Model sets, Meyer sets and quasicrystals

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- ~~Model sets, Meyer sets and quasicrystals~~
- Quasicrystals, cut-and-project sets and Meyer sets

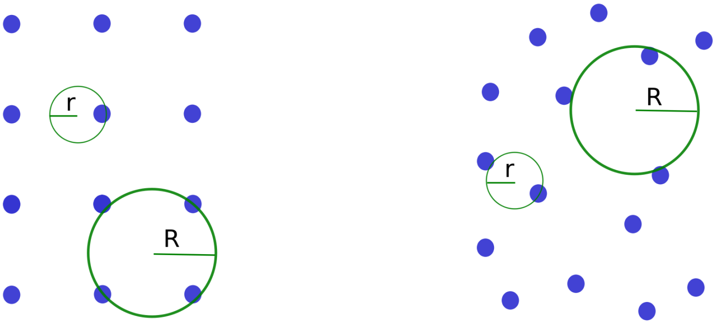
Delone sets

A point set in some metric space X is called a *Delone set* if

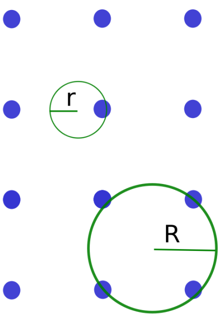
- There is $R > 0$ such that each ball of radius R contains at least one point of Λ
- There is $r > 0$ such that each ball of radius r contains at most one point of Λ

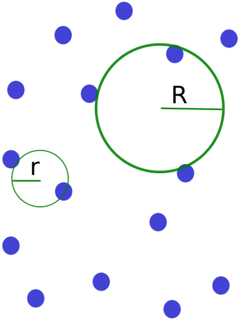
In particular a Delone set is infinite (if X is unbounded)

Delone sets



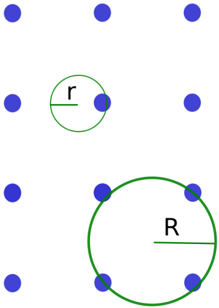
Delone sets





ordered	disordered
point lattice	random
crystallographic	no symmetry at all

Delone sets



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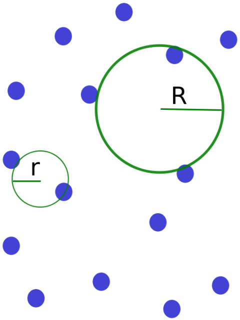


ordered

point lattice

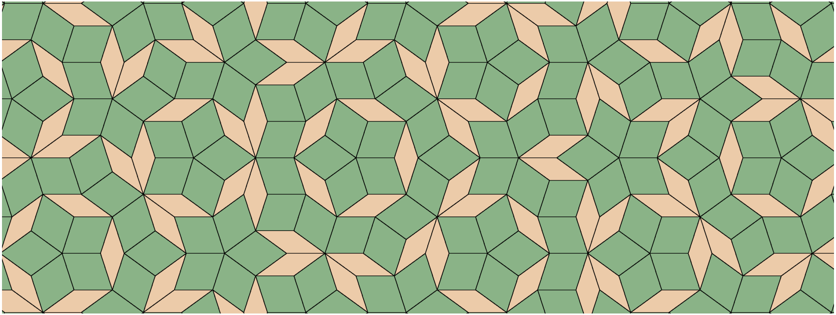
crystallographic

disordered

random

no symmetry at all

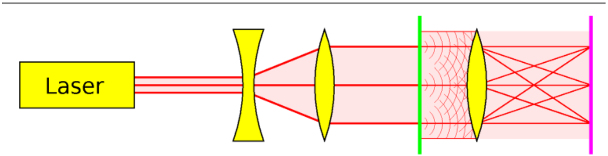
Delone sets that are not crystals, but showing a high degree of order: **Quasicrystals**.



(Here a Penrose tiling. The vertices form a Delone set.)

