Highly symmetric fundamental domains for lattices

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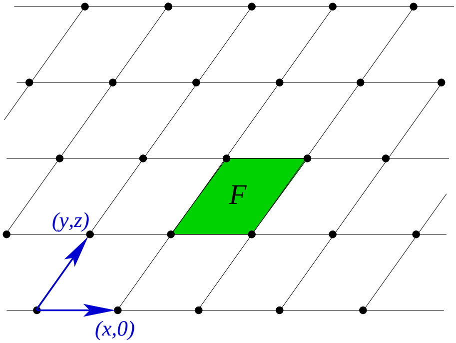
Point lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

Fundamental cell of Γ: $\mathbb{R}^d/\Gamma$.



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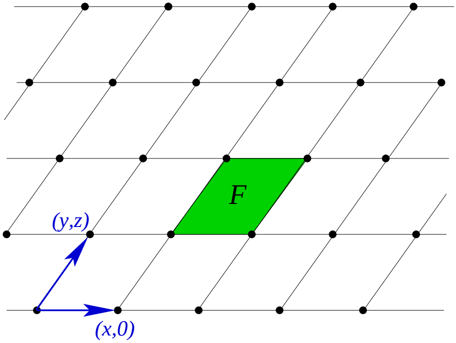
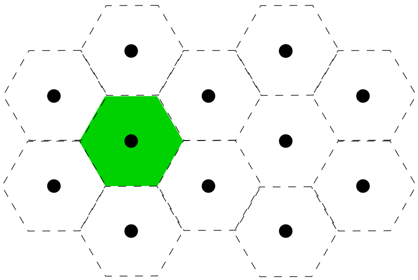
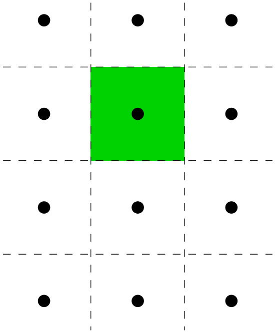


Point group $P(\Gamma)$ of Γ: All linear isometries f with $f(\Gamma) = \Gamma$.

Trivially, each lattice Γ has a fundamental cell which symmetry group is $P(\Gamma)$.

For instance, take the Voronoi cell of a lattice point x. (That is the set of points closer to x than to each other lattice point.)





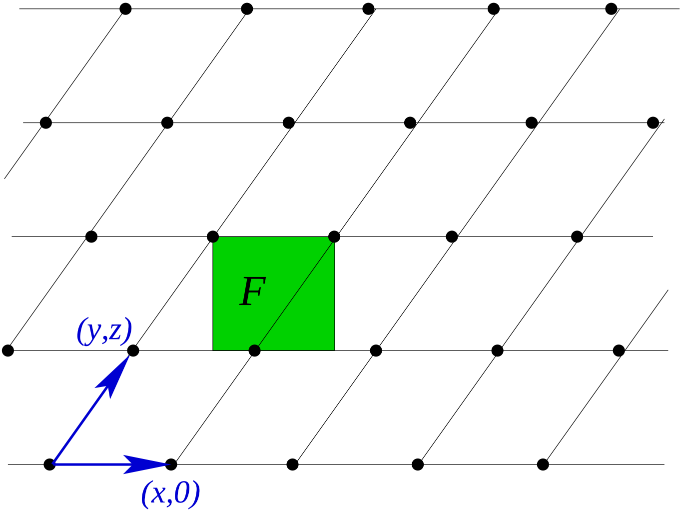
Main result

Theorem (Elser, Fr.)

Let $\Gamma \subset \mathbb{R}^2$ be a lattice, but not a rhombic lattice. Then there is a fundamental cell F of Γ which symmetry group is strictly larger than $P(\Gamma)$: $\quad [S(F) : P(\Gamma)] = 2.$

'Rhombic lattice' means here: one with basis vectors of equal length, but neither square lattice nor hexagonal lattice.

