Tilings, quasicrystals and pinwheels

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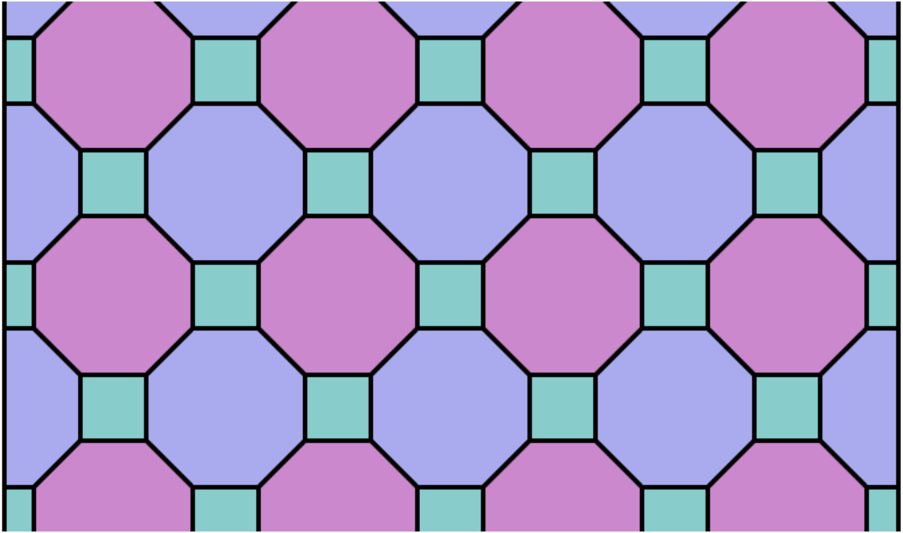
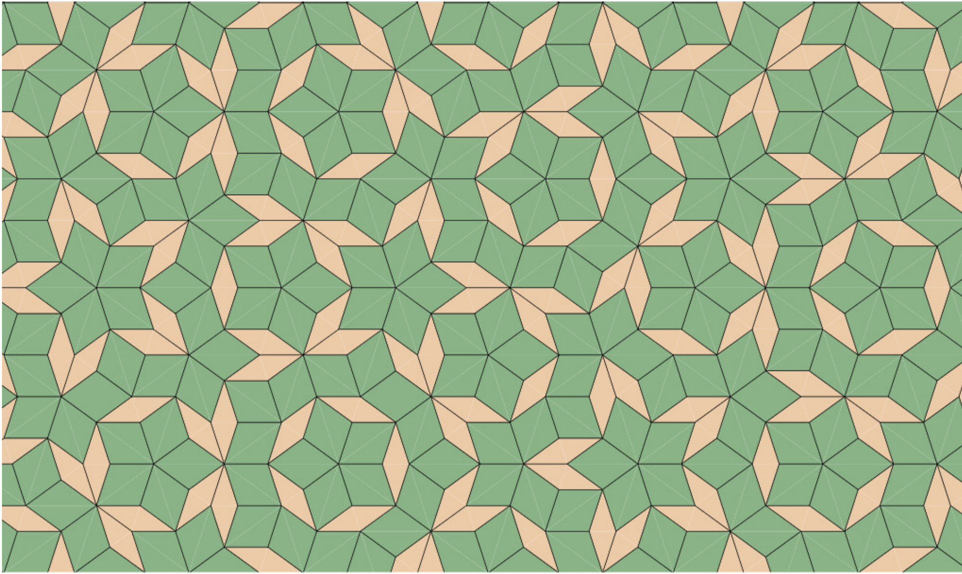
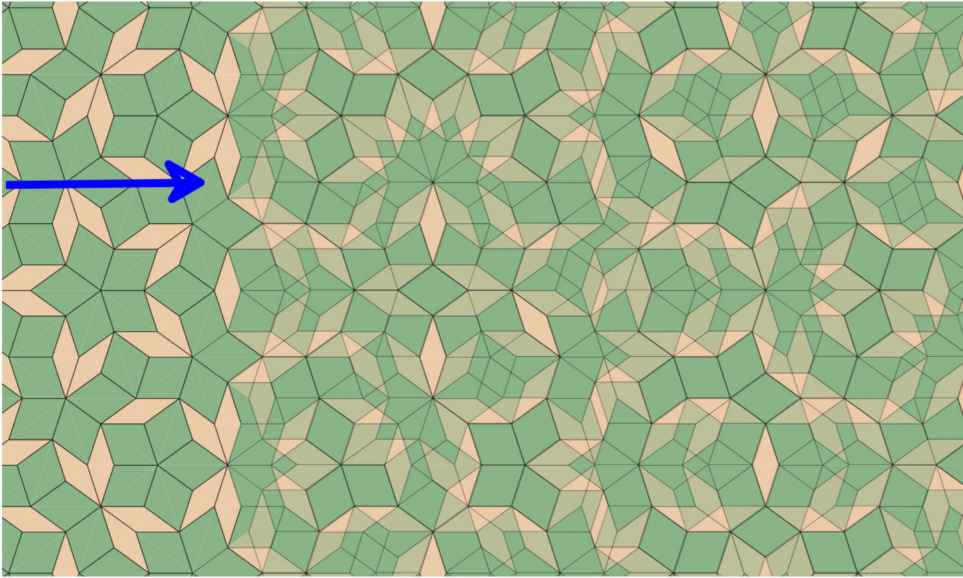
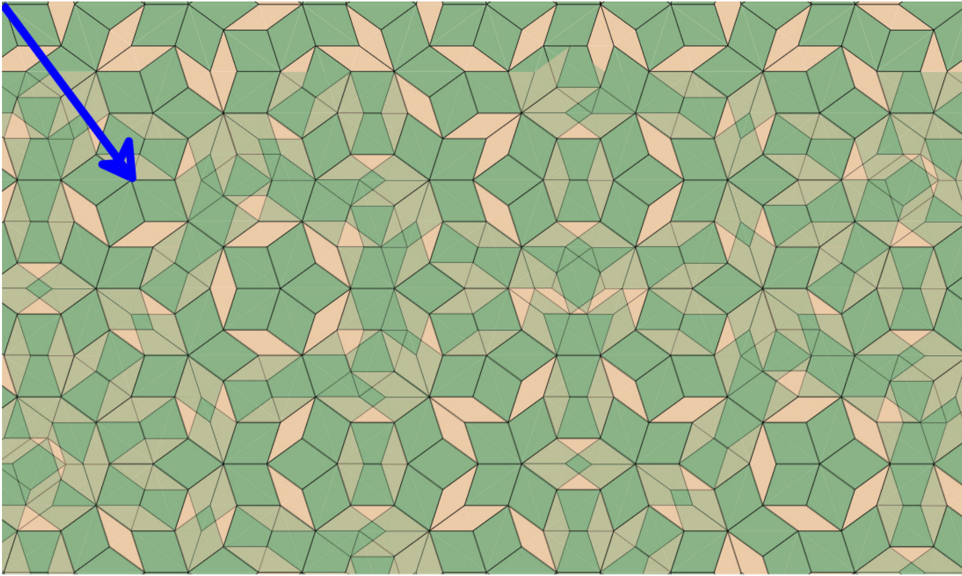
29. Jan. 2014

