Highly symmetric fundamental cells for planar lattices

Dirk Frettlöh

Fakultät für Mathematik (SFB 701)
Universität Bielefeld

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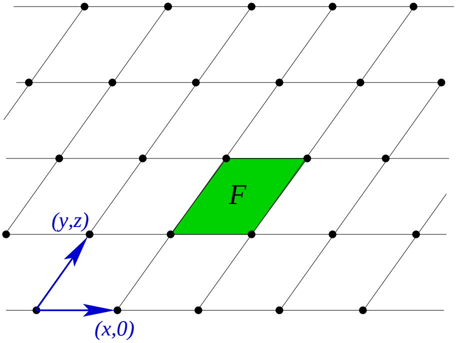
Point lattice Γ in $\mathbb{R}^d$: the $\mathbb{Z}$-span of d linearly independent vectors.

Fundamental cell of Γ: $\overline{\mathbb{R}^d/\Gamma}$.



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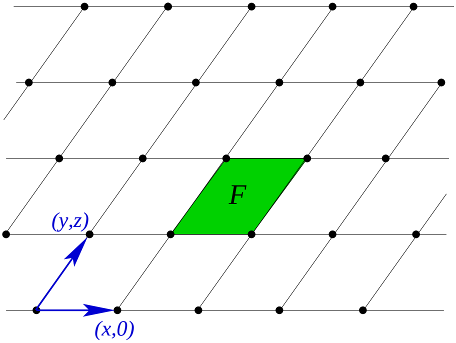
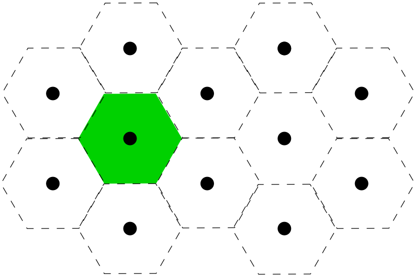
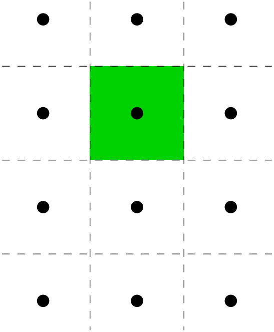


Point group $P(\Gamma)$ of Γ: All linear isometries f with $f(\Gamma) = \Gamma$.

Trivially, each lattice Γ has a fundamental cell which symmetry group is $P(\Gamma)$.

For instance, take the Voronoi cell of a lattice point x. (That is the set of points closer to x than to each other lattice point.)





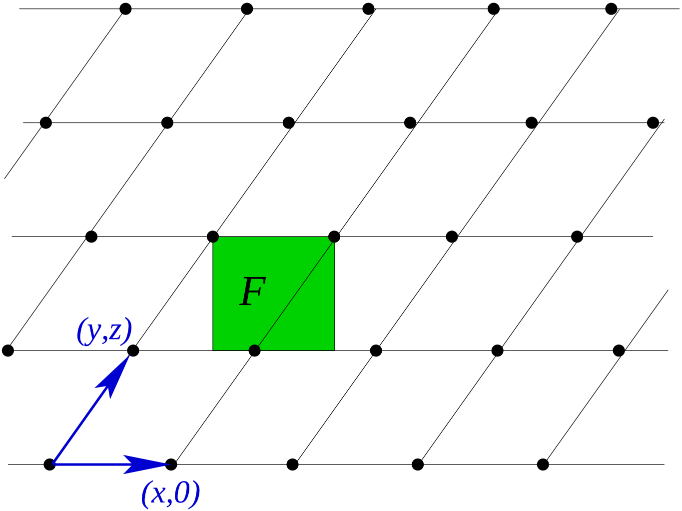
Main result

Theorem (Elser, Fr.)

Let $\Gamma \subset \mathbb{R}^2$ *be a lattice, but not a rhombic lattice. Then there is a fundamental cell* F *of* Γ *whose symmetry group* $S(F)$ *is strictly larger than* $P(\Gamma)$*:* $\quad [S(F) : P(\Gamma)] = 2$.

'Rhombic lattice' means here: one with basis vectors of equal length, but neither a square lattice nor a hexagonal lattice.