Generic lattice:



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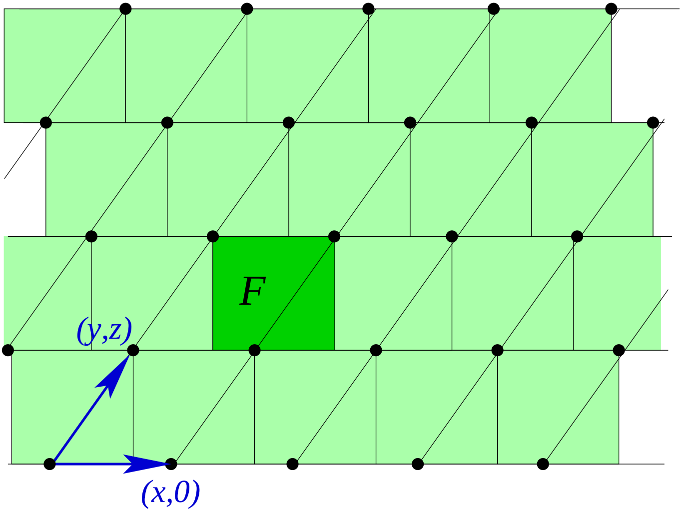
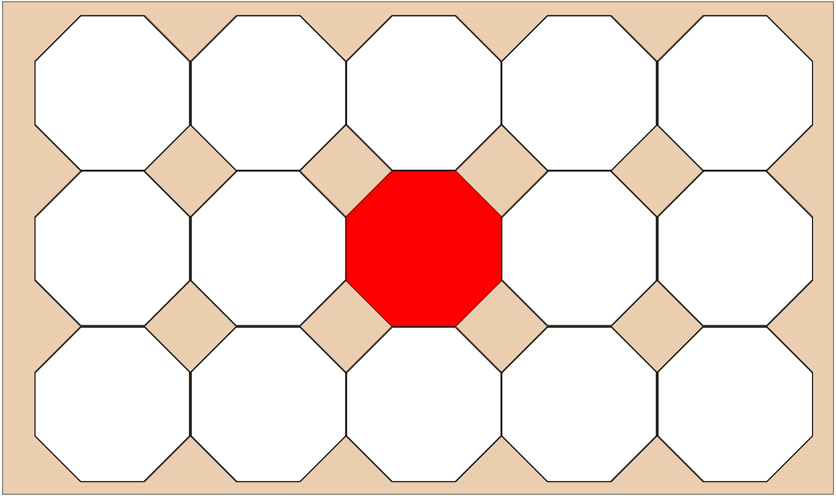
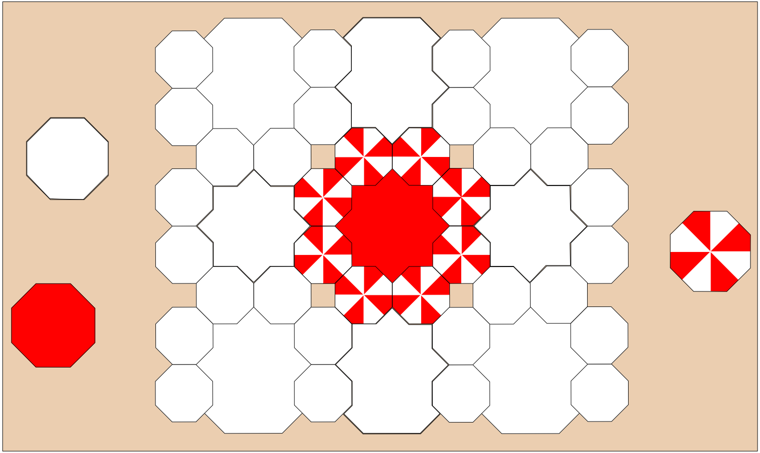
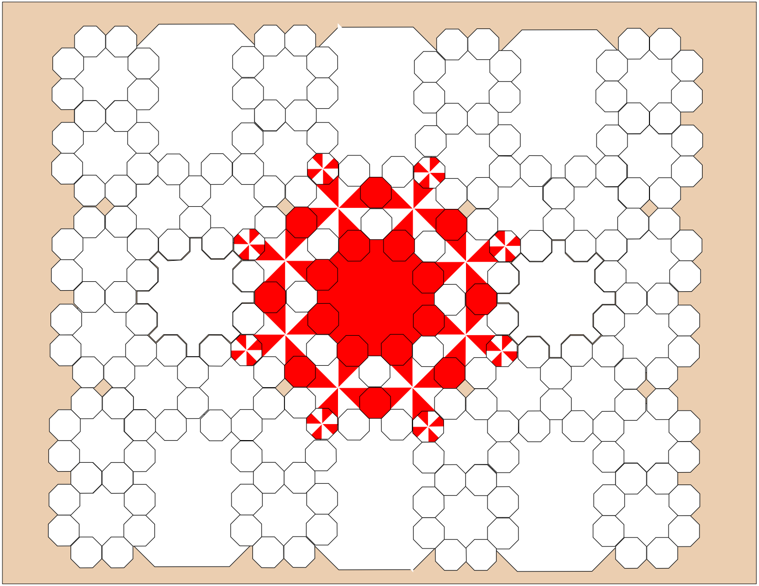
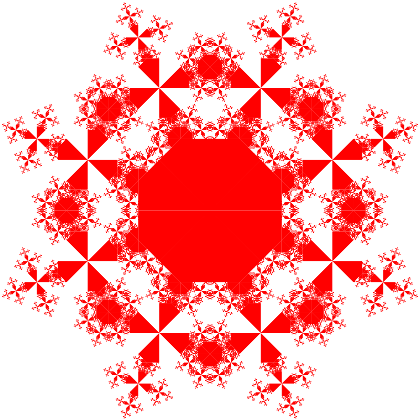


