Counting colour symmetries of regular tilings

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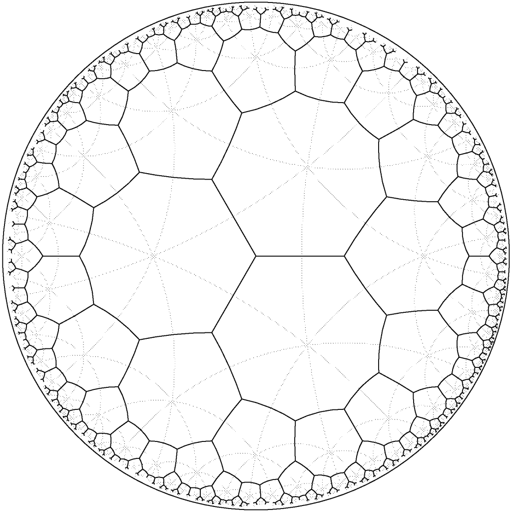
Regular tiling (p^q): edge-to-edge tiling by regular p-gons, where q tiles meet at each vertex.

In $\mathbb{R}^2$: three regular tilings: (4^4), (3^6), (6^3).

In $\mathbb{S}^2$: five regular tilings: (3^3), (4^3), (3^4), (5^3), (3^5).

In $\mathbb{H}^2$: Infinitely many regular tilings: (p^q), where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.

Regular hyperbolic tiling (8^3):



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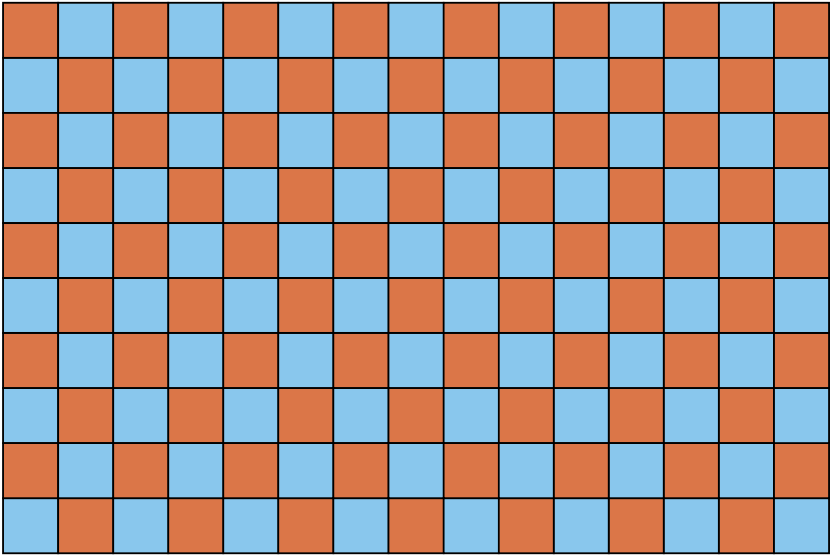
Perfect colouring Those colourings of some pattern X, where each $f \in$ Sym(X) acts as a global permutation of colours.

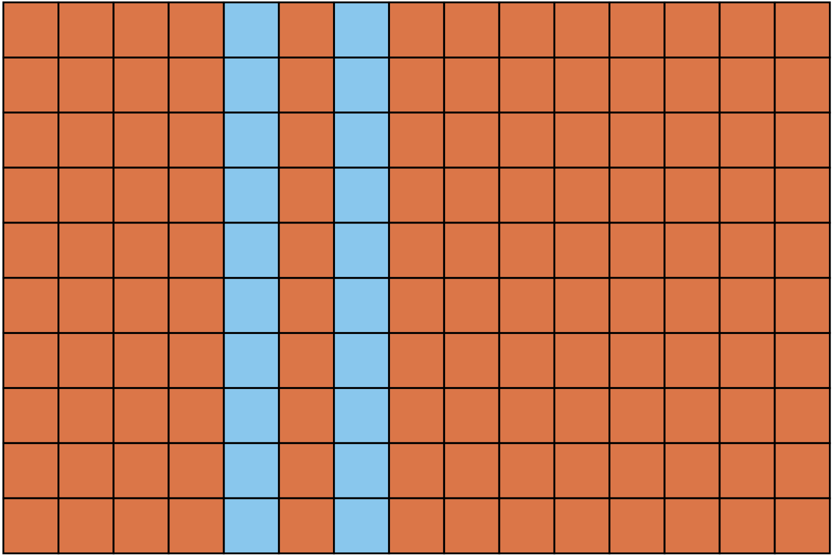
chirally perfect dito for orientation preserving symmetries

(Sometimes a perfect colouring is called colour symmetry.)

