Tilings with tiles in finitely many and infinitely many orientations

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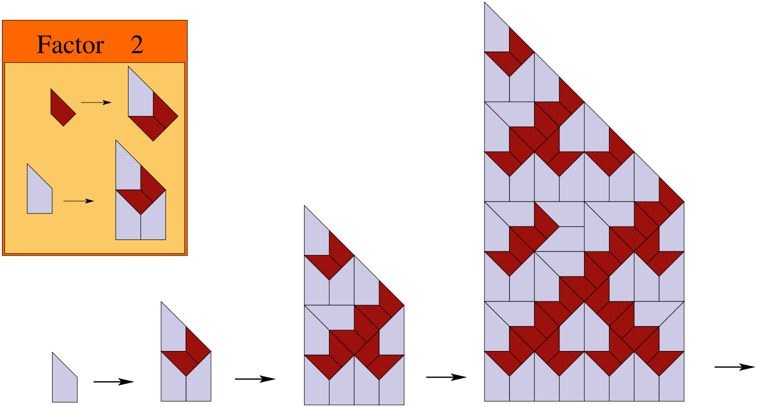
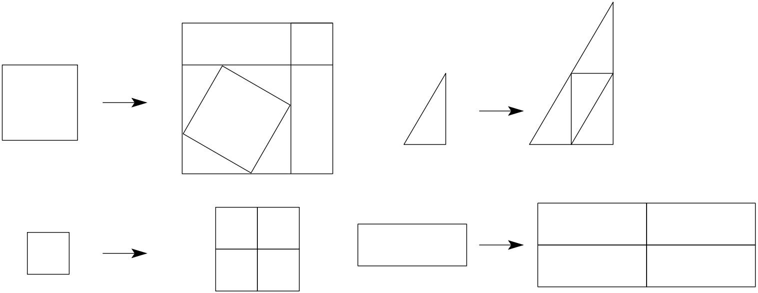
1. Tilings with tiles in infinitely many orientations
2. Tilings with tiles in finitely many orientations

Substitution tilings:



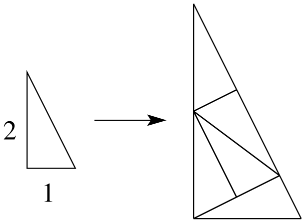
Usually, tiles occur in finitely many different orientations only.

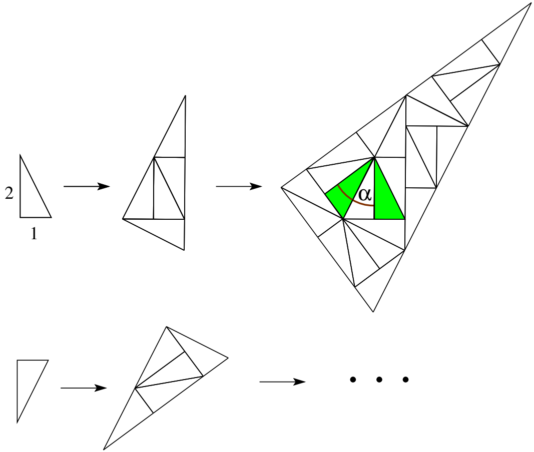
Not always. Cesi's example (1990):



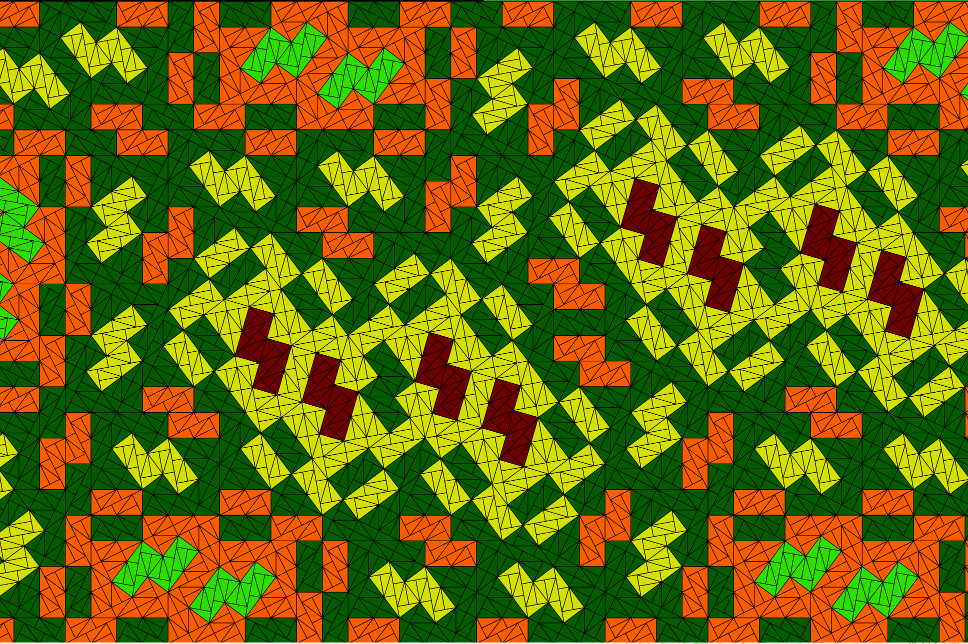
A substitution σ is *primitive*, if for any tile T there is $k \geq 1$ such that $\sigma^k(T)$ contains all tile types.