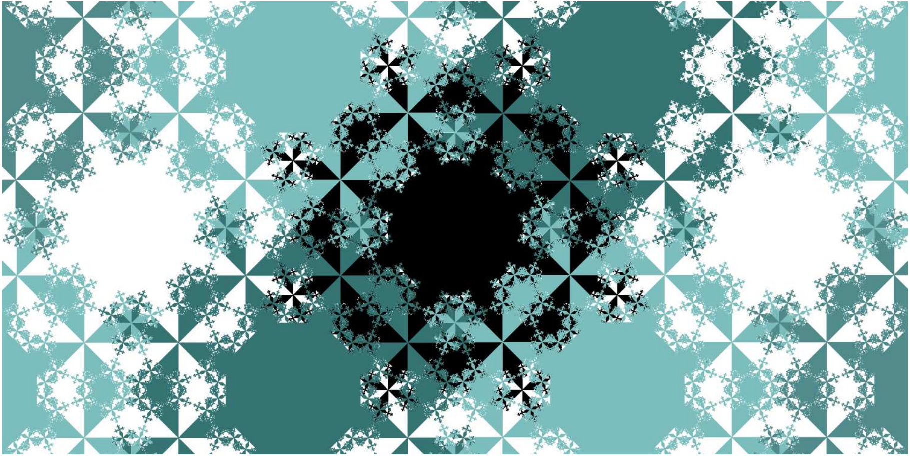
Square lattice (Veit Elser):



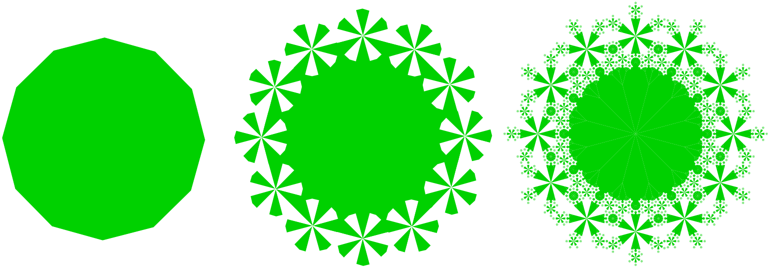




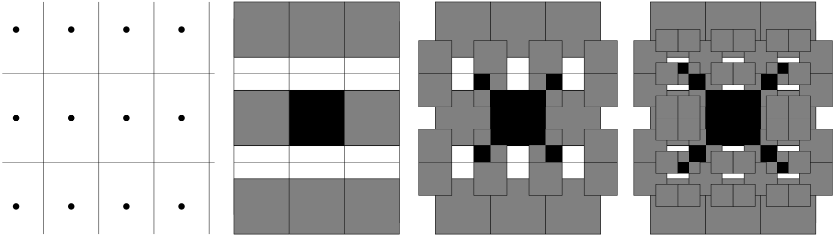


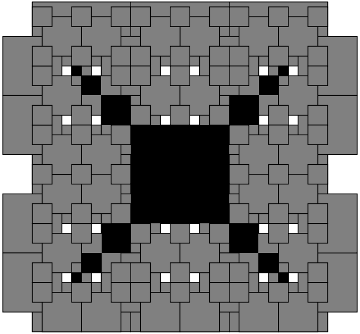


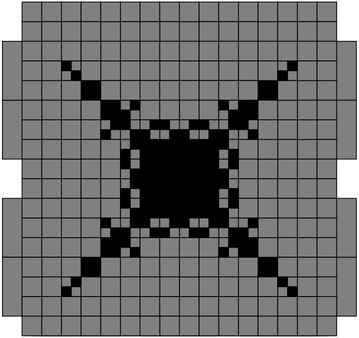
Hexagonal lattice (Elser-Cockayne, Baake-Klitzing-Schlottmann):