Diffraction

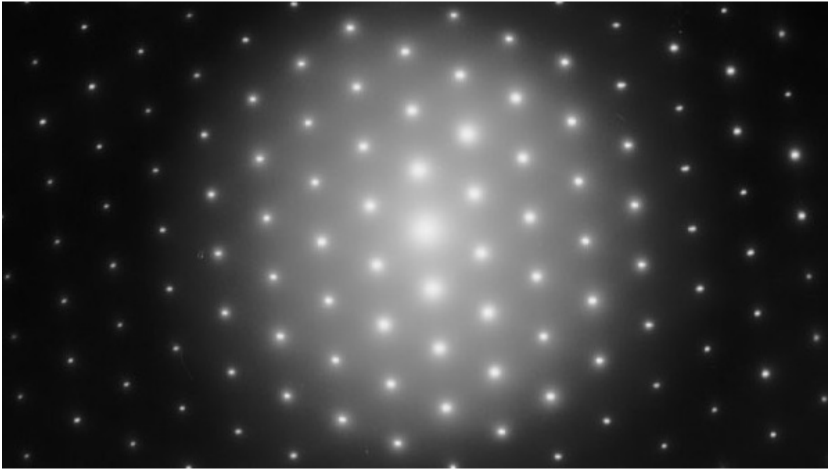
Physical diffraction experiment:



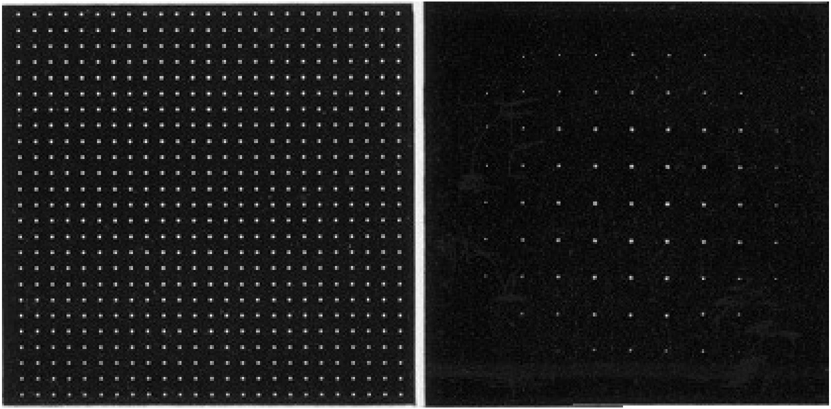
Mathematical diffraction experiment:

- Tiling $\rightsquigarrow$ discrete point set Λ.
- $\gamma_\Lambda = \lim_{r\to\infty} \frac{1}{\text{vol } B_r} \sum_{x,y \in \Lambda\cap B_r} \delta_{x-y}$.
- Fouriertransform $\widehat{\gamma}_\Lambda$ is the *diffraction spectrum*.

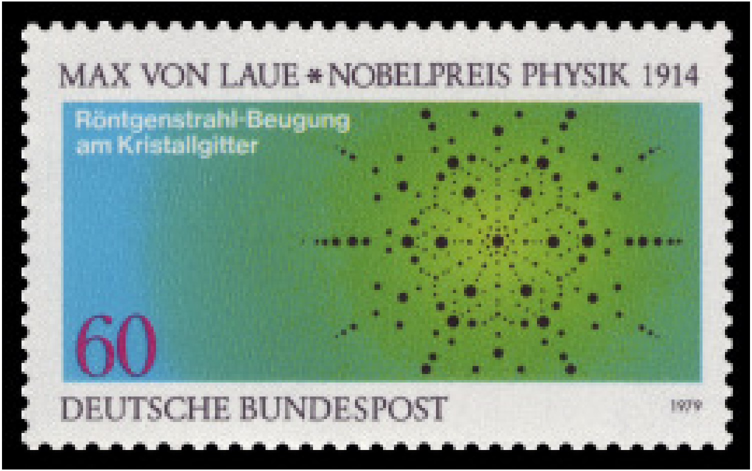
Crystal diffraction (physical experiment):



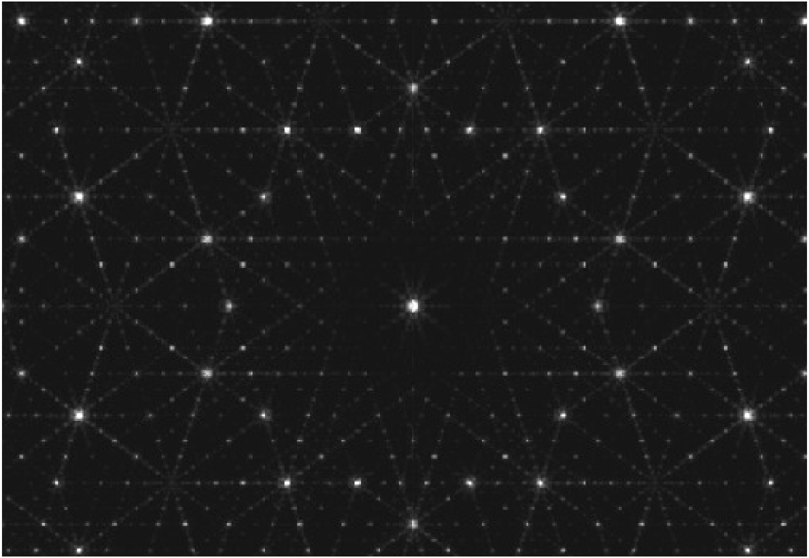
Crystal diffraction (mathematical computation):



