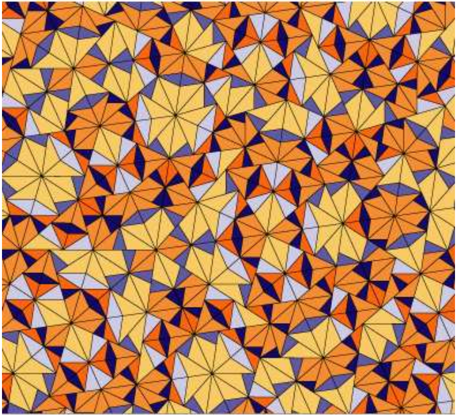
Sadun's generalized Pinwheels (1998):



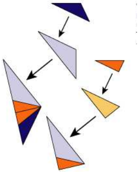
Yields infinitely many proper tile-substitutions.

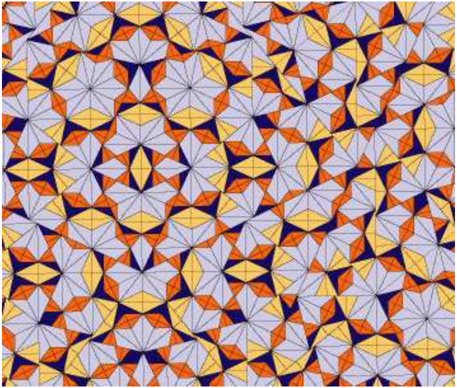
Harriss' Cubic Pinwheel (2004 ± 1):



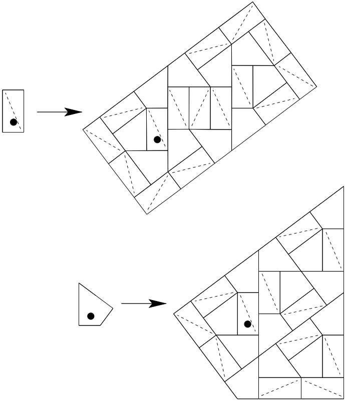


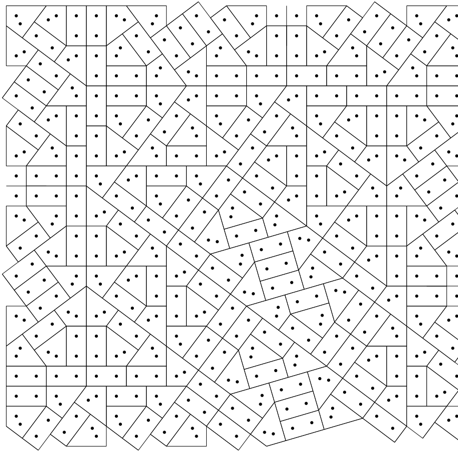
Harriss' Quartic Pinwheel (2004 ± 1):



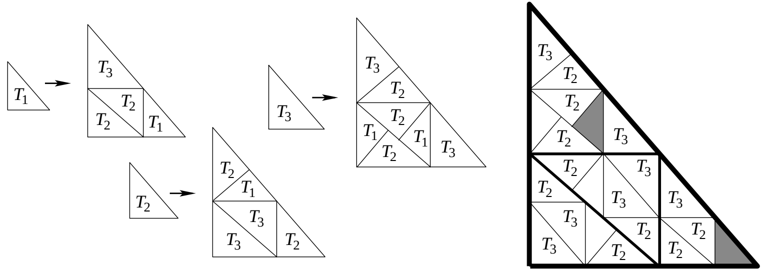


Kite domino (equivalent with pinwheel):

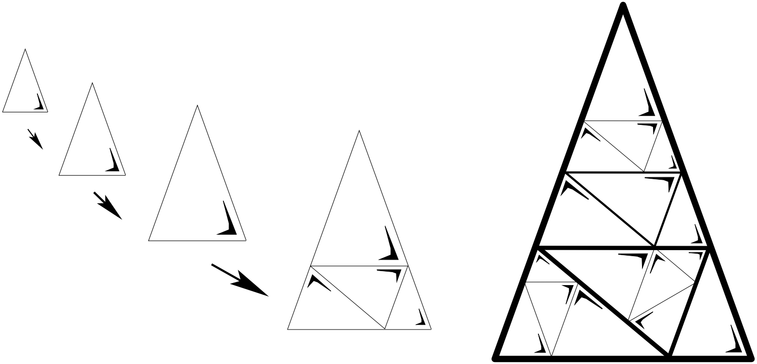




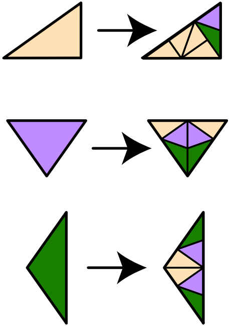
Pythia (m,j), here: $m = 3, j = 1$.