Tilings

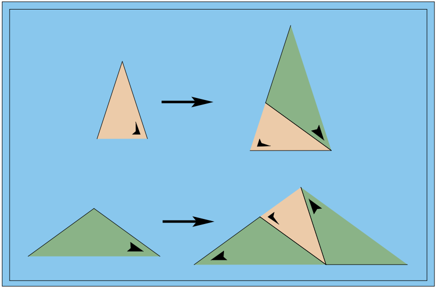


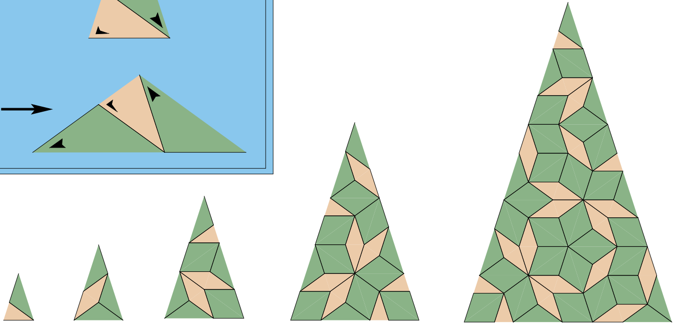






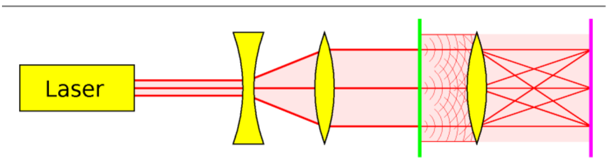
Substitution tilings:





Quasicrystals

Physical diffraction experiment:



Mathematical diffraction experiment:

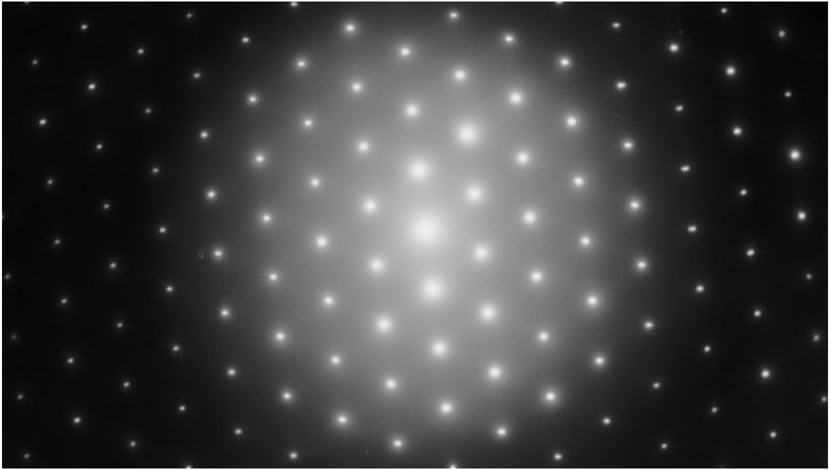
- Tiling $\rightsquigarrow$ discrete point set Λ.
- $\gamma_\Lambda = \lim_{r\to\infty} \frac{1}{\text{vol}\, B_r} \sum_{x,y\in\Lambda\cap B_r} \delta_{x-y}$.
- Fouriertransform $\widehat{\gamma}_\Lambda$ is the diffraction spectrum.

Since $\widehat{\gamma} := \widehat{\gamma}_\Lambda$ is again a measure, it decomposes into three parts: (by Lebesgue's decomposition theorem)

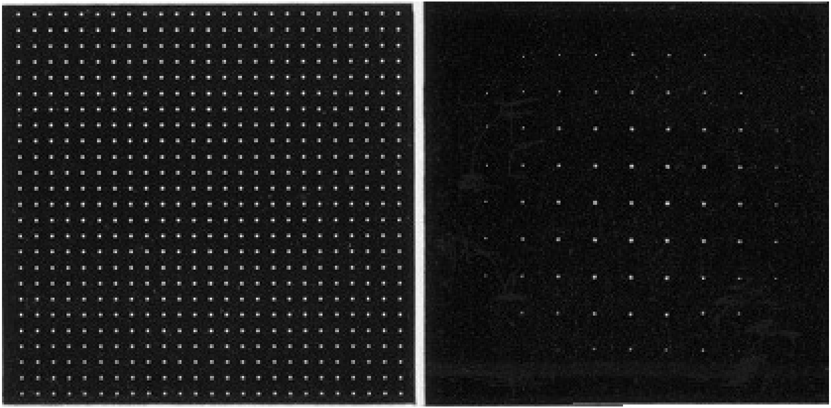
$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}$$

(pp: pure point, ac: absolutely continuous, sc: singular continuous)

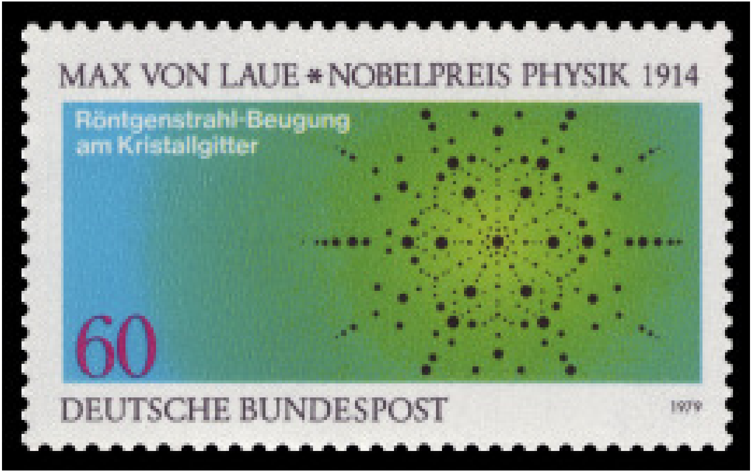
Crystal diffraction (physical experiment):



Crystal diffraction (mathematical computation):



