Bi-Lipschitz equivalence and bounded distance equivalence of Delone sets

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joint work with Alexey Garber

- Basics
- Dimension 1
- Higher dimensions

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each ball of radius R contains at least one point of Λ (*relatively dense*)

(Aka "separated nets". Can also live in $\mathbb{H}^d$, $(\mathbb{Q}_p)^d$...)

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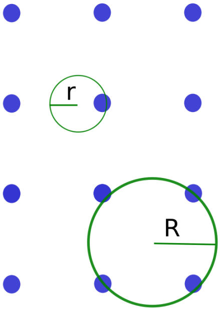
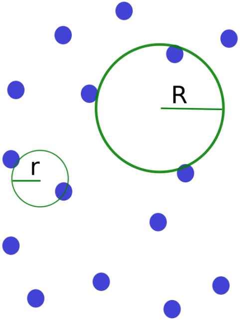


crystallographic



disordered

Two relations between Delone sets:

$\Lambda \overset{\mathrm{bil}}{\sim} \Lambda'$ *(bilipschitz equivalent)*:

There is $f : \Lambda \to \Lambda'$ bijective with

$$\exists c > 0 \quad \forall x, y \in \Lambda \quad \frac{1}{c}|x - y| \leq |f(x) - f(y)| \leq c|x - y|$$

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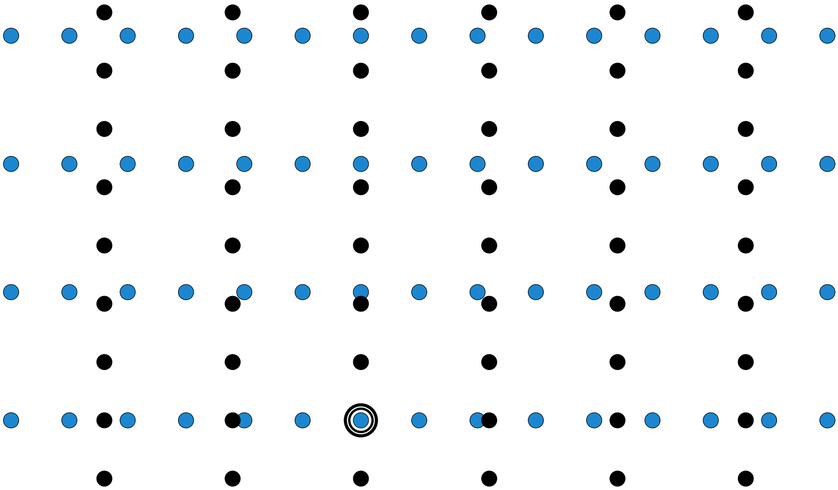
$\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$ *(bounded distance equivalent)*:

There is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$

Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$?



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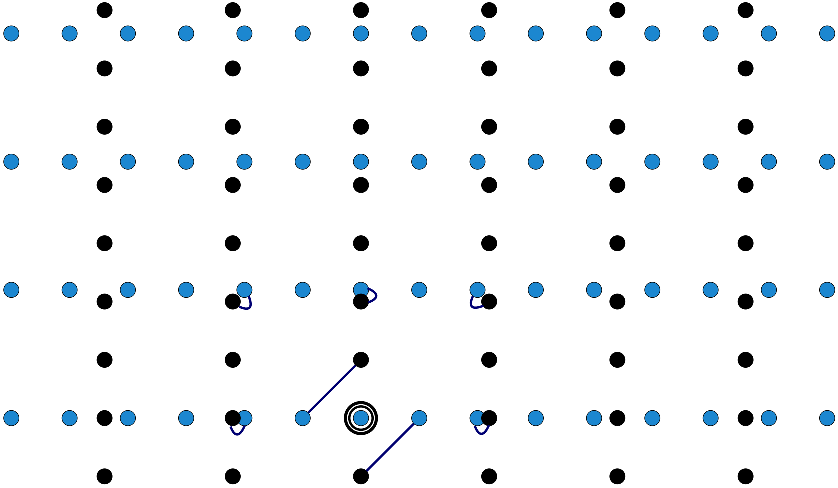