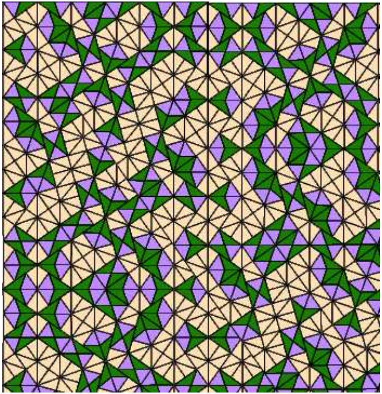


Tipi (m,j), here: $m = 3, j = 1$.



Dale Walton: several single examples





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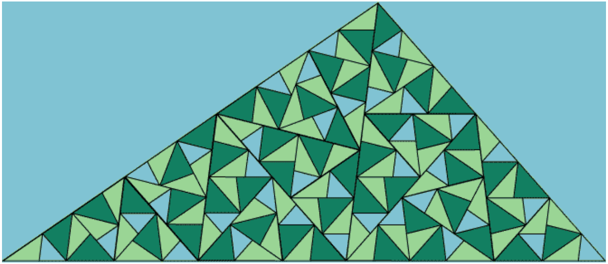
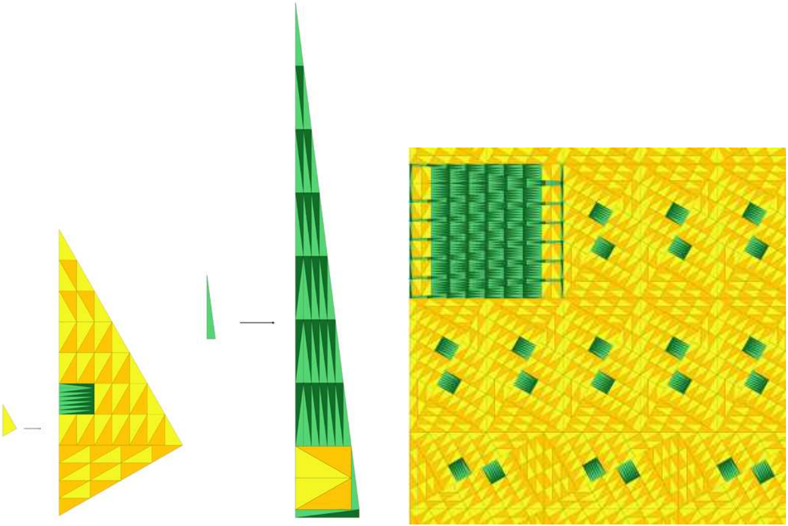


The Uberpinwheel: orientation indexed by *two* parameters



2. Statistical circular symmetry

Definition

A tiling has TIMOR *(**T**iles in **I**nfinitely **M**any **OR**ientations), if some tile type occurs in infinitely many orientations.*

Clear: If $\mathcal{T}$ is a primitive substitution tiling which has TIMOR, then *each* prototile occurs in infinitely many orientations.

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True for the pinwheel. Even more is known:

Theorem (Radin '95, see also Moody-Postnikoff-Strungaru '06)

The pinwheel tiling is of statistical circular symmetry, i.e. (roughly spoken) the orientations are equidistributed on the circle.

Recall: $(\alpha_j)_j$ is *equidistributed* in $[0, 2\pi[$, if for all $0 \le x < y < 2\pi$ holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[x,y]}(\alpha_j) = \frac{x-y}{2\pi}$$

Because the sum is not absolutely convergent, the order matters!

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Definition

A substitution tiling $\mathcal{T} = \{T_1, T_2, \ldots\}$ is of statistical circular symmetry, if

- *for each n exists $\ell \ge n$ such that $\{T_1, \ldots, T_\ell\}$ is congruent to some supertile $\sigma^k(T_i)$, and*
- *for all $0 \le x < y < 2\pi$ holds:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[x,y]}(\angle(T_j)) = \frac{x-y}{2\pi}$$

Probably, this Def can be made simpler for primitive substitution tilings (order tiles wrt distance to 0).