Generic lattice:



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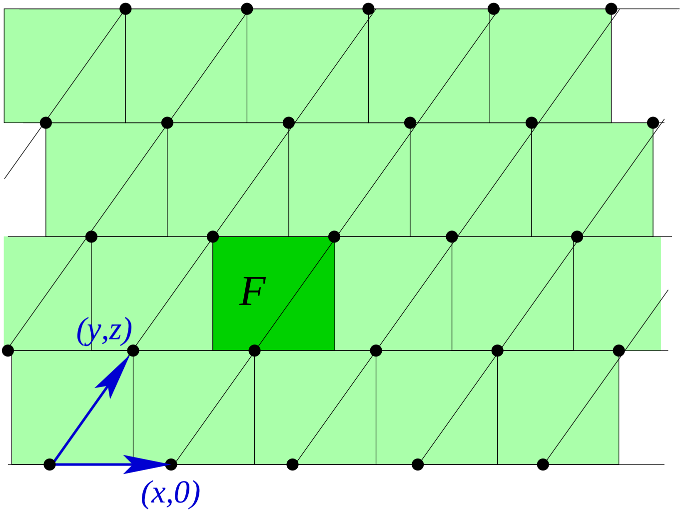
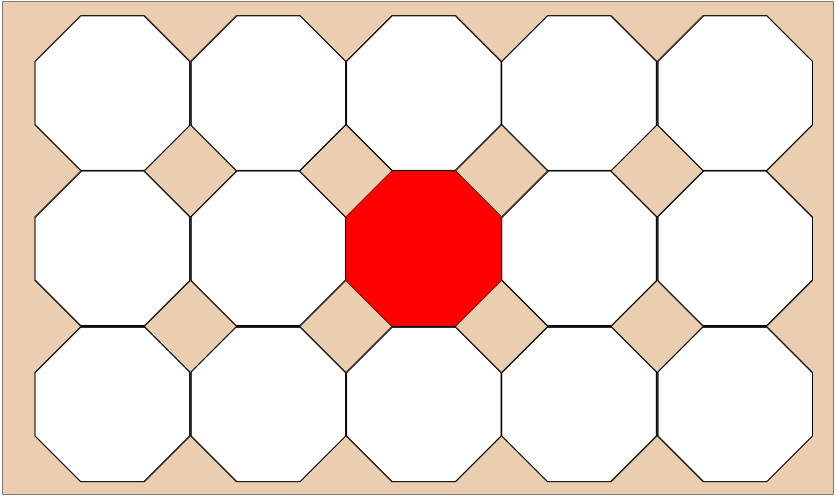
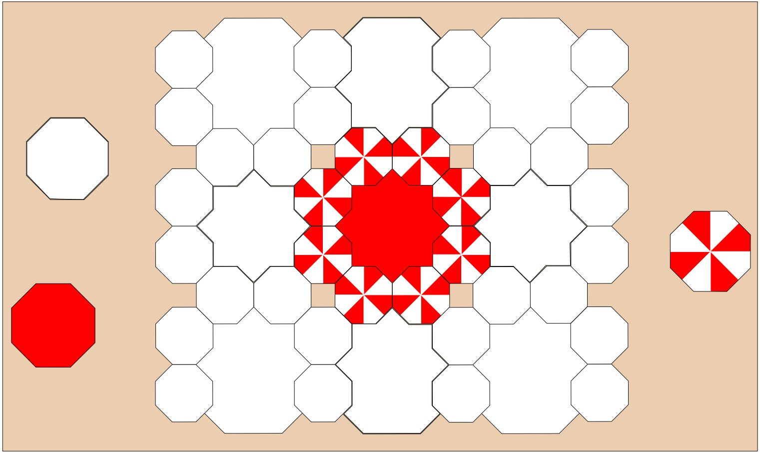
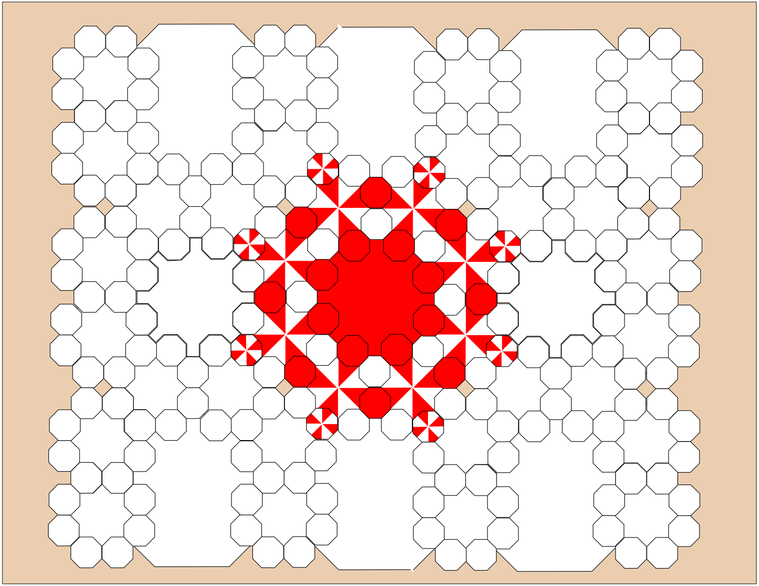
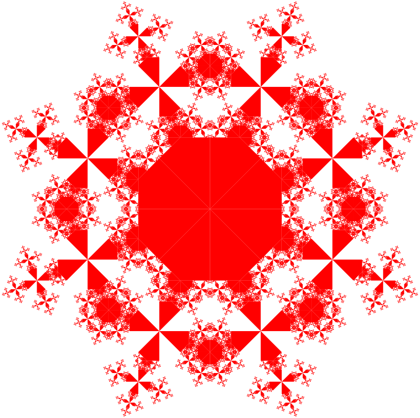