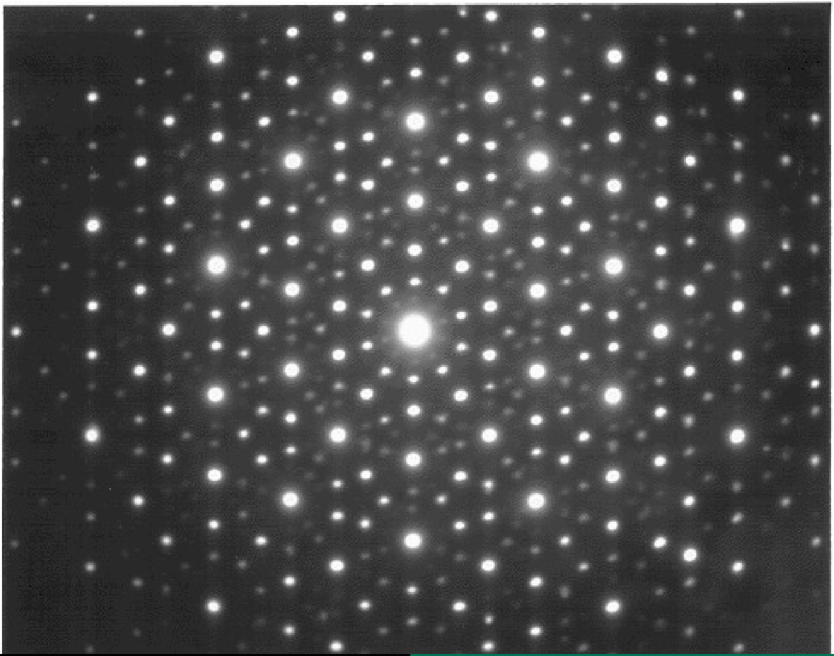
Crystal diffraction: Noble prize 1914



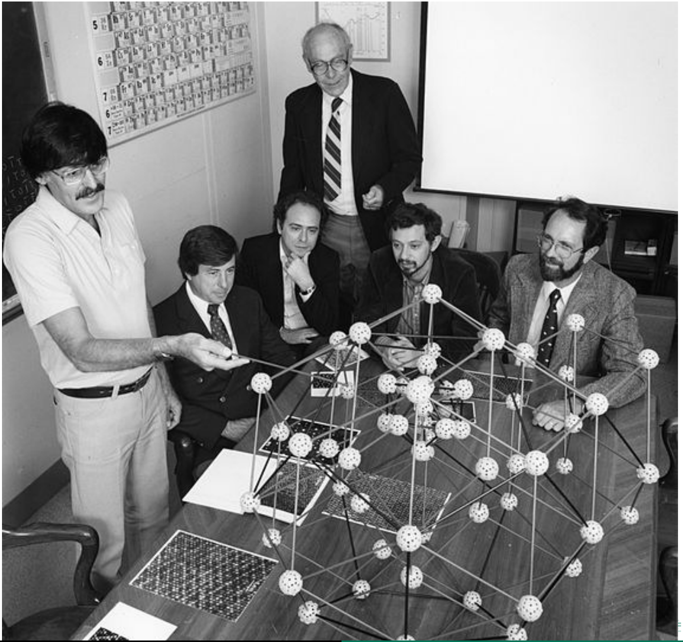
Quasicrystal diffraction (mathematical: Penrose)