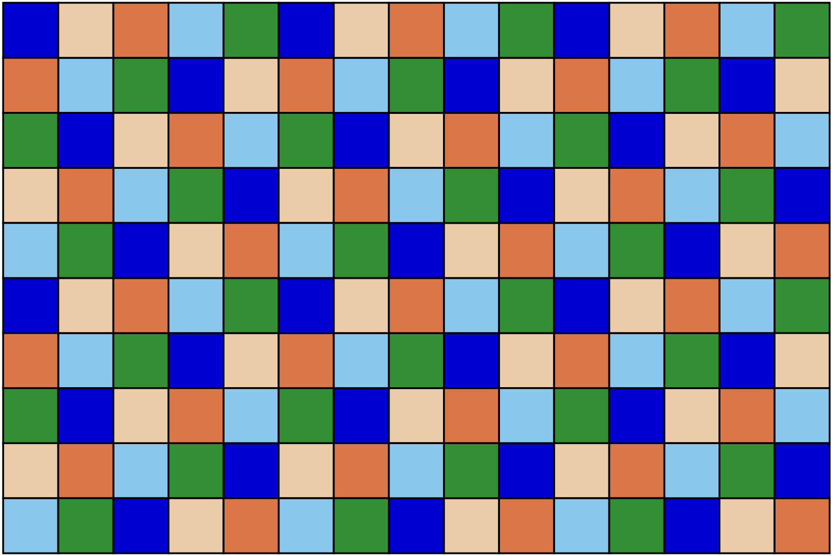
Perfect colouring of (4^4) with two colours:



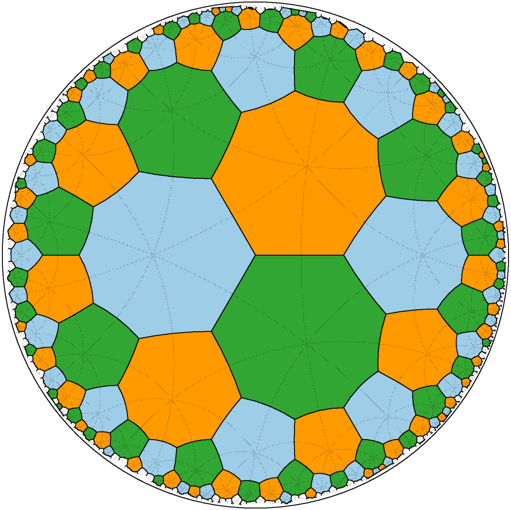
Not a perfect colouring of (4^4):



Chirally perfect colouring of (4^4) with five colours:



Perfect colouring of (8^3) with three colours:



Questions: Given a regular tiling (p^q),

1. for which number of colours does there exist a perfect colouring?
2. how many for a certain number of colours?
3. what is the structure of the generated permutation group?

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Some answers:

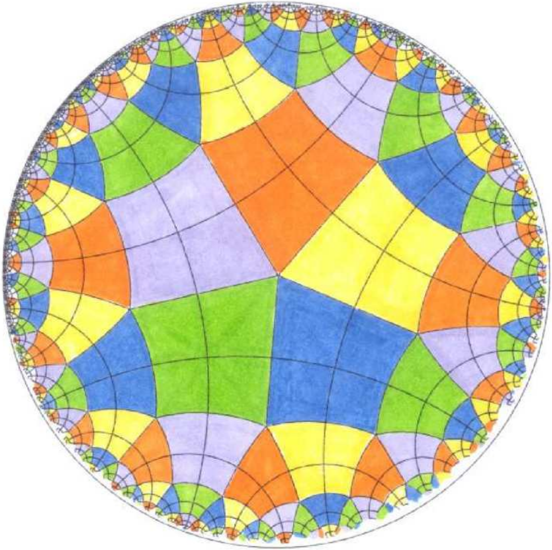
Perfect colourings:

(4^4)	$1, 2, 4, 8, 9, 16, 18, 25, 32, 36, \ldots$
(3^6)	$1, 2, 4, 6, 8, 16, 18, 24, 25, 32, \ldots$
(6^3)	$1, 3, 4, 9, 12, 16, 25, 27, 36, \ldots$
(7^3)	$1, 8, 15, 22, 24, 30, 36^2, 44, 50^5, \ldots$
(3^7)	$1, 22, 28^5, 37, 42^4, 44, 49^7, 50^3, \ldots$
(8^3)	$1, 3, 6, 12, 17, 21^4, 24, 25^5, 27^3, 29^4, 31^4, 33^6, 37^6, 39^8, \ldots$
(3^8)	$1, 2, 4, 8, 10^2, 12, 14, 16^2, 18, 20^4, 24^3, 25^5, 26, 28^{12}, 29, 30^2, \ldots$
(5^4)	$1, 2, 6, 11, 12, 16^2, 21^3, 22^5, 24, 26^9, 28, \ldots$
(4^5)	$1, 5^2, 10^4, 11, 15^7, 16, 20^9, 21^3, 22, 25^{27}, 26, 27^3, 30^{38}, \ldots$
(6^4)	$1, 2, 4, 6, 8, 10^2, 12^7, 13^4, 14, 15^2, 16^{13}, 18^{13}, 19^{10}, 20^{23}, 21^{10} \ldots$
(4^6)	$1, 2, 3, 5, 6^3, 9^4, 10^1, 11^2, 12^7, 13^5, 14^2, 15^{16}, 16^2, 17^9, 18^{26}, \ldots$