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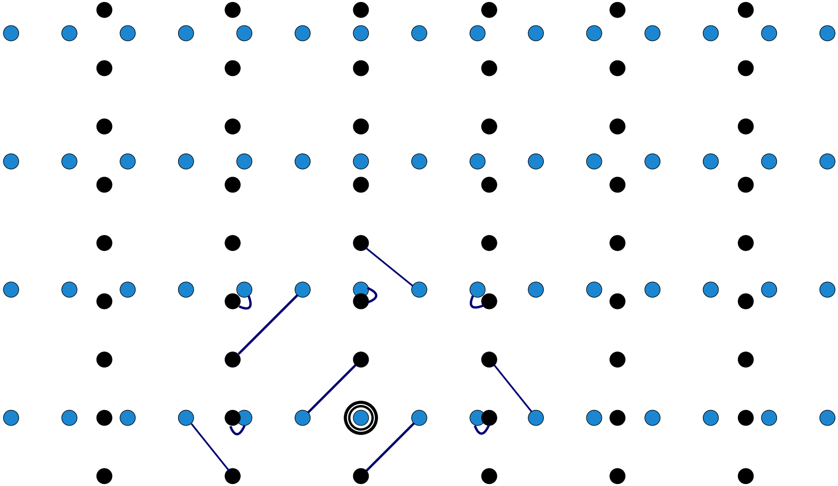

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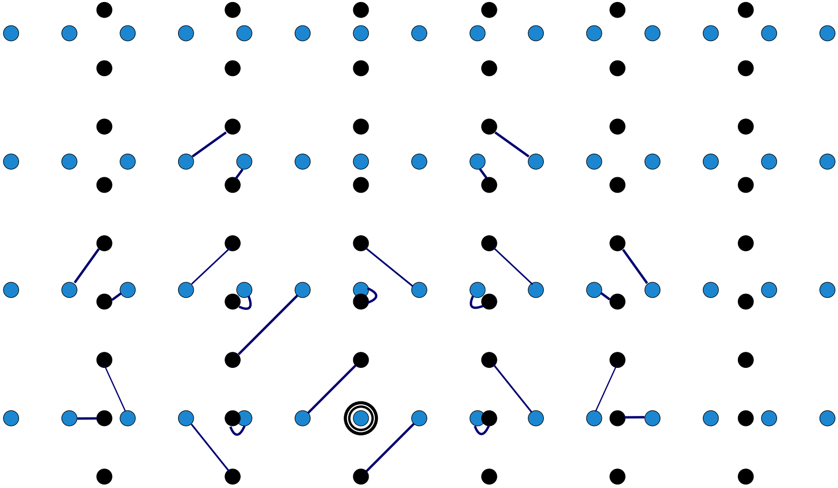

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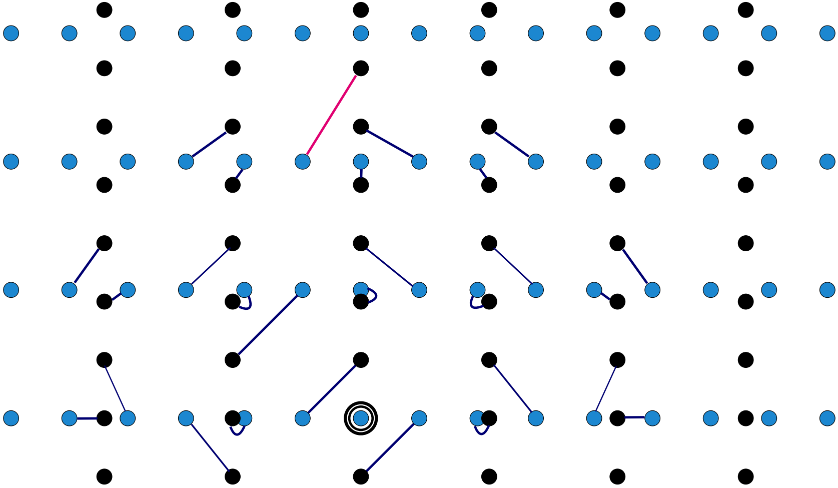

Some basic results

Lemma (1)

Bilipschitz equivalence and bounded distance equivalence are equivalence relations.