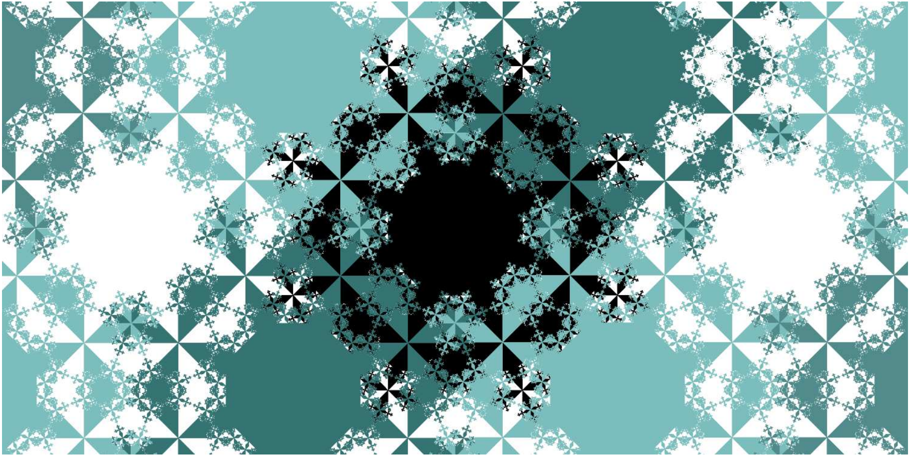
Square lattice (Veit Elser):



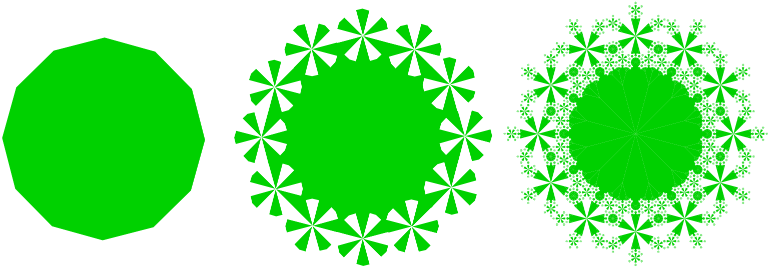




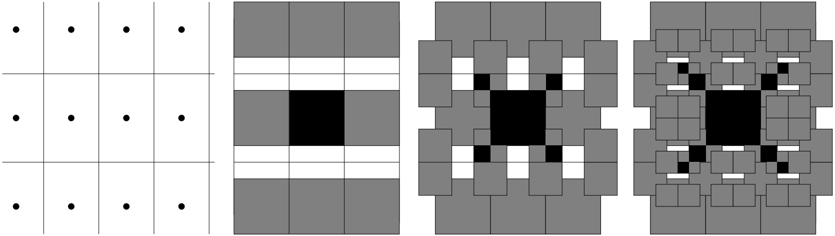


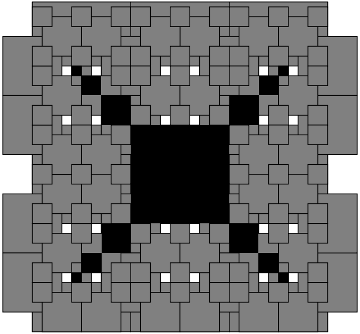


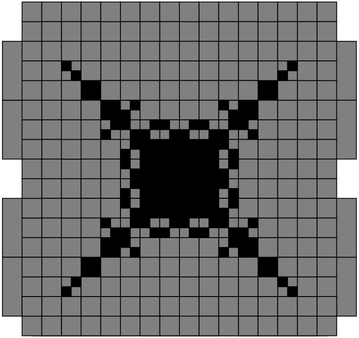
Hexagonal lattice (Elser-Cockayne, Baake-Klitzing-Schlottmann):