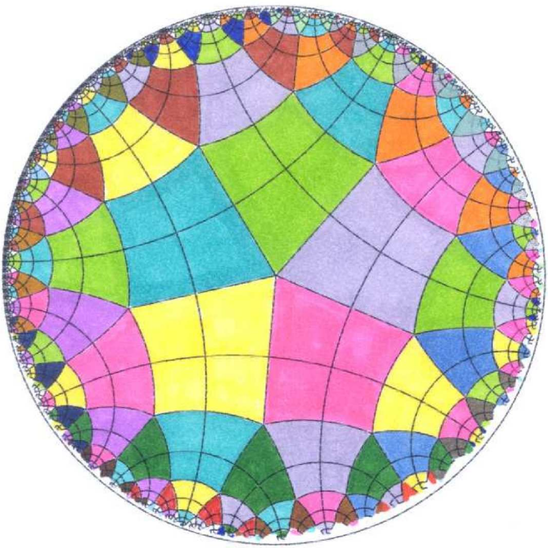
Chirally perfect colourings:

(4^4)	$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25^2, 26, 29, 32, \ldots$
(3^6)	$1, 2, 4, 6, 7, 8, 13, 14, 16, 18, 19, 24, 25, 26, 28, 31, \ldots$
(6^3)	$1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, \ldots$
(7^3)	$1, 8, 9, 15^2, 22^7, 24, \ldots$
(3^7)	$1, 7, 8, 14^6, 21^2, 22^7, \ldots$
(8^3)	$1, 3, 6, 9, 10, 12, 13^2, 15, 17^5, 18^5, 19^5, \ldots$
(3^8)	$1, 2, 4, 8^4, 10^3, 12, 13^2, 14^2, 16^{12}, 17^5, 18, 19^5, \ldots$
(5^4)	$1, 2, 6^2, 11^3, 12^6, 16^{12}, 17^4, \ldots$
(4^5)	$1, 5^2, 6, 10^6, 11^3, 15^{15}, 16^2, 17^4, \ldots$
(6^4)	$1, 2, 4^2, 6, 7^2, 8^3, 9^2, 10^6, 12^{11}, \ldots$
(4^6)	$1, 2, 3, 5, 6^4, 7^2, 8, 9^8, 10^3, 11^5, 12^{15}, \ldots$

Perfect colouring of (4^5) with five colours (R. Lück, Stuttgart):



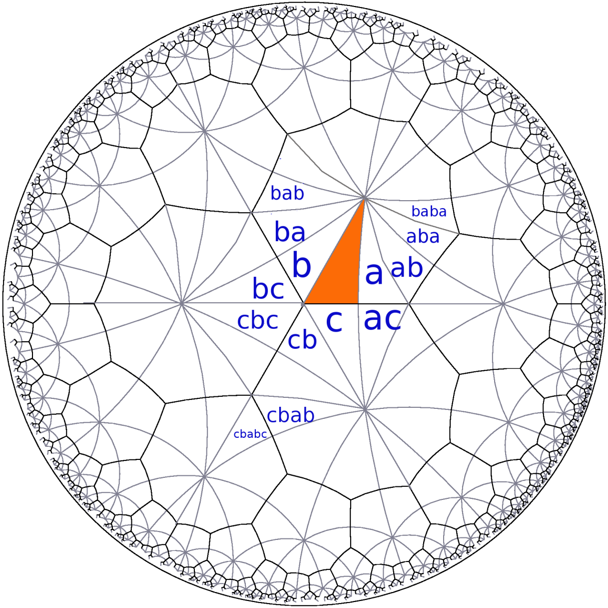
Perfect colouring of (4^5) with 25 colours (R. Lück, Stuttgart):



How to obtain these values?

The (full) symmetry group of a regular tiling (p^q) is a Coxeter group:

$$G_{p,q} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (ac)^2 = (bc)^q = \mathrm{id} \rangle$$


bab
baba
ba
aba
b
a
ab
bc
cbc
c
ac
cb
cbab
cbabc

Left coset colouring of (p^q):

Let F be the fundamental triangle.

- Choose a subgroup S of $G_{p,q}$ such that $a, b \in S$
- Assign colour 1 to each $f\,F$ $(f \in S)$
- Analoguosly, assign colour i to the i-th coset S_i of S

Yields a colouring with $[G_{p,q} : S]$ colours.

How to count perfect colourings now?

- Show that each of these colourings is perfect (simple)
- Show that each perfect colouring is obtained in this way
- Count subgroups of index k in $G_{p,q}$ (hard)

Using `GAP` yields the tables above.

Since `GAP` identifies subgroups if they are conjugate, we obtain indeed all *different* colourings.

In a similar way one can count chirally perfect colourings.

- Consider the rotation group $\bar{G}_{p,q} = \langle ab, ac \rangle_{G_{p,q}}$.
- Use left coset colouring in $\bar{G}_{p,q}$.
- Check for conjugacy in $G_{p,q}$.

The last step requires some programming in `GAP`.

Conclusion

We've seen a method to count perfect colourings of regular tilings. What next?

- Algebraic properties of S. For instance, some S are generated by three generators, some S require four generators.
- Algebraic properties of the induced permutation group P. For a start, P acts transitively on the colours. Which P can arise in this way? Can we obtain a symmetric group?