Bounded distance equivalence in substitution tilings

Dirk Frettlöh

Joint work with Yaar Solomon (Beer Sheba, Israel)
and Yotam Smilansky (?, USA)

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Soft packings, nested clusters, and condensed matter

Oaxaca September 2019

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each open ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each closed ball of radius R contains at least one point of Λ (*relatively dense*)

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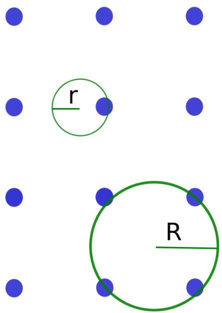
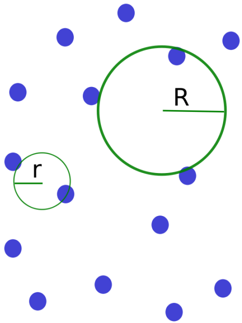


periodic crystal



disordered

An equivalence relation for Delone sets:

Definition

Two Delone sets Λ, Λ' are bounded distance equivalent, if there is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad \|x - g(x)\| < C$$

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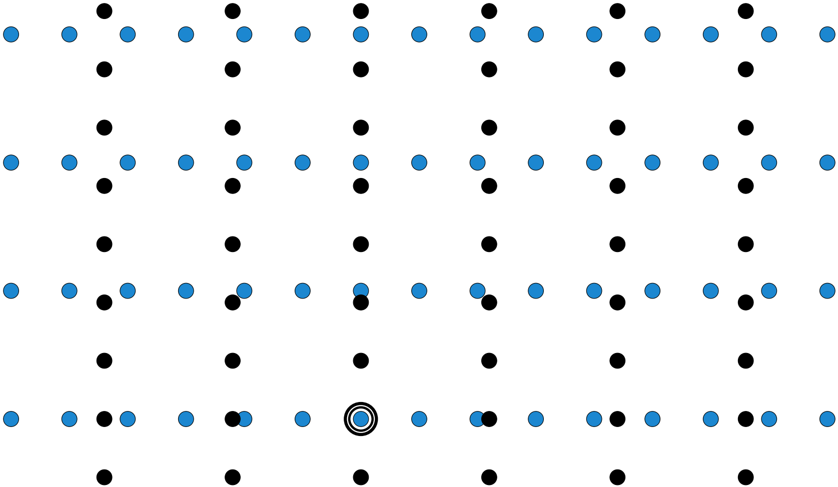
In other words: there is a perfect matching between Λ and Λ' such that matched points have distance $< C$.

Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?



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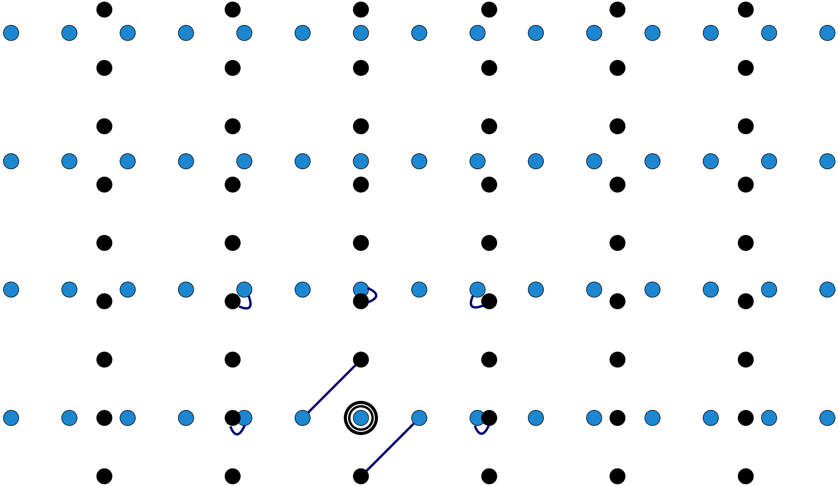