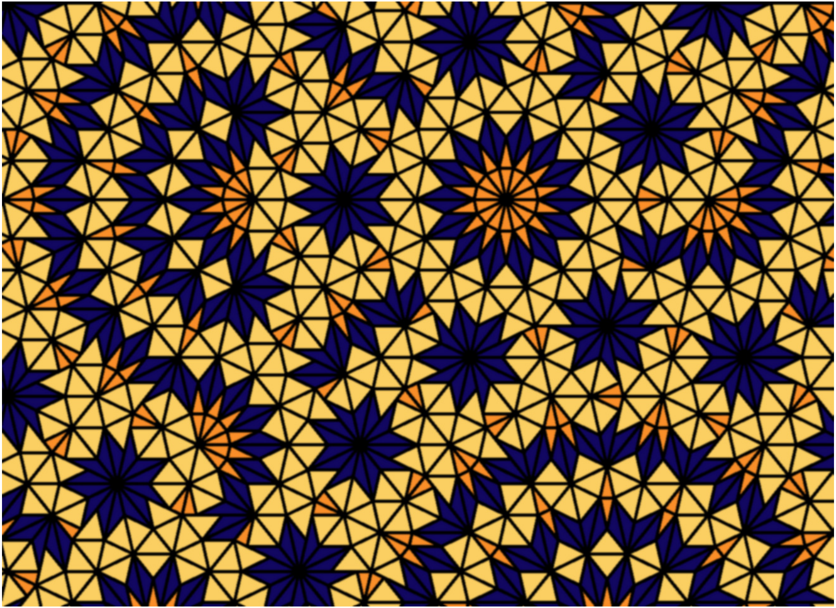


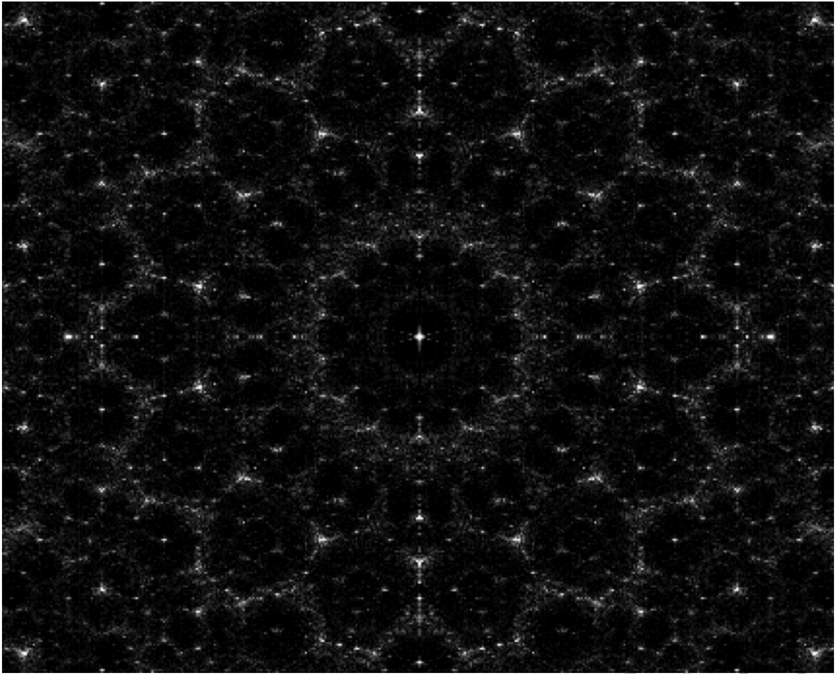
Quasicrystal diffraction (physical experiment: some metallic alloy)



Quasicrystal diffraction: Noble prize 2011 for Danny Shechtman







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For a perfect (mathematical, infinite) crystal by Poisson's summation formula:

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For a perfect (mathematical, infinite) quasicrystal (e.g. Penrose pattern):

$$\widehat{\gamma} = \widehat{\gamma}_{pp}$$

$*\quad *$
$*$

Today there is a well-developed theory of mathematical quasicrystals.