Conway's Pinwheel substitution (1991):

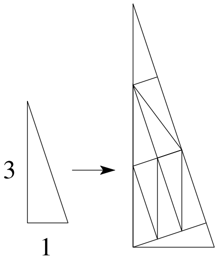




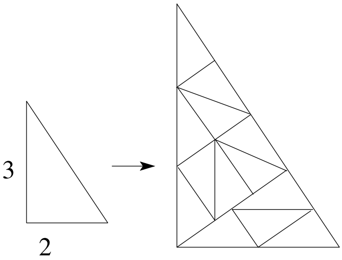
The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



Obvious generalizations: Pinwheel (n, k)



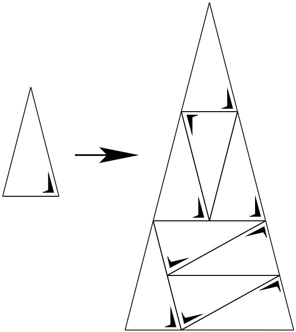
$n = 3, k = 1$

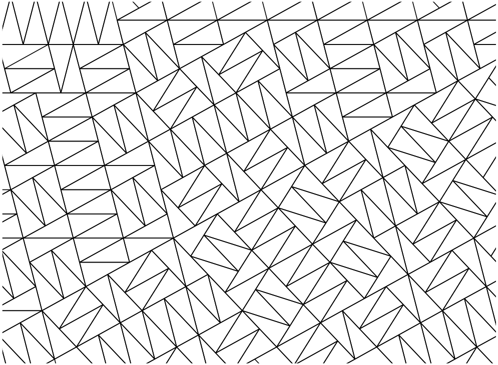


$n = 3, k = 2$

etc.

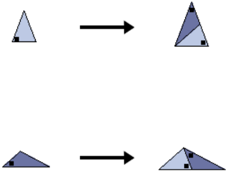
Unknown (< 1996, communicated to me by Danzer):

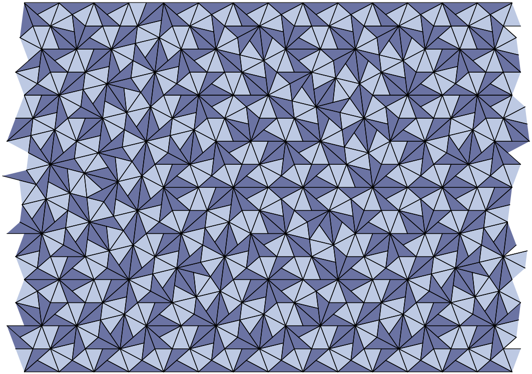




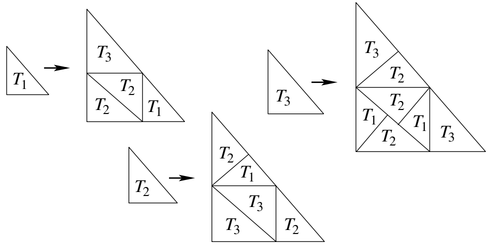
(+ obvious generalizations)

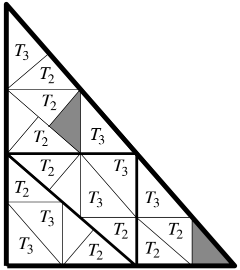
C. Goodman-Strauss, L. Danzer (ca. 1996):





Pythia (m, j), here: $m = 3$, $j = 1$.





For all examples: the orientations are dense in $[0, 2\pi[$.