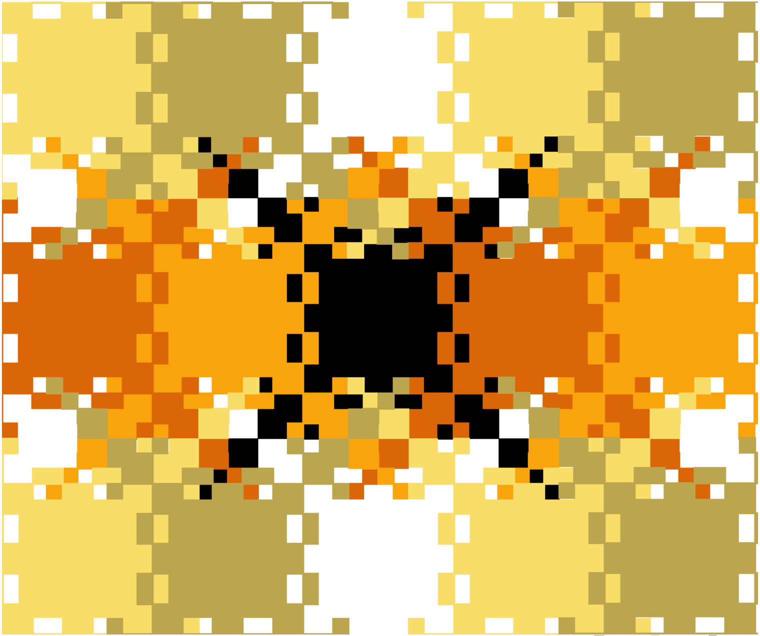


Rectangular lattice









Application: Minimal matchings

Consider the square lattice $\mathbb{Z}^2$, and $R_{45}\mathbb{Z}^2$, the square lattice rotated by 45°.

Problem: Find a perfect matching between $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$ with maximal distance not larger than $C > 0$. How small can C be?

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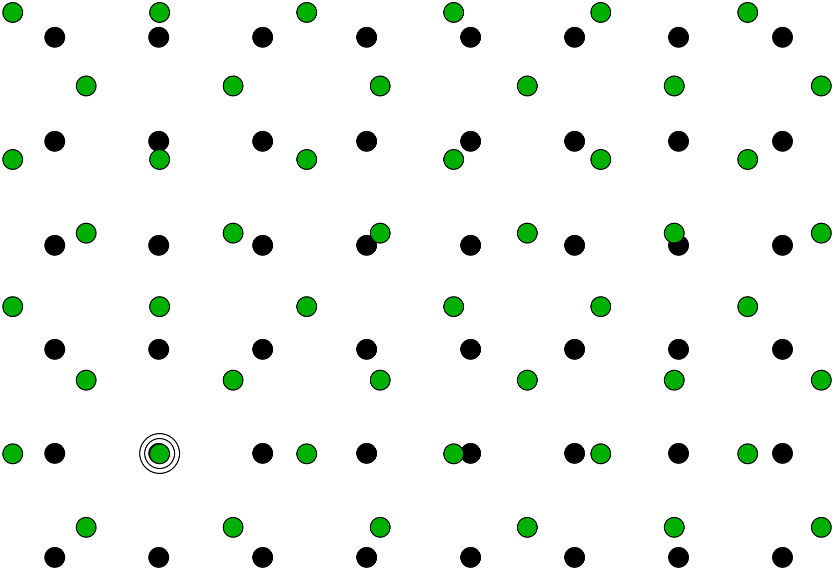
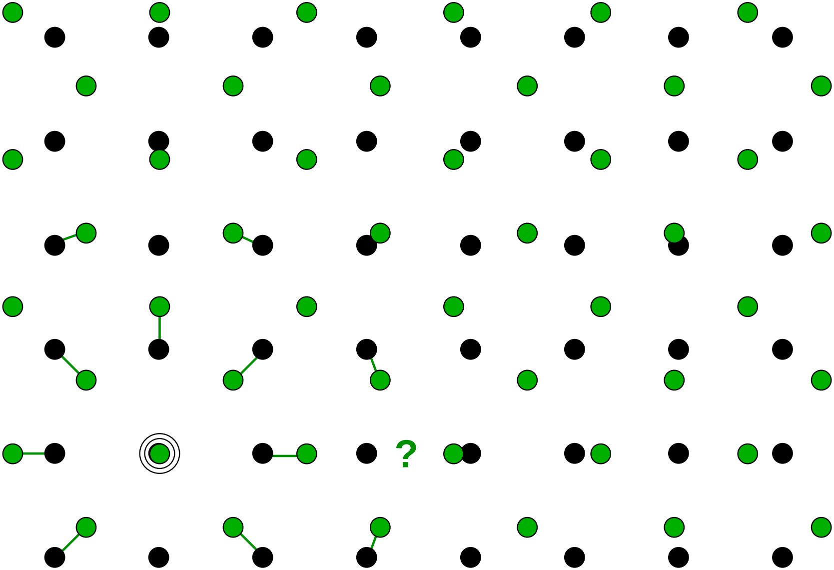
That is, find $f : \mathbb{Z}^2 \to R_{45}\mathbb{Z}^2$, where f is bijective and