(Even though there is no precise mathematical definition of "quasicrystal")

$*\quad *$
$*$

Two central results:

Cut-and-project sets

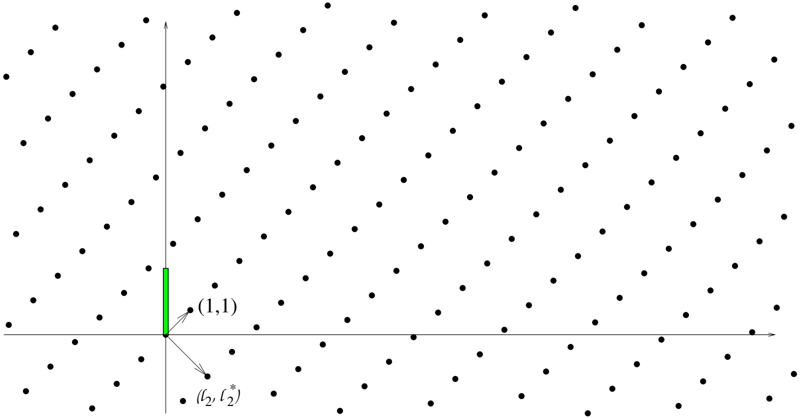
Theorem (deBruijn 1981)

The set of vertices of the Penrose tiling is a cut-and-project set.

Theorem (Hof 1995, Schlottmann 2000)

Each cut-and-project set is pure point diffractive.

Generalisation of lattices: *cut-and-project sets*. ("model sets")



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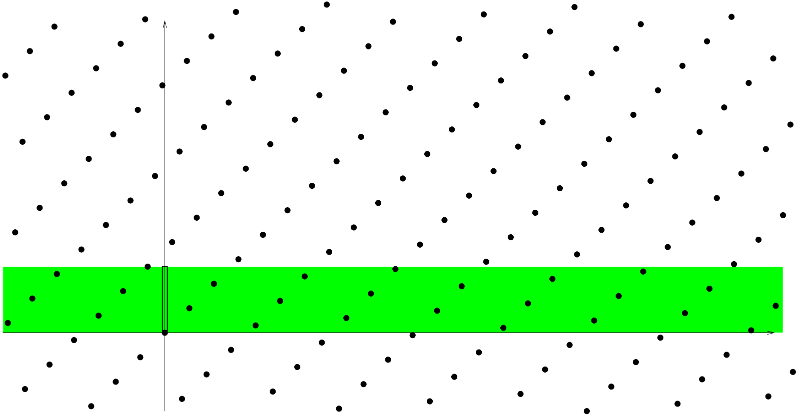

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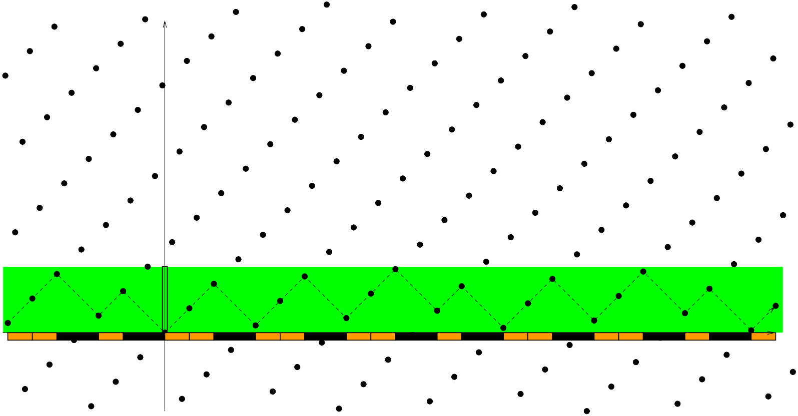

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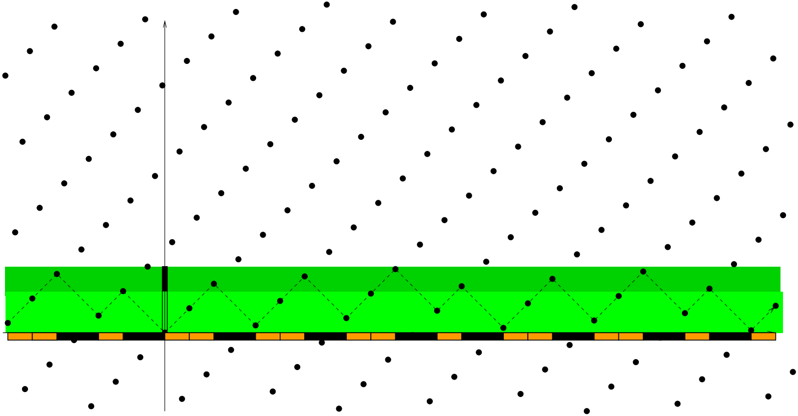
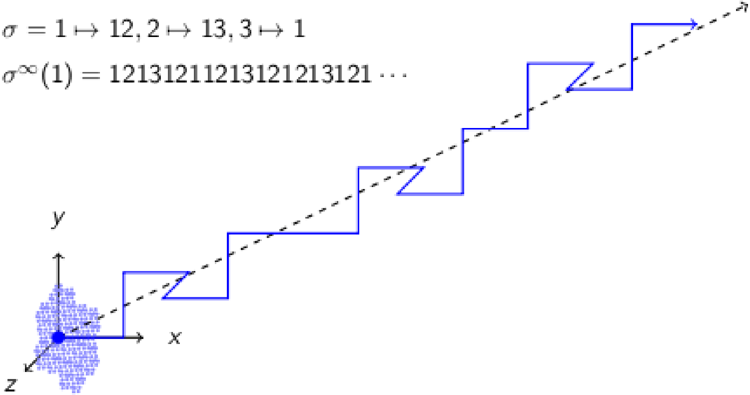