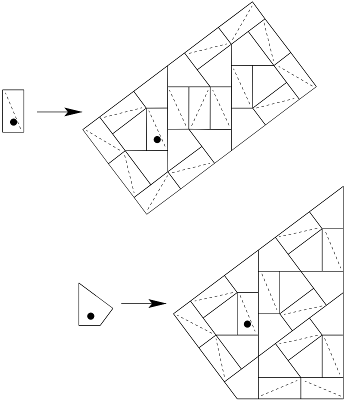
Even more: The orientations are equidistributed in $[0, 2\pi[$.

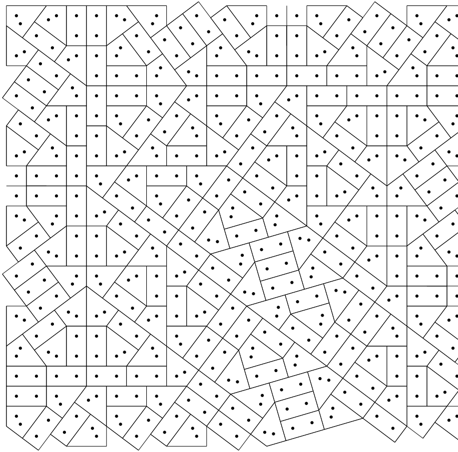
Theorem (F. '08)

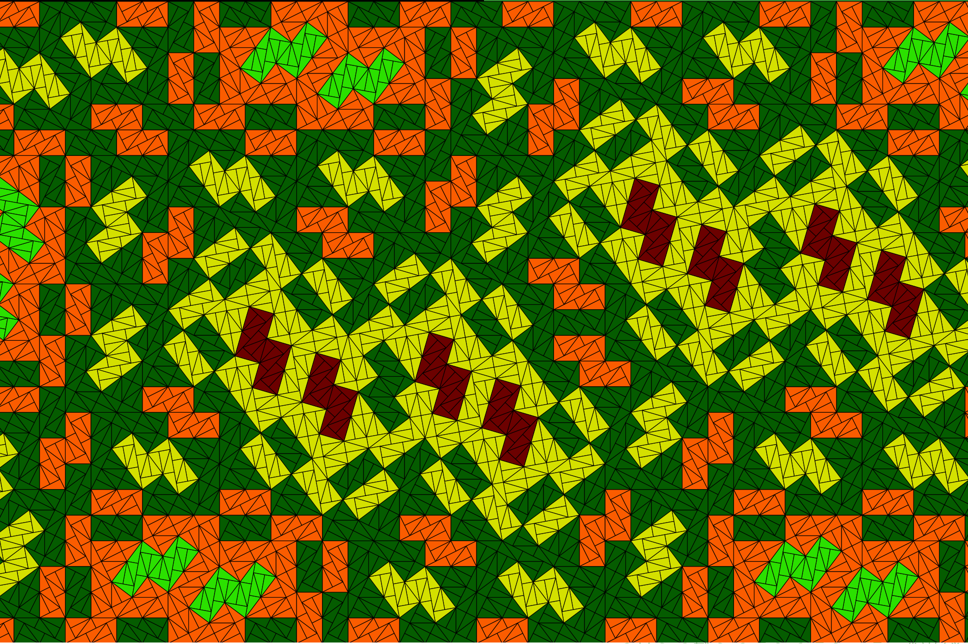
In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in $[0, 2\pi[$.

So far: tiles are always triangles. One exception:

Kite Domino (equivalent with Pinwheel):







Can we find examples with rhombic tiles for instance?

Answer: No.

Theorem (F.-Harriss '12+)

Let $\mathcal{T}$ be a tiling with finitely many prototiles (i.e., finitely many different tile shapes). Let all prototiles be centrally symmetric convex polygons. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

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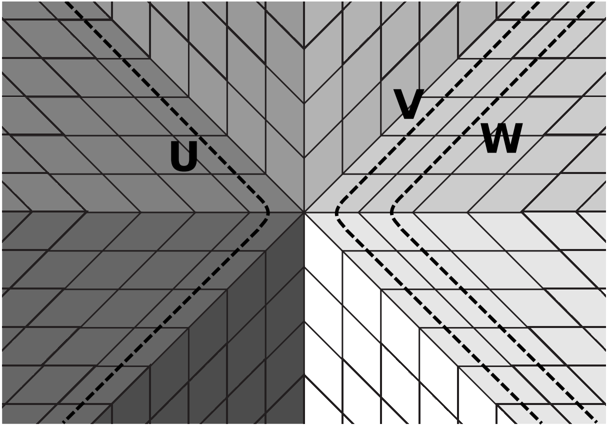
Theorem (F.-Harriss '12+)

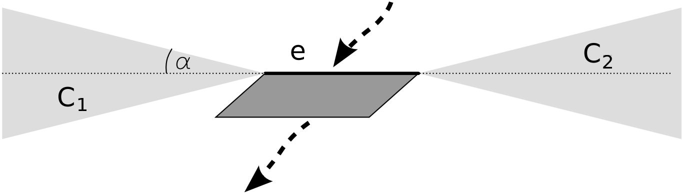
Let $\mathcal{T}$ be a tiling with finitely many parallelograms as prototiles. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

Assume all tiles are vertex-to-vertex.