Theorem (F. '08)

Each primitive substitution tiling with TIMOR is of statistical circular symmetry.

Proof uses just Perron's theorem, Weyl's Lemma and a technical result:

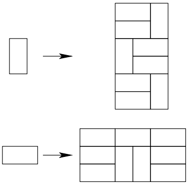
$$\text{“Bad angles”} \Leftrightarrow \text{TIMOR}$$

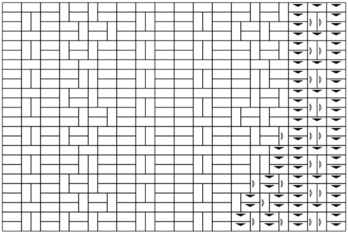
(Clear: Bad angles $\Rightarrow$ TIMOR)

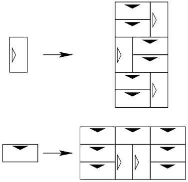
Btw:

Theorem (F. '08)

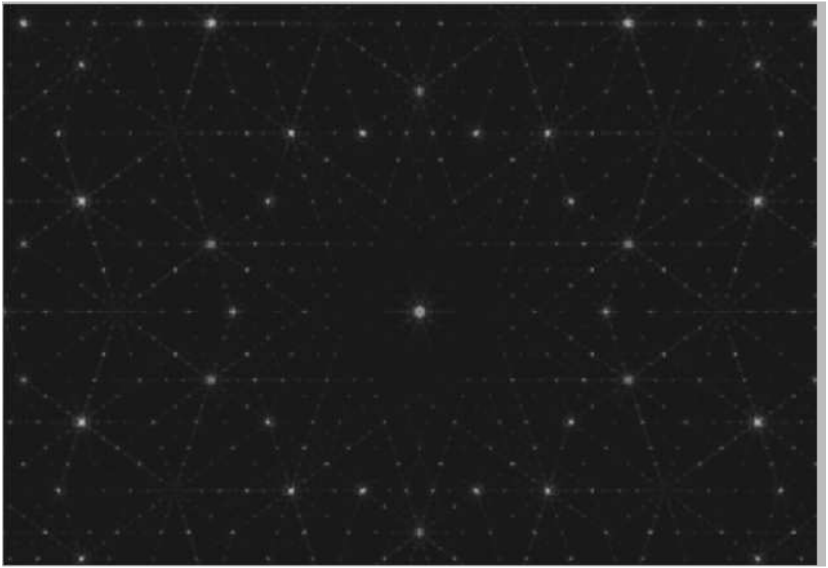
In each primitive substitution tiling, each prototile occurs with the same frequency in each of its orientations.







3. Diffraction



Mathematical description:

- Tiling $\rightsquigarrow$ discrete point set Λ.
- Autocorrelation $\gamma_\Lambda = \lim_{r \to \infty} \frac{1}{\mathrm{vol}\, B_r} \sum_{x,y \in \Lambda \cap B_r} \delta_{x-y}$.
- Fouriertransform $\widehat{\gamma}_\Lambda$ of the autocorrelation is the *diffraction spectrum*.