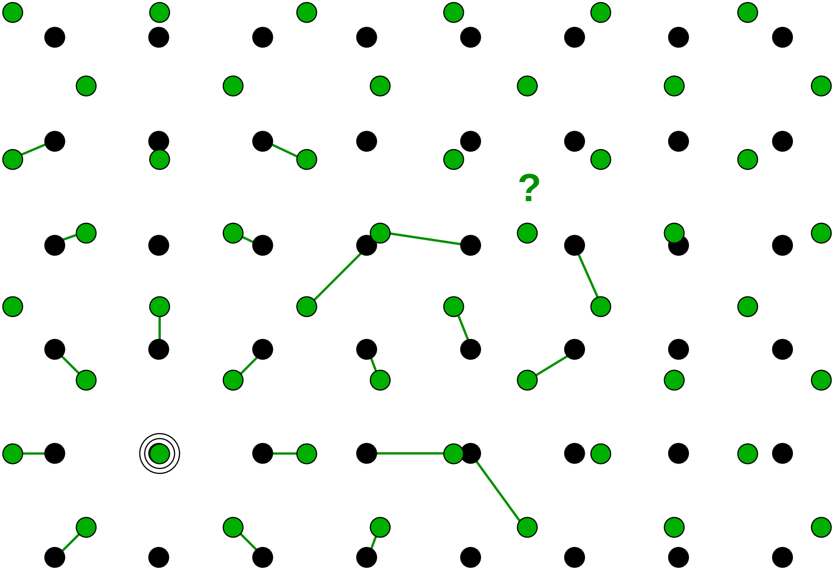
$$\forall x \in \mathbb{Z}^2 : \quad d(x, f(x)) \leq C$$

for C as small as possible.

(It is easy to see that $C \geq \frac{\sqrt{2}}{2} = 0.7071\ldots$.)







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How?

- Consider $\mathbb{Z}^2 + F$. Each $x + F$ ($x \in \mathbb{Z}^2$) contains exactly one point of $\mathbb{Z}^2$ in its centre.
- F is also fundamental domain for $R_{45}\mathbb{Z}^2$. Thus each $x + F$ ($x \in \mathbb{Z}^2$) contains exactly one point $x' \in R_{45}\mathbb{Z}^2$.
- Let $f(x) = x'$.

This (and its analogues) yields good matchings for

- $\mathbb{Z}^2$ and $R_{45}\mathbb{Z}^2$: $C = 0.92387\ldots.$
- The hexagonal lattice H and $R_{30}H$: $C = 0.78867\ldots$
- A rectangular lattice P and $R_{90}P$: $C \le \frac{1}{\sqrt{2}}\frac{\sqrt{5}+1}{2}b.$

  If $b/a \in \mathbb{Z}$: $C = \frac{1}{\sqrt{2}}b.$

Here, b is the length of the longer lattice basis vector of P.

What next?

- Rhombic lattices
- Higher dimensions
- Hyperbolic spaces
- Iterated function systems
- Dimension of the boundaries
- Connectivity
- Uniqueness of the fundamental cell
- Better matchings
- Cut-and-Project Sets
- ...


Thank you.