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Lemma (2)

Let Λ, Λ' *be Delone sets in* $\mathbb{R}^d$. *If* $\Lambda \stackrel{\mathrm{bd}}{\sim} \Lambda'$, *then* $\Lambda \stackrel{\mathrm{bil}}{\sim} \Lambda'$.

Warmup: Dimension 1

Lemma (3)

Let Λ, Λ' be Delone sets in $\mathbb{R}$ (with Euclidean metric).
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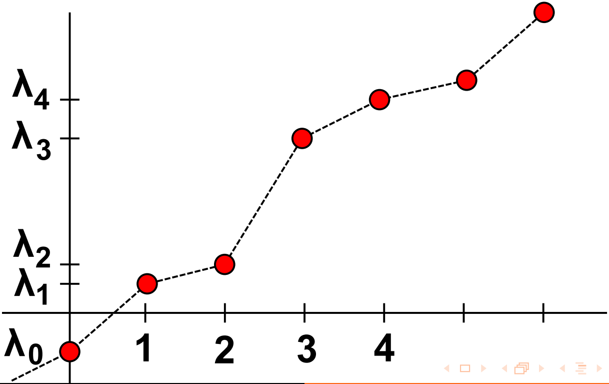
Proof (by image): Show $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}$.

Let $\Lambda = \{\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots\}, \quad \lambda_i < \lambda_{i+1}$. Plot (i, λ_i):



Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$? Always? No:

Examples:

- $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \stackrel{\text{bd}}{\not\sim} \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$
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Density matters. Preliminary definition:

$$\operatorname{dens}(\Lambda) := \lim_{r \to \infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]),$$

if it exists.

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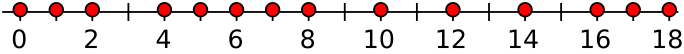
if it exists. Does not need to exist:



Oscillates between $\frac{2}{3}$ and $\frac{5}{6}$.

Question: If dens(Λ)=dens(Λ'), is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?

Lemma

Let Λ, Λ' be periodic. Then dens(Λ)=dens(Λ') implies $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$.
(True even in $\mathbb{R}^d$ for $d \geq 2$)

Interesting examples are non-periodic.

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Interesting examples are non-periodic.

Theorem (Kesten 1966)

Let $\xi \in [0,1]$, $0 \leq a < b \leq 1$ and define

$$\Lambda := \{k \in \mathbb{Z} \mid a \leq \big(k\xi \bmod 1\big) < b\}.$$

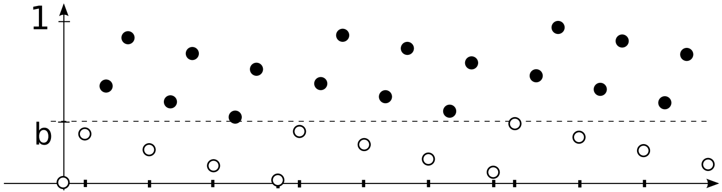
Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - n(b-a)$ is bounded, if and only if $b - a = k\xi \mod 1$ for some $k \in \mathbb{Z}$.

(if-part: Hecke 1921, Ostrowski 1927)

Choose $\xi \in [0,1]$ irrational, let $0 < b \leq 1$ and define

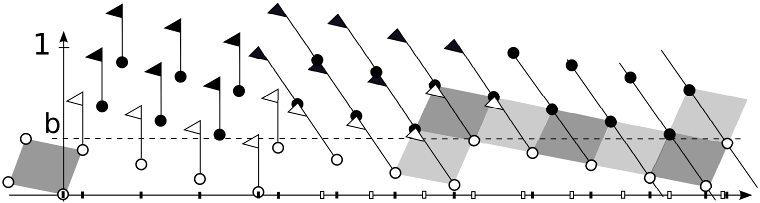
$$\Lambda_b := \{k \in \mathbb{Z} \mid 0 \leq \big(k\xi \mod 1\big) < b\}.$$

Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - nb$ is bounded, if and only if $b = k\xi \mod 1$ for some $k \in \mathbb{Z}$.