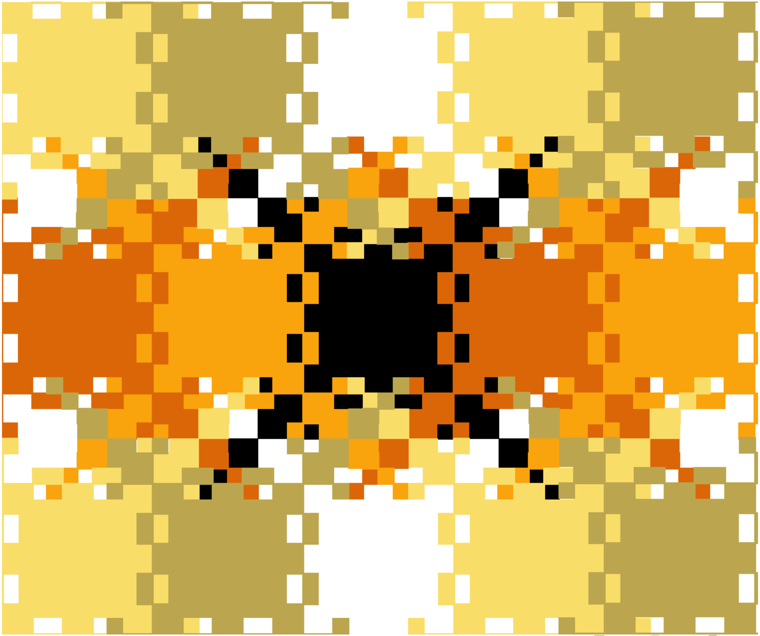


Rectangular lattice









Application: Minimal matchings

Consider the square lattice $\mathbb{Z}^2$, and $R_{45}\mathbb{Z}^2$, the square lattice rotated by 45°.

Problem: Find a perfect matching between $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$ with maximal distance not larger than $C > 0$. How small can C be?

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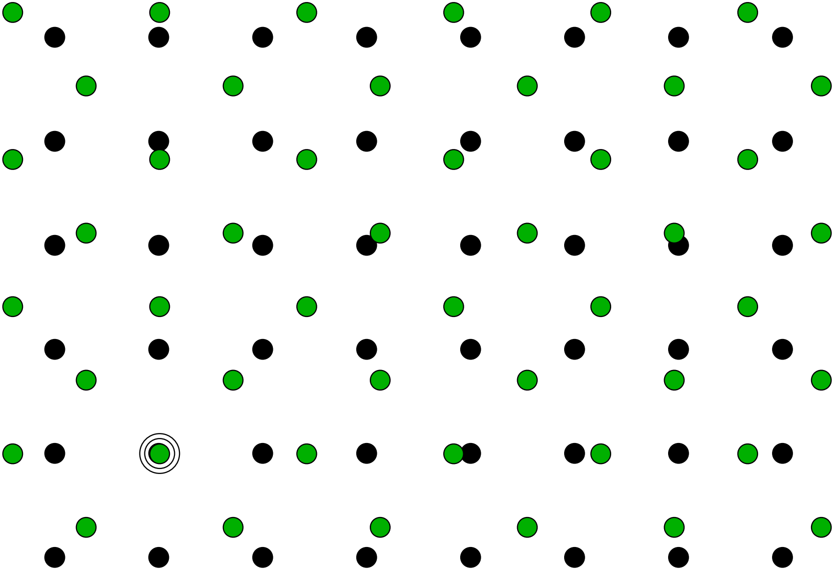
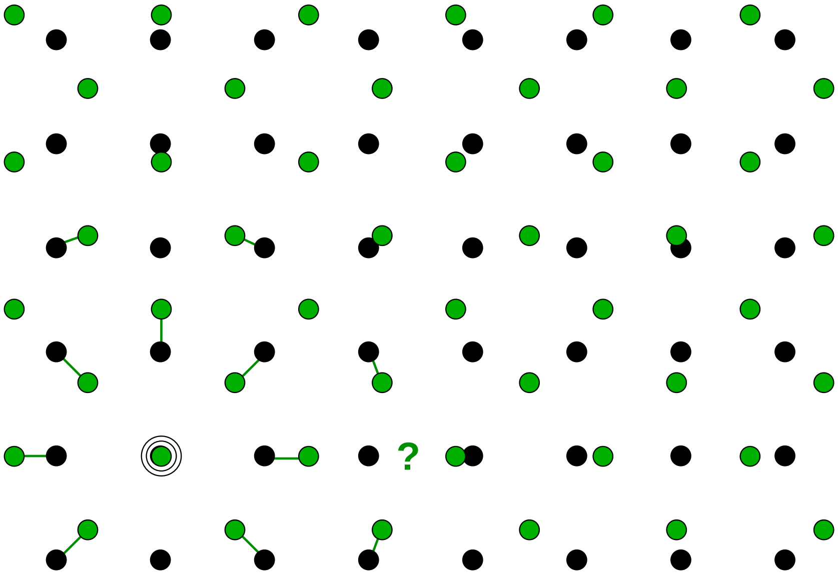
That is, find $f : \mathbb{Z}^2 \longrightarrow R_{45}\mathbb{Z}^2$, where f is bijective and