$\sigma = 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

$\sigma^\infty(1) = 1213121121312121 3121 \cdots$



Generalisation of lattices: *cut-and-project sets*. ("model sets")

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

- Λ a *lattice* in $\mathbb{R}^d \times H$ (i.e. cocompact discrete subgroup)
- π_1, π_2 *projections*
  - $\pi_1|_\Lambda$ injective
  - $\pi_2(\Lambda)$ dense
- W *compact*
  - cl(int(W))= W
  - $\mu(\partial(W)) = 0$

Then $V = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set*.

$* \quad *$
$*$

...but the story started already in the 70s.

$* \quad *$
$*$

Meyer sets

For a lattice $\Lambda \subset \mathbb{R}^d$ holds:

- $\Lambda - \Lambda = \Lambda$ (where $\Lambda - \Lambda = \{x - y \,|\, x, y \in \Lambda\}$)
- In particular, $\Lambda - \Lambda$ is Delone.

Meyer sets

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- $\Lambda - \Lambda = \Lambda$ (where $\Lambda - \Lambda = \{x - y \mid x, y \in \Lambda\}$)
- In particular, $\Lambda - \Lambda$ is Delone.

Generalisation: A *Meyer set* is a Delone set Λ such that

- $\Lambda - \Lambda \subseteq \Lambda + F$ for some finite F.

Theorem (Meyer 1972)

Λ is a Meyer set iff Λ^{ε} is relatively dense for all $\varepsilon > 0$.
$\Lambda^{\varepsilon} := \{k \in \mathbb{R}^d \,|\, \forall x \in \Lambda : \; |e^{2\pi i x \cdot k} - 1| \leq \varepsilon\}$

Theorem (Meyer 1972)

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Meyer studied cut-and-project sets already before 1972!
Y. Meyer: *Algebraic numbers and harmonic analysis* (1972)

Theorem (Lagarias 1995)

Λ is a Meyer set iff $\Lambda - \Lambda$ is Delone.

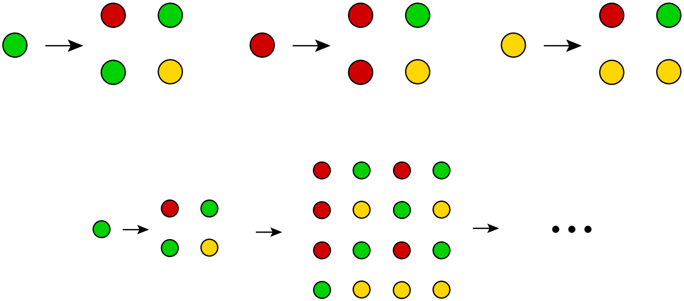
Typical questions today:

- Q1: Is each Meyer set pure point diffractive?
- Q2: Is each pure point diffractive set Meyer / Delone?
- ...

Q1: Is each Meyer set pure point diffractive?

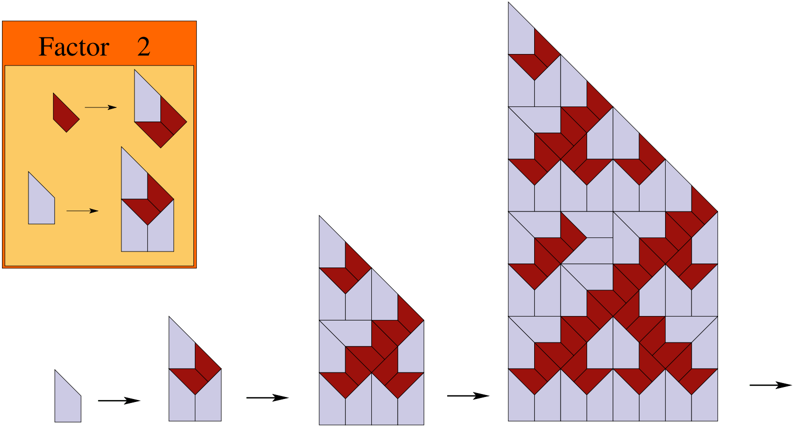
A: No. (Solomyak, Lee-Moody, F-Sing)