Crystal diffraction: Noble prize 1914



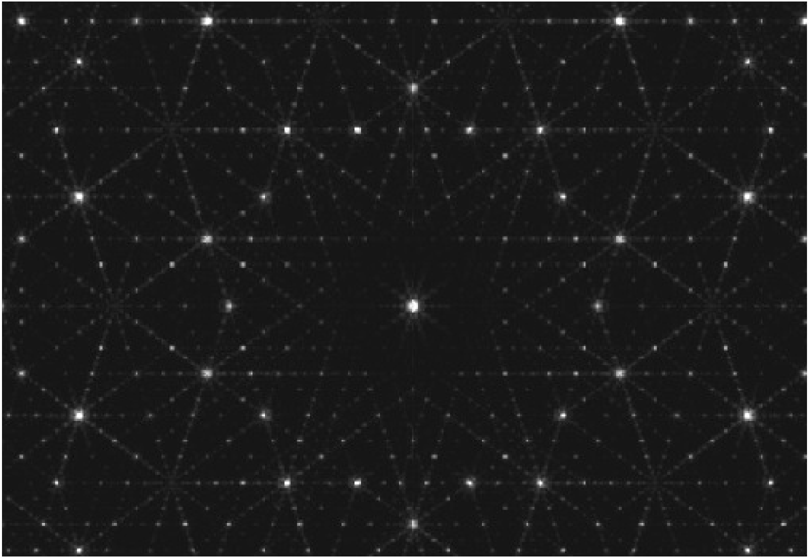
Ideal (perfect, infinite) crystals have pure point diffraction:

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \sout{\widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}}$$

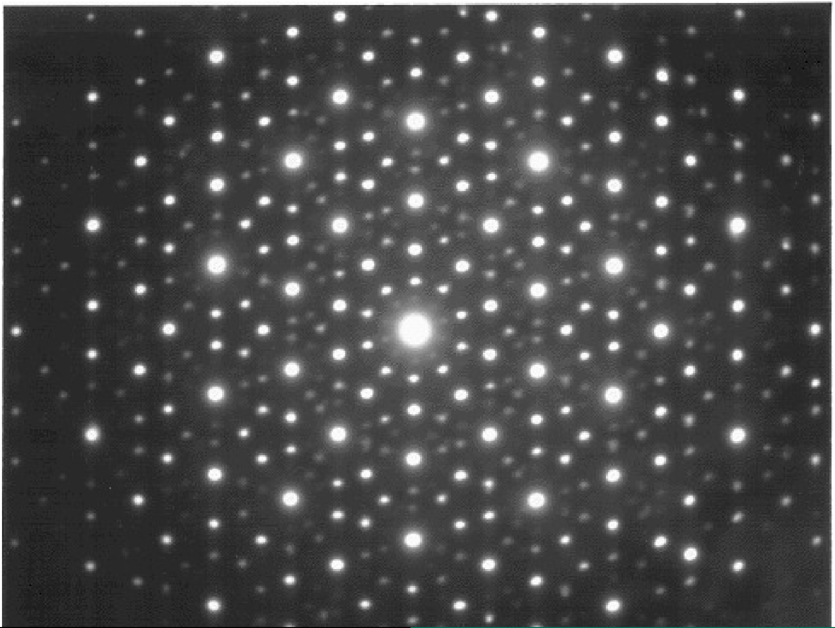
Ideal (mathematical, infinite) quasicrystals have also pure point diffraction:

$$\widehat{\gamma} = \widehat{\gamma}_{pp}$$

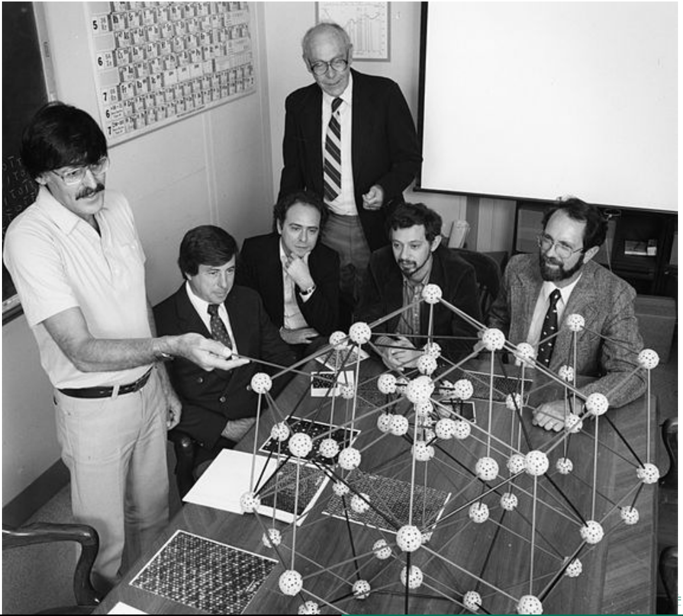
Quasicrystal diffraction (mathematical: Penrose)