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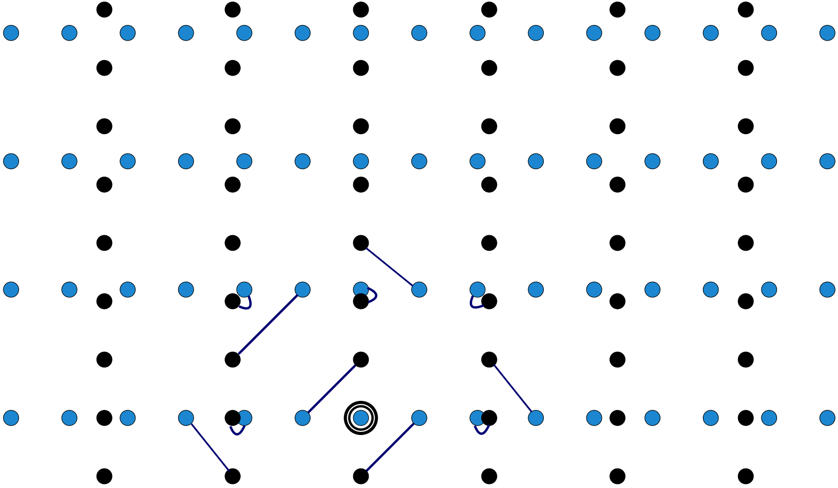

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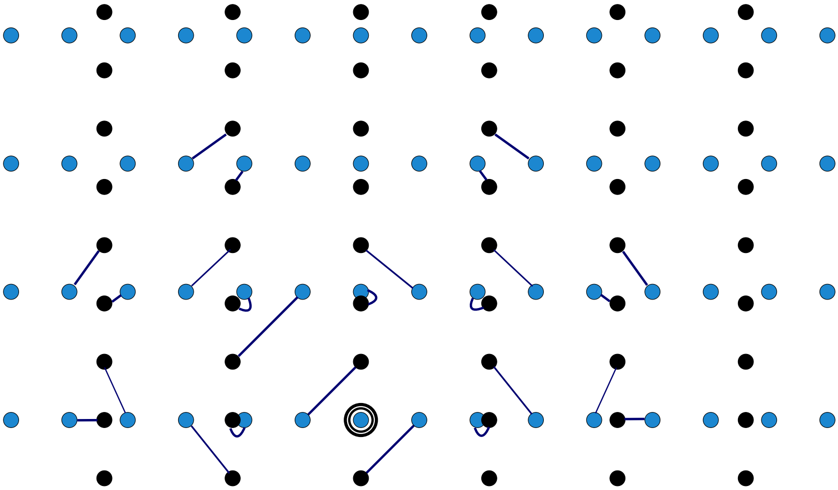

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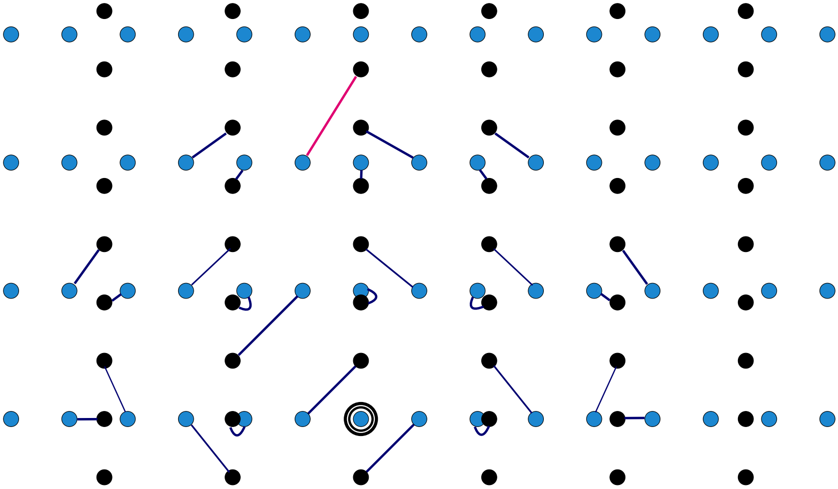

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Of course, density matters: $\mathbb{Z}^2 \overset{\text{bd}}{\not\sim} 2\mathbb{Z}^2$.