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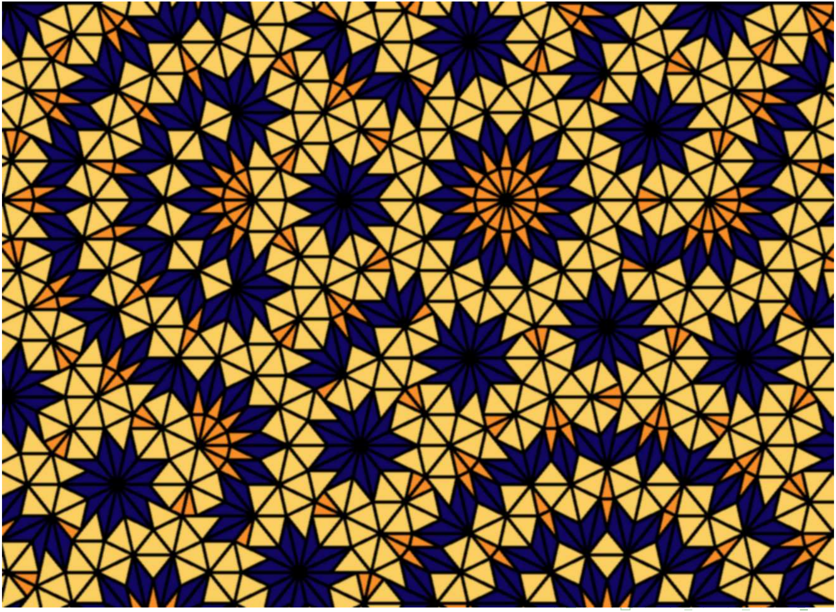
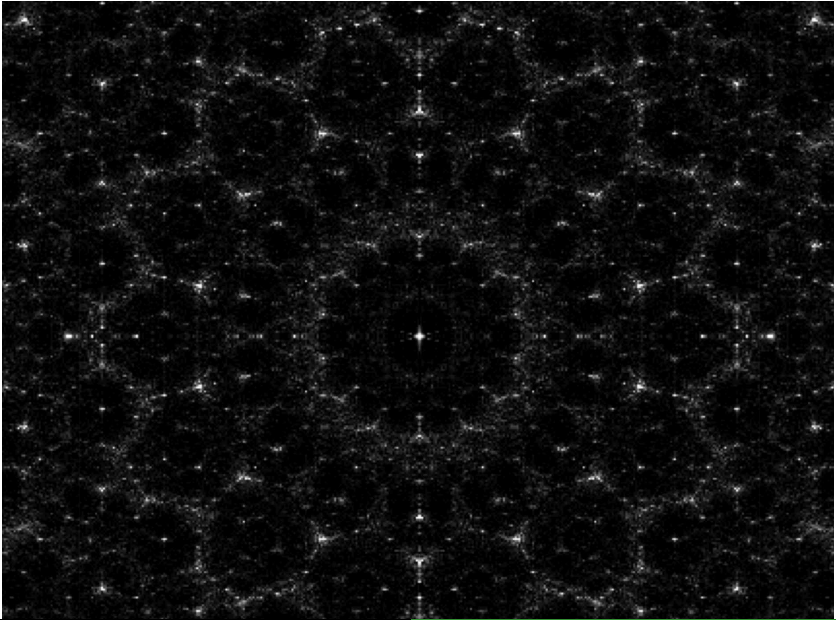
Since $\widehat{\gamma} := \widehat{\gamma}_\Lambda$ is again a measure, it decomposes into three parts:

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}$$

(pp: pure point, ac: absolutely continuous, sc: singular continuous)





$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}$$

For an ideal (mathematical, infinite) quasicrystal:

$$\widehat{\gamma} = \widehat{\gamma}_{pp}$$

For primitive substitution tilings with TIMOR:

$$\widehat{\gamma} = \delta_0 + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac},$$

and $\widehat{\gamma}$ is circular symmetric.

This follows from statistical circular symmetry:

Then, the autocorrelation is of perfect circular symmetry, and circular symmetry of the diffraction spectrum follows.

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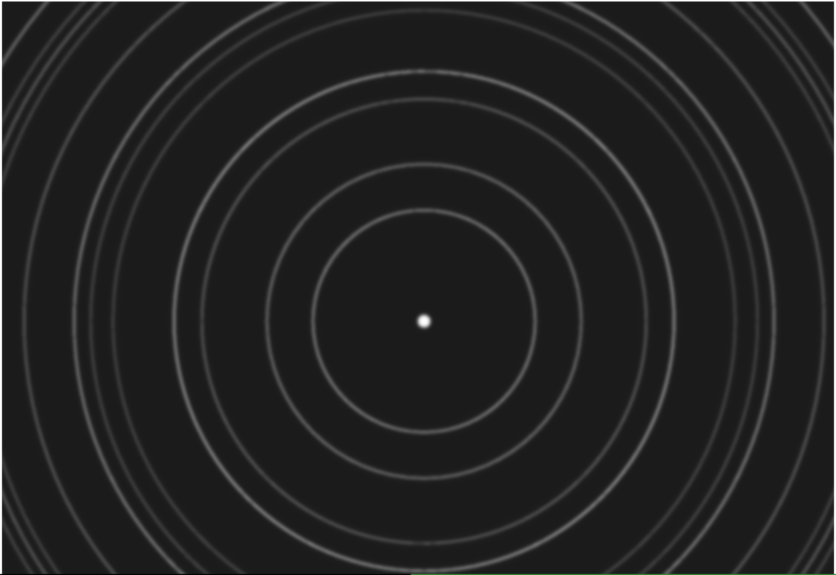
Circular symmetry of a measure implies no pure point part apart from 0.