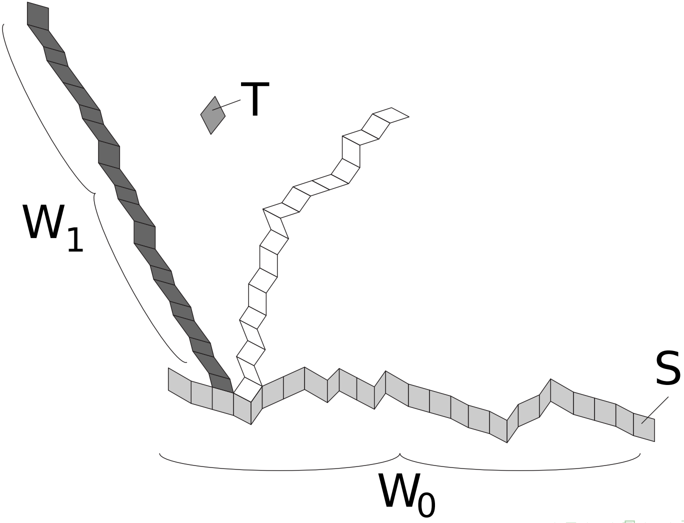
A *worm* is a sequence of tiles $\ldots, T_{-1}, T_0, T_1, T_2, \ldots$ where T_k and T_{k+1} share a common edge, and all shared edges are parallel.

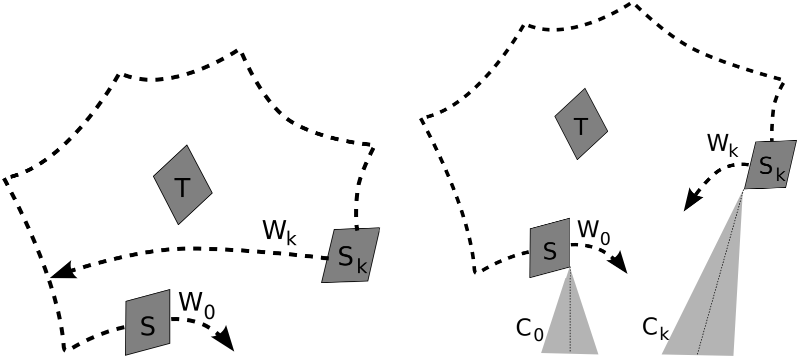
Cone Lemma: A worm defined by edge e cannot enter C_1 or C_2.
(α the minimal interior angle in the prototiles)



Loop Lemma: A worm has no loop.

Travel Lemma: Any two tiles can be connected by a finite sequence of finite worm pieces. (At most $\lceil \frac{2\pi}{\alpha} \rceil$ many.)




T
W_k
S_k
S
W_0
T
W_k
S_k
S
W_0
C_0
C_k

Proof of theorem (parallelogram version): Fix some tile S. Every tile T can be connected to S by at most $\lceil \frac{2\pi}{\alpha} \rceil$ worm pieces. That is, with $\lceil \frac{2\pi}{\alpha} \rceil$ turns.

(Non-vertex-to-vertex case can be handled.) $\square$

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Theorem

Let $\mathcal{T}$ be a tiling with finitely many prototiles. Let all prototiles be centrally symmetric convex polygons. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

Can we drop "finitely many"?

No. Even if we assume: infimum of interior angles > 0. (Exercise)

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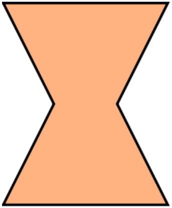
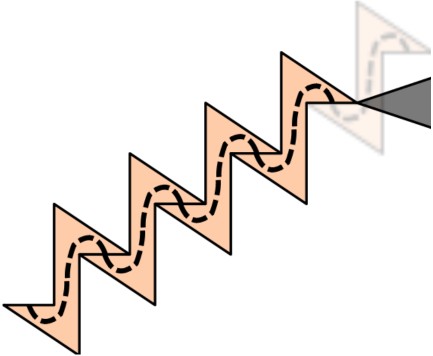
Can we drop "convex"?

Hmm...






Thank you.