Theorem (Duneau-Oguey '91)

For any two d-periodic Delone sets Λ, Λ' in $\mathbb{R}^d$ holds:

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Note: A Delone set Λ in $\mathbb{R}^d$ is *d-periodic* if its period lattice

$$T(\Lambda) = \{t \in \mathbb{R}^d \mid t + \Lambda = \Lambda\}$$

has d linear independent directions. Aka periodic crystal

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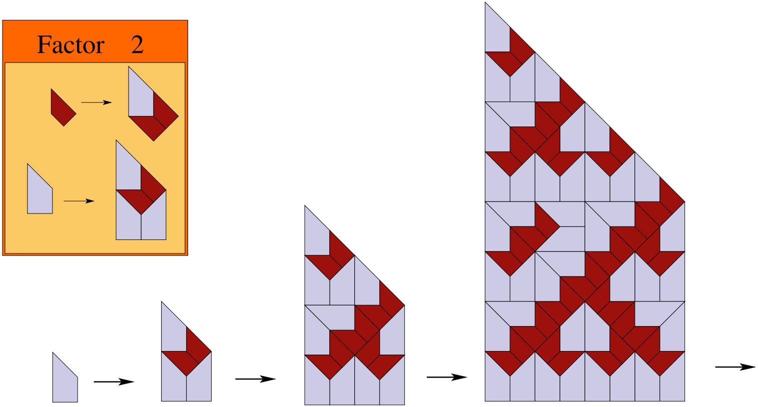
Hence more interesting examples are non-periodic.

A simple way to generate interesting (non-periodic, but highly ordered) Delone sets goes via *substitution tilings*.

Substitution tiling with *substitution factor* 2, and two *prototiles*:



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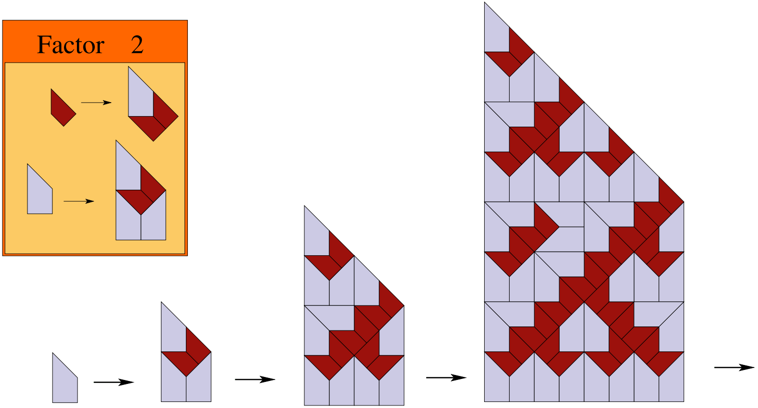
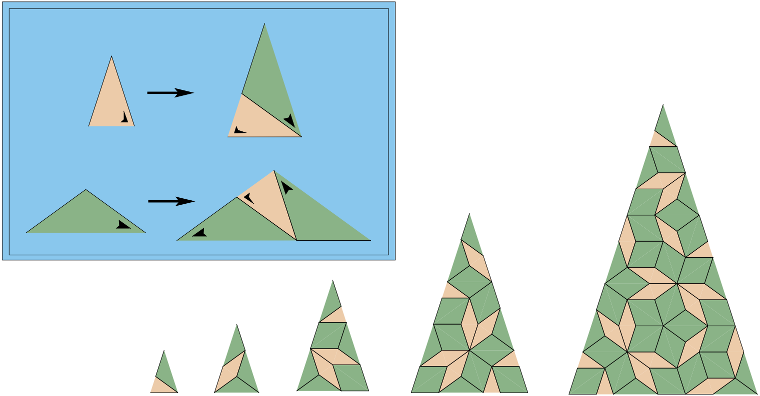


Substitution matrix here $M_\sigma = \left(\begin{smallmatrix} 2 & 2 \\ 1 & 3 \end{smallmatrix}\right)$.

The famous Penrose tiling can also be generated by a tile substitution, with *substitution factor* $\theta = \frac{1}{2}(1+\sqrt{5})$:



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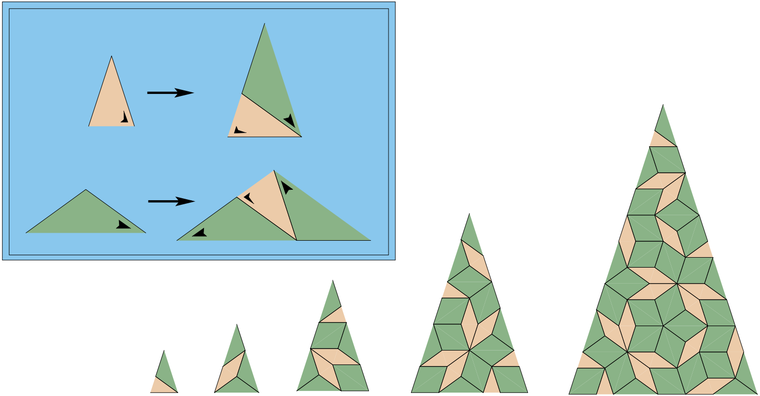