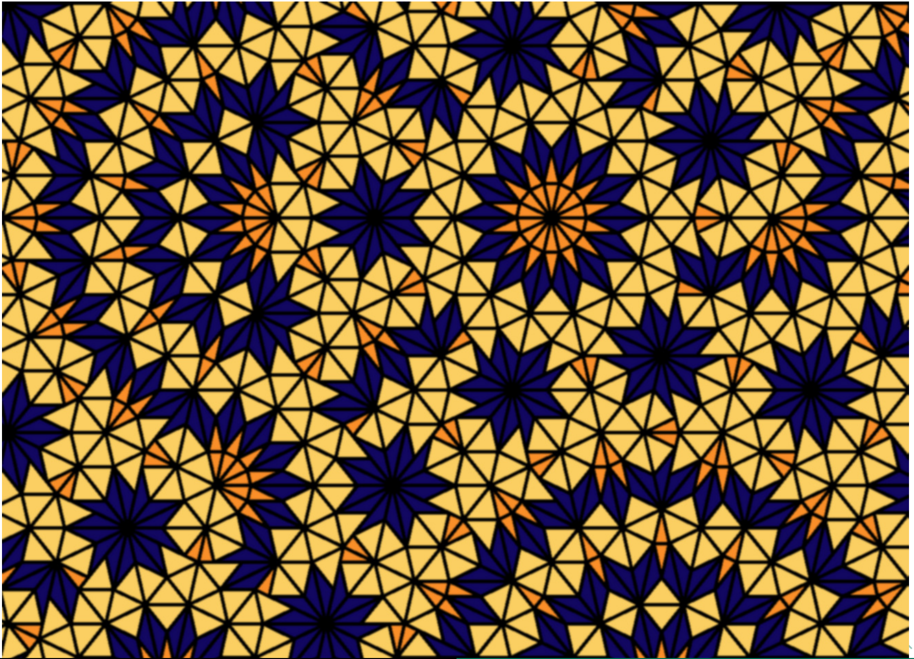


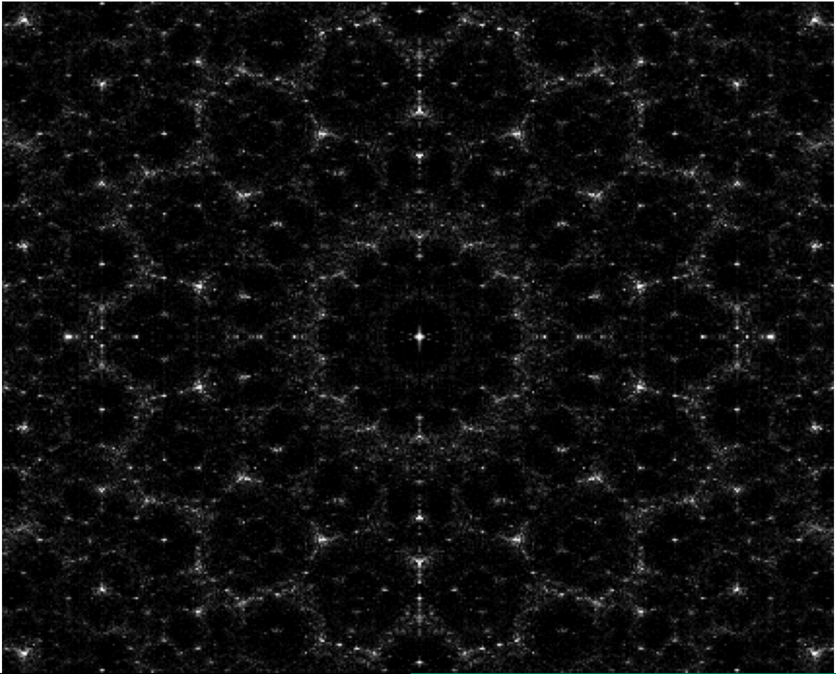
Quasicrystal diffraction (physical experiment: some metallic alloy)



Quasicrystal diffraction: Noble prize 2011 for Danny Shechtman

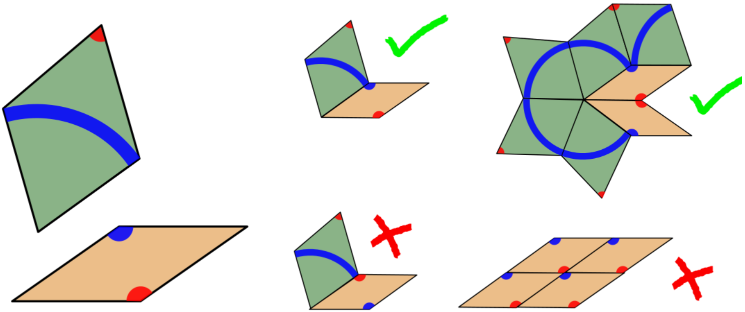




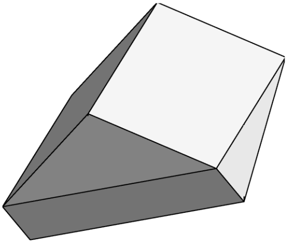


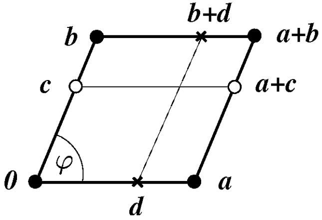
Local rules

Penrose tiling: force aperiodicity by local rules



Conway's biprism, Schmitt-Conway-Danzer tile:





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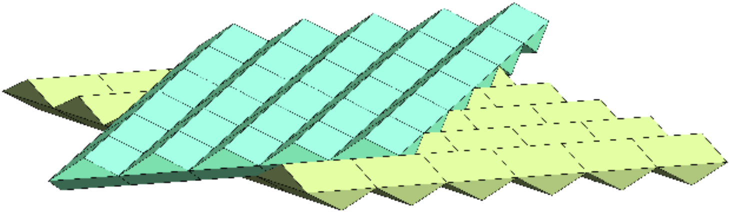
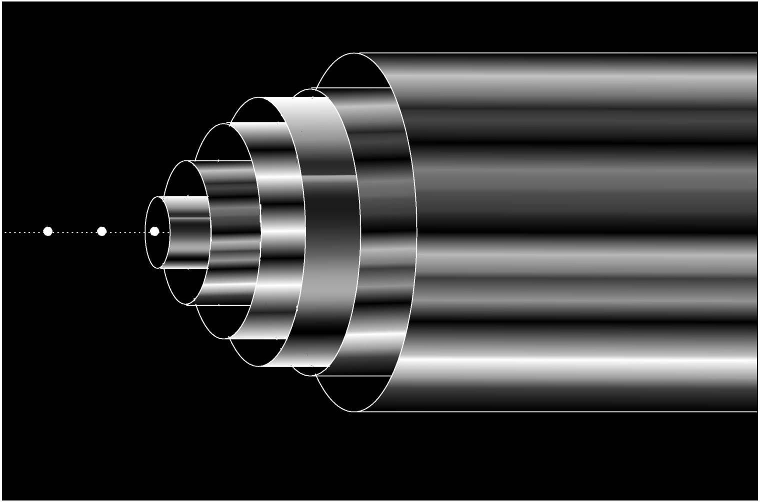
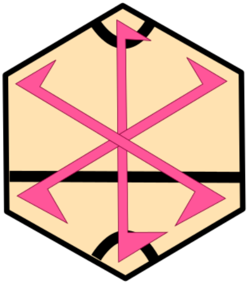


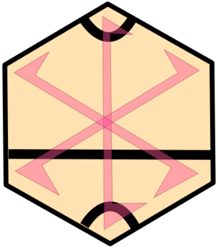
Its diffraction: (Baake-F 2005)

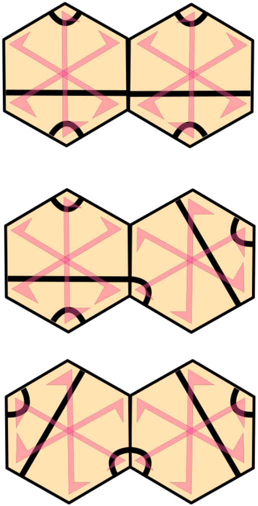


Joan Taylor's monotile:

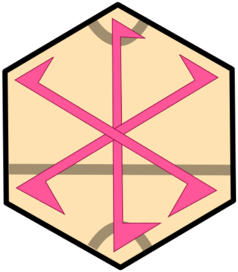


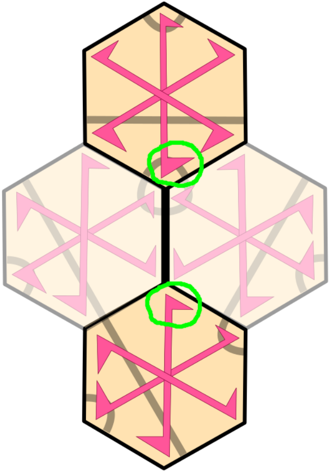
Schwarze Regel:

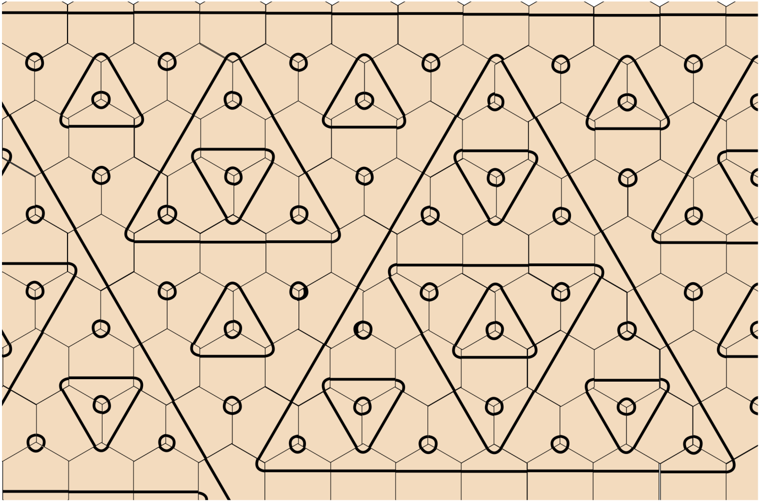


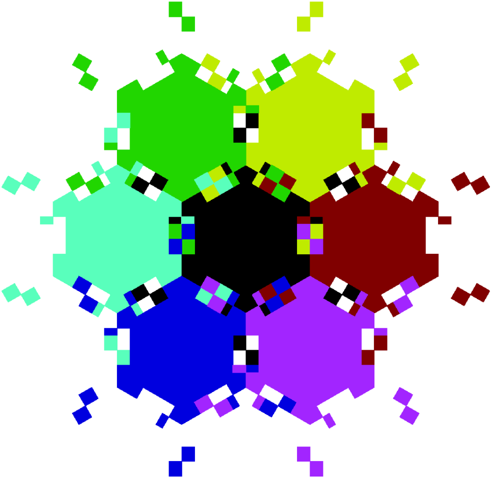


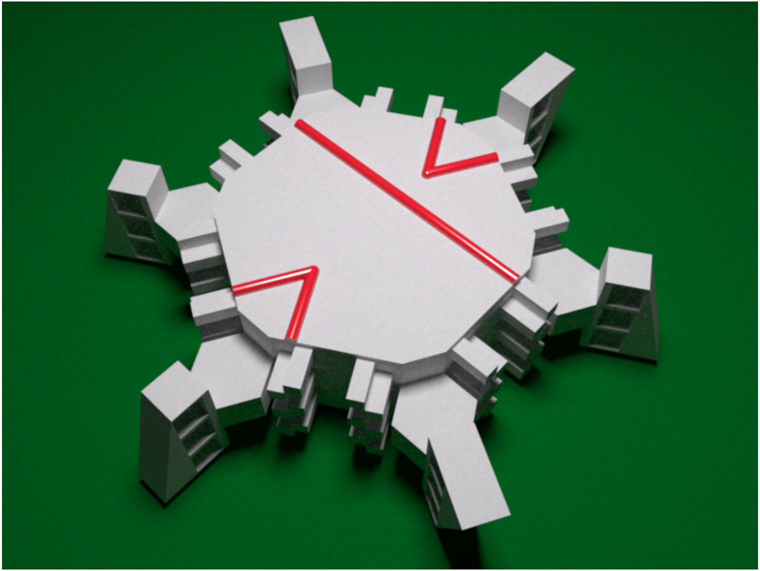
Pinke Regel:

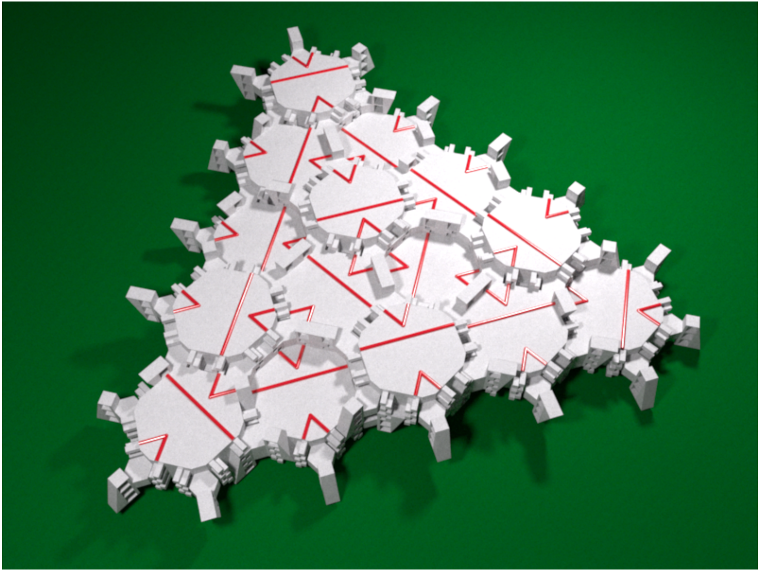






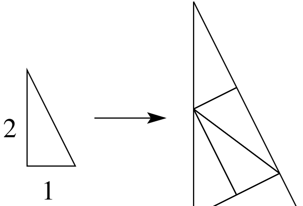


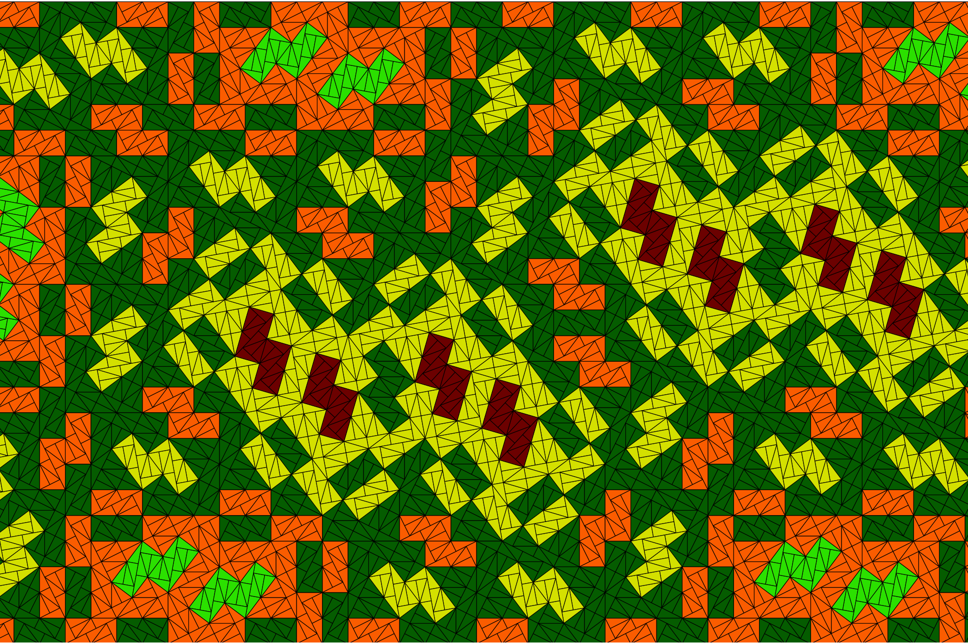


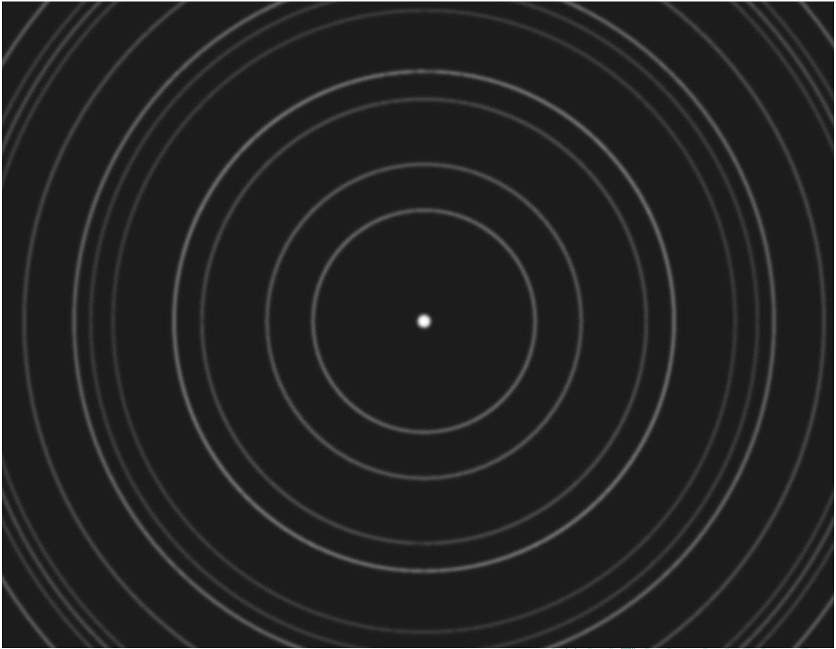


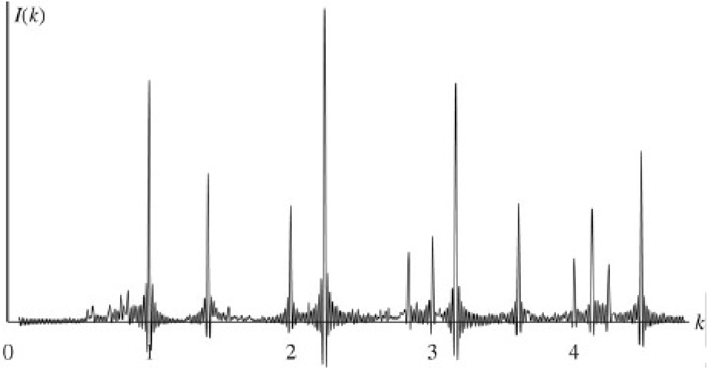
Pinwheels

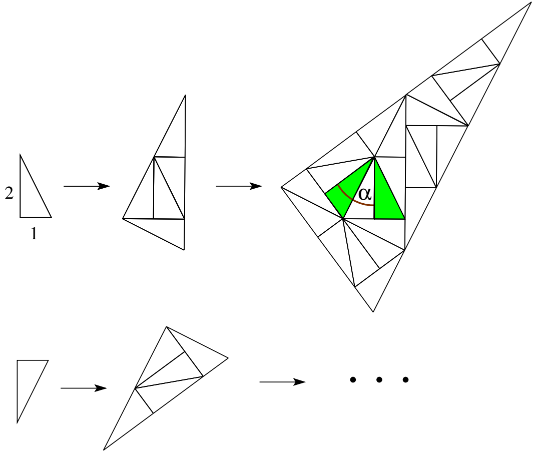
Conway's Pinwheel substitution (1991):








I(k)
k
0
1
2
3
4



The angle α is irrational; that is, $\alpha \notin \pi\mathbb{Q}$.

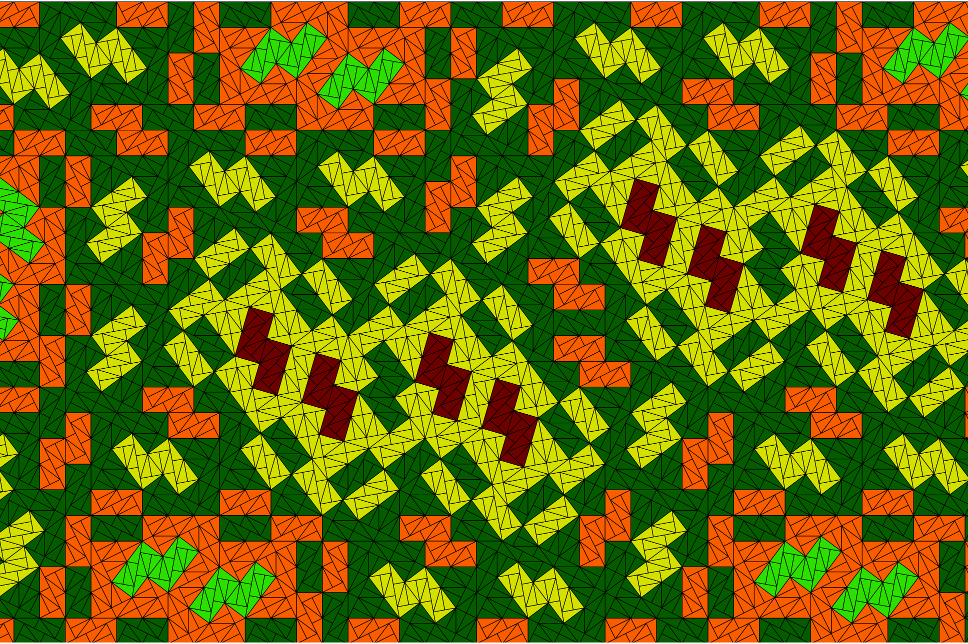
For pinwheel tilings: Orientations are dense in $[0, 2\pi[$.