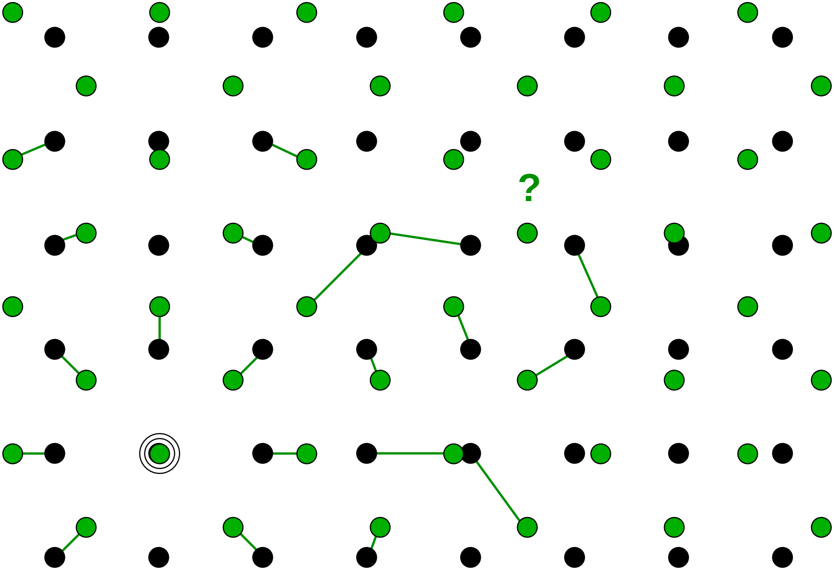
$$\forall x \in \mathbb{Z}^2 : \quad d(x, f(x)) \le C$$

for C as small as possible.

(It is easy to see that $C \ge \frac{\sqrt{2}}{2} = 0.7071\ldots$.)







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How?

- Consider $\mathbb{Z}^2 + F$. Each $x + F$ $(x \in \mathbb{Z}^2)$ contains exactly one point of $\mathbb{Z}^2$ in its centre.
- F is also fundamental domain for $R_{45}\mathbb{Z}^2$. Thus each $x + F$ $(x \in \mathbb{Z}^2)$ contains exactly one point $x' \in R_{45}\mathbb{Z}^2$.
- Let $f(x) = x'$.

This (and its analogues) yields good matchings for

- $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$: $C = 0.92387\ldots.$
- The hexagonal lattice H and $R_{30}H$: $C = 0.78867\ldots$
- A rectangular lattice P and $R_{90}P$: $C \leq \frac{1}{\sqrt{2}}\frac{\sqrt{5}+1}{2}b.$

Here, b is the length of the longer lattice basis vector of P.

What next?

- Rhombic lattices
- Higher dimensions
- IFS: Not even known for Elser's example
- Dimension of the boundary


Thank you.