The image shows $\{(k, k\xi \text{ mod } 1) \mid k = 0, 1, 2, \ldots\}$.

Proof (by image) of if-part: (F-Gähler 2011, Duneau-Oguey 1990):



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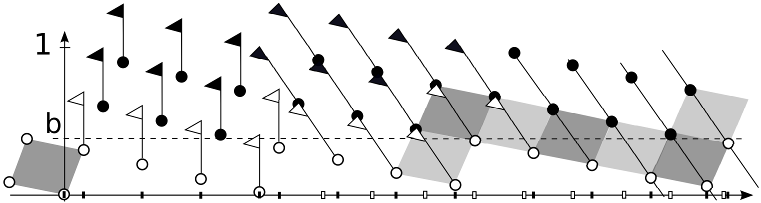

In particular Kesten yields Delone sets Λ_b that are not bounded distance equivalent to any $c\mathbb{Z}$. Even when $\text{dens}(\Lambda_b)$ exists!

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Any two Delone sets in $\mathbb{H}^d$ $(d \geq 2)$ are bounded distance equivalent, hence bilipschitz equivalent.

Higher dimensional spaces

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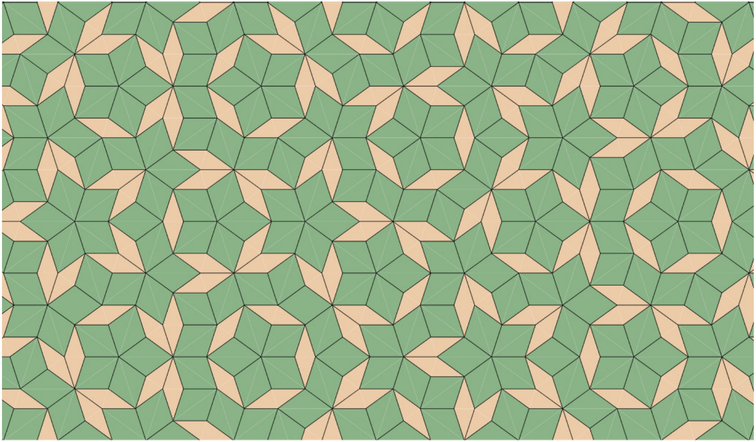
Cool! Let us study some problems in this field. E.g.

1. Are the vertices of the Penrose tiling bounded distance equivalent to some lattice?
2. How many equivalence classes wrt $\overset{\text{bd}}{\sim}$ resp. $\overset{\text{bil}}{\sim}$?

Recall: Interesting examples are non-periodic.
Like the Penrose tiling:



Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^2$.

Repetitive: there is R such that congruent copies of any local patch in Λ occur in every ball of radius R.
Linear repetitive: R depends linearly on the diameter of the patch.

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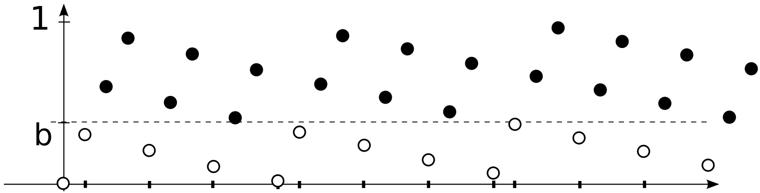

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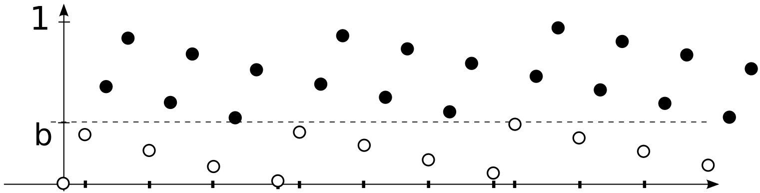