Even more: orientations are equidistributed in $[0, 2\pi[$.

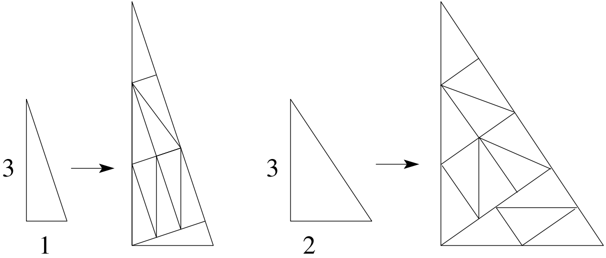
This is true not only for the pinwheel tiling:

Theorem (F. '08)

In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in $[0, 2\pi[$.



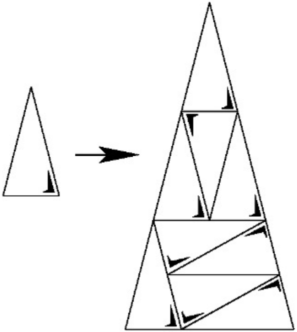
There are many examples: Pinwheel (n, k)

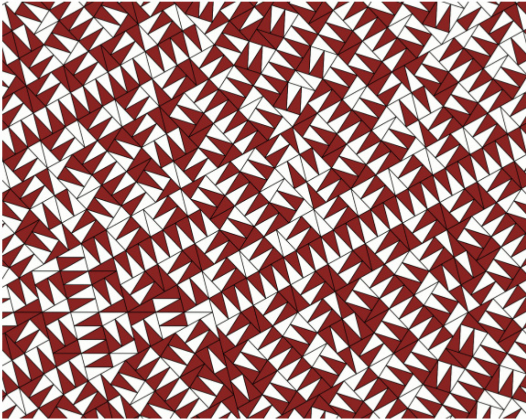


$n = 3, k = 1$ $n = 3, k = 2$

etc.

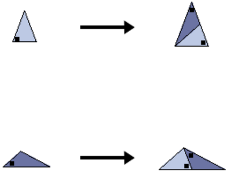
Unknown ($\leq$1996, Penrose? Danzer?):

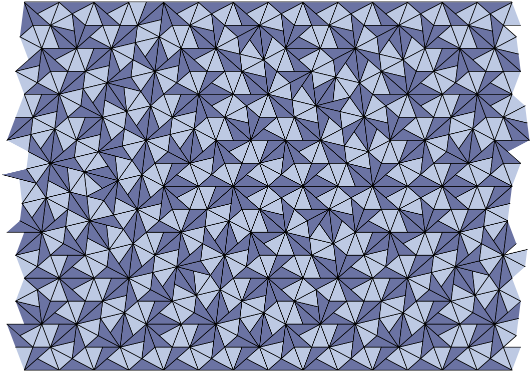




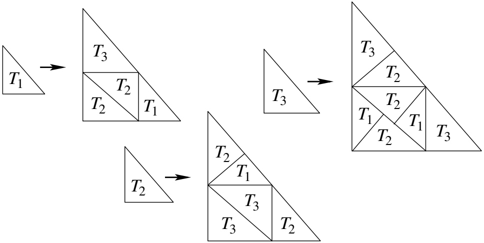
(+ obvious generalizations)

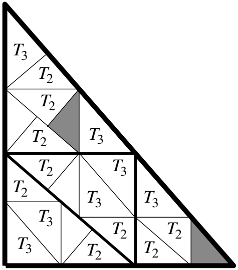
C. Goodman-Strauss, L. Danzer (ca. 1996):





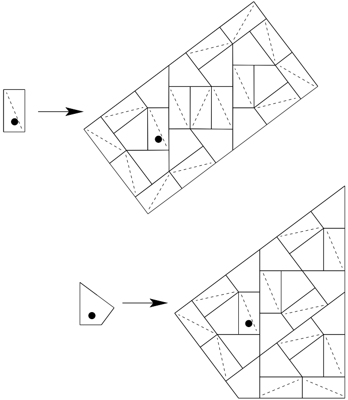
Pythia (m, j), here: $m = 3$, $j = 1$.

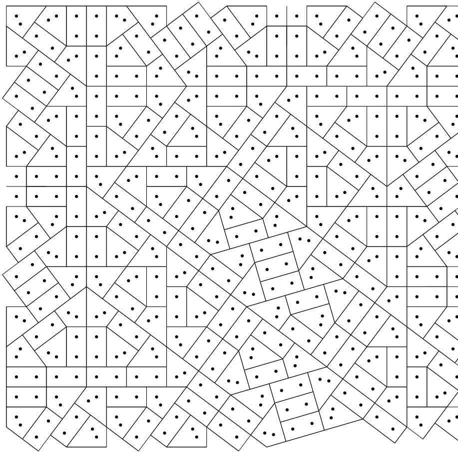


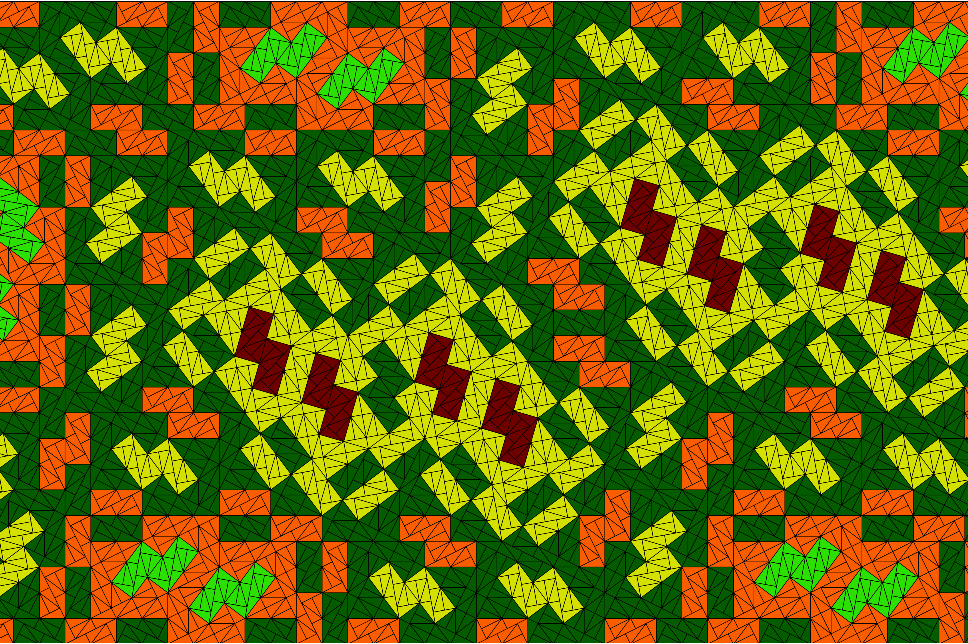


So far: tiles are always triangles. One exception:

Kite Domino (equivalent with Pinwheel):







Can we find examples with rhombic tiles for instance?