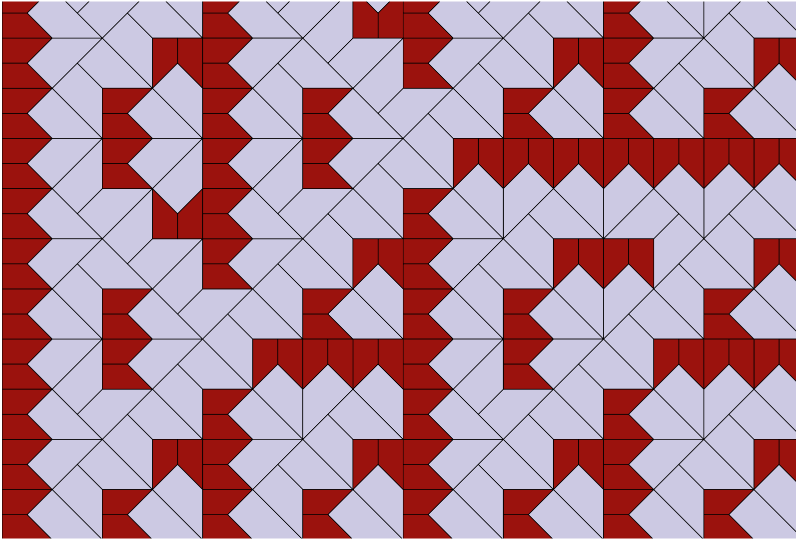
Lattice substitution systems: Essentially coloured lattices, with no translational symmetry.



Theorem (Solomyak 1998, Lee-Moody 2002)

Let L be a lattice substitution system.
L is a cut-and-project set **iff** *L is pure point diffractive (i.e., the red points are, the green points are...).*


Factor 2

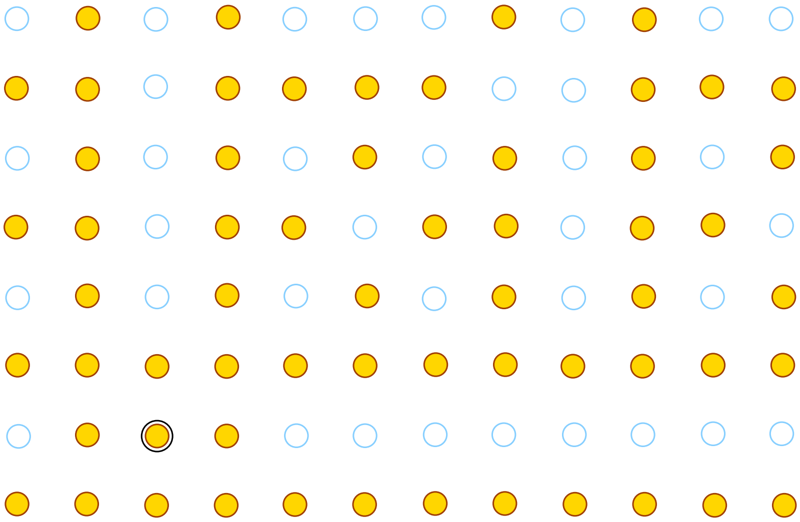


In F-Sing 2007 we used the Theorem of Lee-Moody-Solomyak to show that the vertex set of this tiling is not pure point diffractive.

Q2: Is each pure point diffractive point set Meyer / Delone?

A: No. (Baake-Moody-Pleasants)

Consider the "visible lattice points" of the square lattice.



The visible lattice points:

$$V = \{(n,m) \, | n, m \text{ coprime}\}$$

V has arbitrary large holes, so V is not Delone / Meyer! But

Theorem (Baake-Moody-Pleasants)

V is pure point diffractive.

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Theorem (Baake-Moody-Pleasants)

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$*$ $*$
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Thank you!