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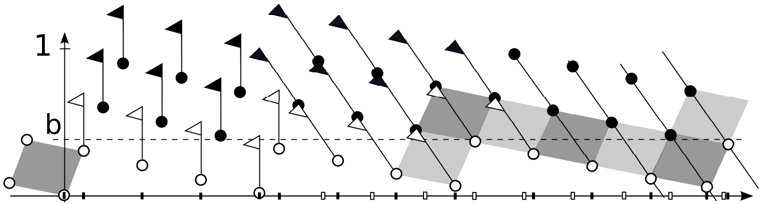
One can state the argument in purely algebraic terms:

- X an $\mathbb{R}$-vector space (here $X = \mathbb{R}^2$),
- $X = V_p + V_i$ (here: horizontal + vertical), $W \subset V_i$ compact set (here $W = [0, b]$),
- π_p projection to V_p (here: $\downarrow$),
- π_i projection to V_i (here: $\leftarrow$),
- Γ discrete cocompact subgroup (here: black and white points)



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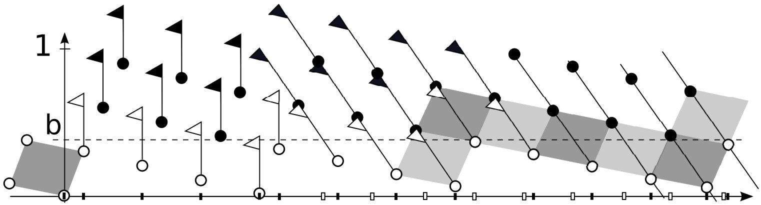

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- $Y = \pi_i^{-1}(W) \cap \Gamma$ (here: white points),
- $\Lambda = \pi_p(Y)$
- Z subgroup of X with $V_p + Z = X$, $Z \cap \Gamma$ compact (here "lattice direction" for projection)
- π_Z corresponding projection etc...

...then $\pi_p(Y) \overset{\mathrm{bd}}{\sim} \pi_Z(Y)$.

Other colleagues had the same idea: Haynes-Kelly-Weiss 2014, Haynes-Koivusalo 2015+.

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Last October I've learned from Alan Haynes that this was done already in

C. Godrèche and C. Oguey:
Construction of average lattices for quasiperiodic structures by the section method, *J. Phys. France* 51 (1990) 21-37

So much on Question 1.

Regarding Question 2:

How many equivalence classes wrt $\overset{\text{bd}}{\sim}$ resp. $\overset{\text{bil}}{\sim}$?

Theorem (Magazinov 2010)

There are $|\mathbb{R}|$ equivalence classes of Delone sets in $\mathbb{R}^d$ wrt $\overset{\text{bil}}{\sim}$.

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Proof: (of 2nd result, sketch) It is easy to show that each Delone set in $\mathbb{R}^d$ is bounded distance equivalent to some subset of $r\mathbb{Z}^d$, where r is the radius of uniformly discreteness.

($|\mathbb{R}|$ many values of r) $\times$ ($|\mathbb{R}|$ many subsets of $\mathbb{Z}^d$) $= |\mathbb{R}|$.

(this shows 'at most $|\mathbb{R}|$ many'. Density yields 'at least $|\mathbb{R}|$ many')

Further research:

- "Only if"-part of Kesten's Theorem
- Let $\Lambda_1 \overset{\text{bil}}{\sim} \Lambda_2$. Is $\Lambda_2 \overset{\text{bil}}{\sim} \Lambda_1 \cup \Lambda_2$? Under which conditions?
- Let $\Lambda \overset{\text{bd}}{\sim} \mathbb{Z}^2$, $\Lambda = \Lambda_1 \dot\cup \Lambda_2$, $\Lambda_1 \overset{\text{bd}}{\sim} \Lambda_2$. Is $\Lambda_1 \overset{\text{bd}}{\sim} \sqrt{2}\mathbb{Z}^2$?
- ...

More in

D.F., Alexey Garber:
Bounded distance and bilipschitz equivalence of Delone sets, preprint,
www.math.uni-bielefeld.de/∼frettloe/papers/bilip-draft.pdf

and references therein.

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Thank you!