Answer: No.

Theorem (F.-Harriss, 2013)

Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many prototiles (i.e., finitely many different tile shapes). Let all prototiles be centrally symmetric convex polygons (i.e., $P = -P$). Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

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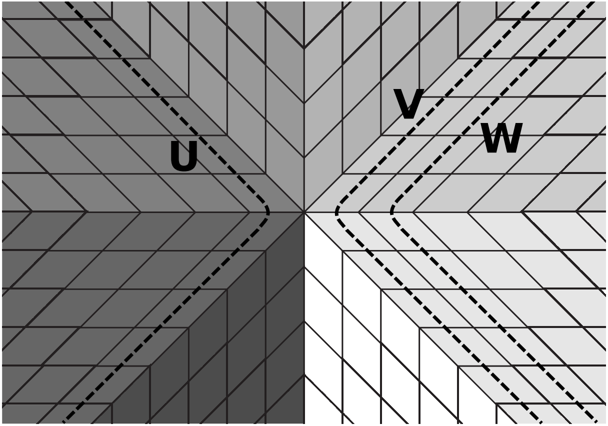
Theorem (F.-Harriss, 2013)

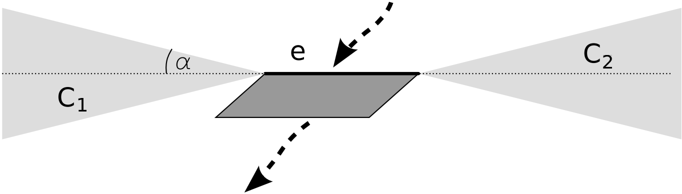
Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many parallelograms as prototiles. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

Assume all tiles are vertex-to-vertex.



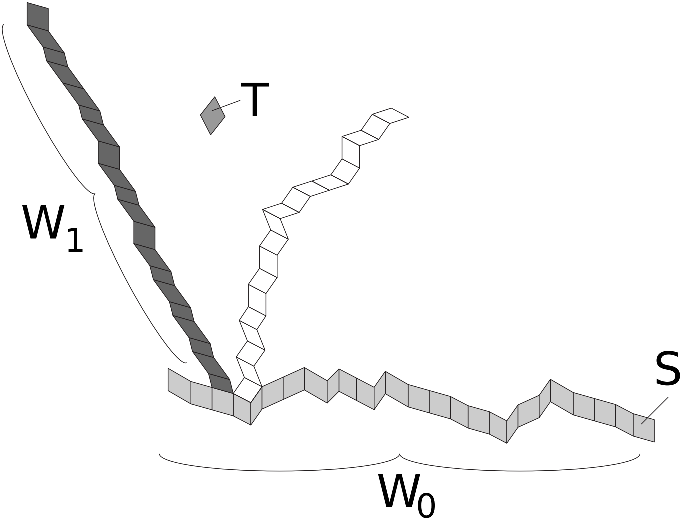
A worm is a sequence of tiles $\ldots, T_{-1}, T_0, T_1, T_2, \ldots$ where T_k and T_{k+1} share a common edge, and all shared edges are parallel.

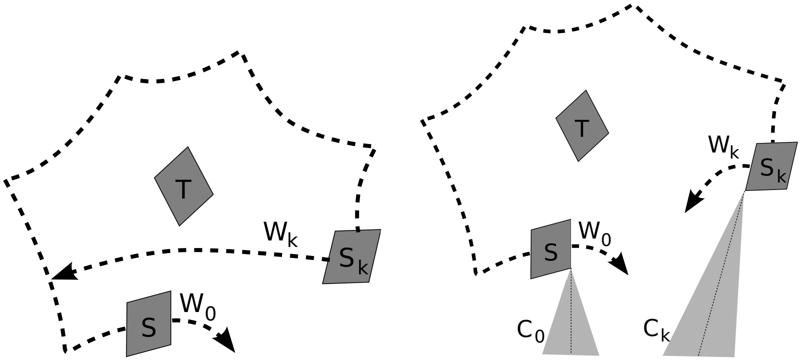
Cone Lemma: A worm defined by edge e cannot enter C_1 or C_2.
(α the minimal interior angle in the prototiles)



Loop Lemma: A worm has no loop.

Travel Lemma: Any two tiles can be connected by a finite sequence of finite worm pieces. (At most $k = \lceil \frac{2\pi}{\alpha} \rceil$ many.)




T
W_k
S_k
S
W_0
T
W_k
S_k
S
W_0
C_0
C_k

Proof of theorem (parallelogram version): Fix some tile S. Every tile T can be connected to S by at most $\lceil \frac{2\pi}{\alpha} \rceil$ worm pieces. That is, with $\lceil \frac{2\pi}{\alpha} \rceil$ turns. $\square$

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Proof of theorem (general): Any centrally symmetric convex polygon can be dissected into parallelograms.
(see e.g. Kannan-Soroker 1992)

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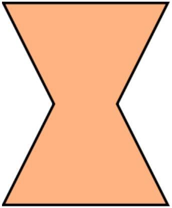
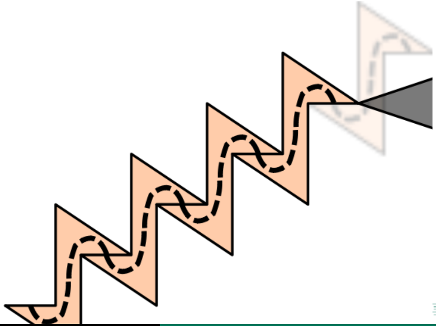
Proof of theorem (general): Any centrally symmetric convex polygon can be dissected into parallelograms.
(see e.g. Kannan-Soroker 1992)

- Probably true in higher dimensions
- Also true for non-convex? Hmm...





Possible application:

Generalise to 3D and non-convex. A physical interpretation:

If the interactions in a solid (bounding forces between atoms, molecules...) are centrally symmetric, the solid shows finitely many orientations.

In fact, there are only very few exceptions (e.g. smectic phases of liquid crystals)


Thank you.