Continuous parts are still mysterious.

Pinwheel diffraction (approximation)



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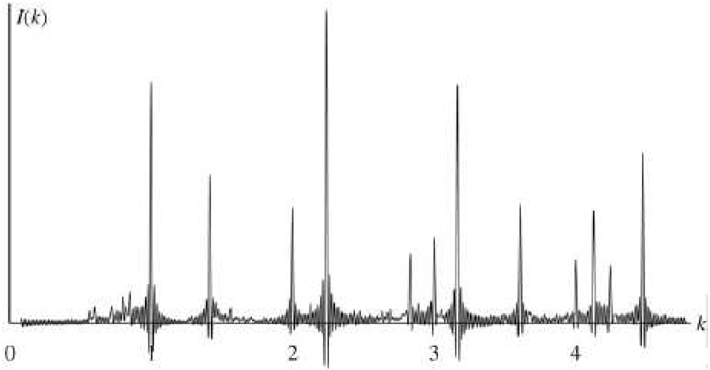

Detailled rigorous results on the diffraction: still missing (but aimed for).

Achieved: Results on powder diffraction, and on the frequency module:

- Frequencies of distances contained in $\frac{1}{264}\mathbb{Z}[\frac{1}{5}]$ [Baake-F-Grimm '07]
- Exact values, up to $\sqrt{5}$ [BFG], up to 25 [Moustafa]

4. Dynamics

The *hull* $\mathbb{X}_{\mathcal{T}}$ of a tiling $\mathcal{T}$:

- wrt *translations*: the closure of $G\mathcal{T}$, where G is the group of translations in $\mathbb{R}^d$
- wrt *Euclidean motions*: the closure of $G\mathcal{T}$, where G is the group of Euclidean motions in $\mathbb{R}^d$

'Closure' wrt an appropriate topology, e.g.

- tiling top
- wiggle top
- local rubber top

- *tiling top*: $\mathcal{T}$ and $\mathcal{T}'$ are ε-close: $\mathcal{T} + x$ and $\mathcal{T}' + y$ agree on $B_{1/\varepsilon}(0)$ for $|x|, |y| < \varepsilon/2$.
- *wiggle top*: $\mathcal{T}$ and $\mathcal{T}'$ are ε-close: $R_\alpha \mathcal{T} + x$ and $\mathcal{T}' + y$ agree on $B_{1/\varepsilon}(0)$ for $|x|, |y| < \varepsilon/2$, $|\alpha| < \varepsilon$.
- *local rubber top* (for discrete point sets): Λ and Λ' are ε-close: Λ and Λ' agree on $B_{1/\varepsilon}(0)$, after moving each point *individually* by an amount $< \varepsilon$.

For primitive substitution tilings without TIMOR: All three yield the same hull.

For those with TIMOR: local top not appropriate!
Then: $\mathbb{X}_{\mathcal{T}}$ not compact, $(\mathbb{X}_{\mathcal{T}}, G)$ not ergodic...

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For primitive substitution tilings of FLC: wiggle top and local rubber top yield the same hull.

Then, in the TIMOR case [Radin, Radin-Wolff,...]:

- $\mathbb{X}_{\mathcal{T}}$ compact
- $(\mathbb{X}_{\mathcal{T}}, E(d))$ minimal
- $(\mathbb{X}_{\mathcal{T}}, E(d))$ uniquely ergodic

where $E(d)$ denotes the Euclidean motions in $\mathbb{R}^d$.

What about $(\mathbb{X}_{\mathcal{T}}, \mathbb{R}^d)$?

Theorem (F. 2008)

Every primitive substitution tiling of FLC with TIMOR (thus statistical circular symmetry) is wiggle-repetetive.

- Repetitive: For all $r > 0$, each r-patch occurs relatively dense
- wiggle-repetetive: For all $r, \varepsilon > 0$, each r-patch occurs relatively dense, up to rotation by ε

This yields: $\mathbb{X}_{\mathcal{T}}$ compact, $(\mathbb{X}_{\mathcal{T}}, \mathbb{R}^d)$ minimal.

Work in progress: show that $(\mathbb{X}_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic

(via uniform *wiggled*-patch frequency)