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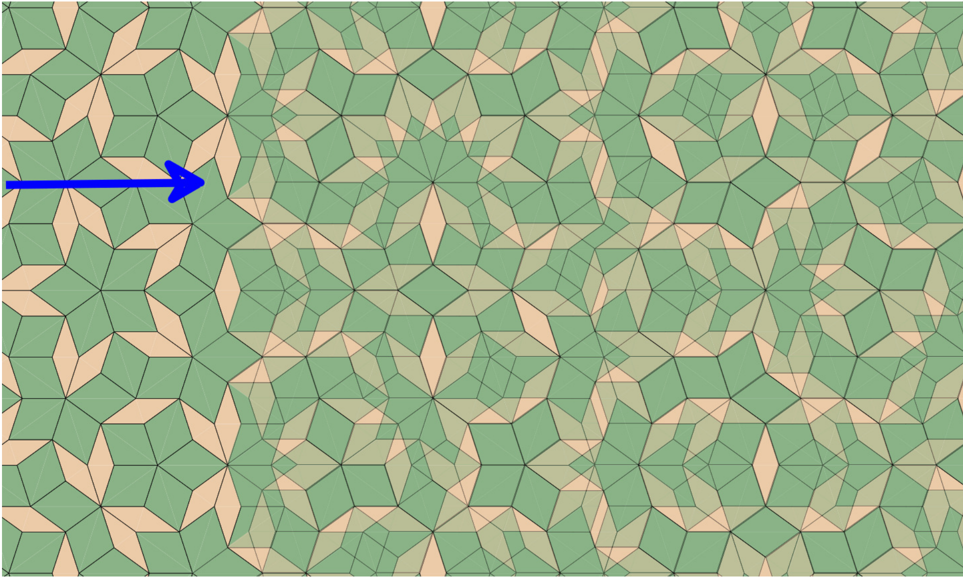
Both examples are non-periodic:



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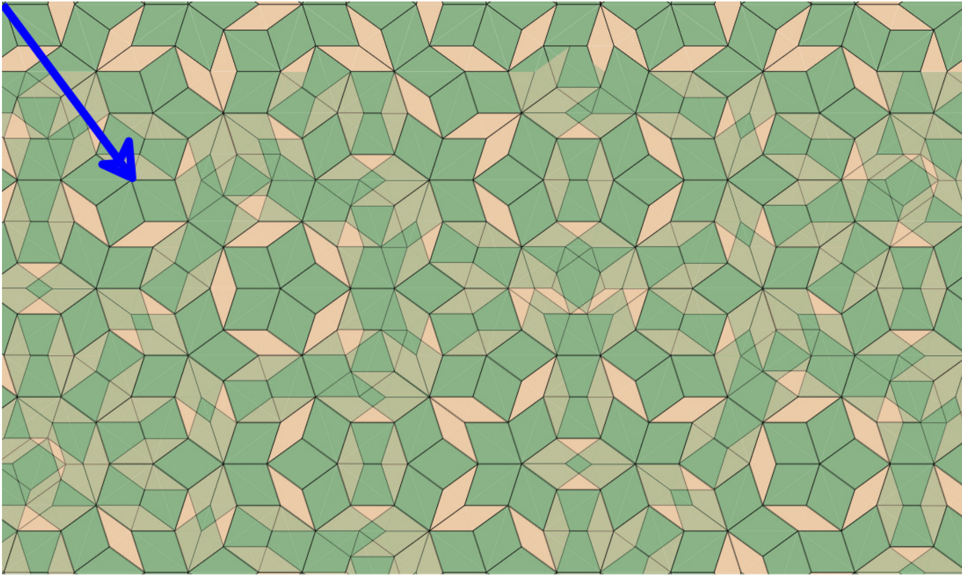

The substitution matrix M_σ contains a lot of information about the tilings
(if the substitution σ is nice, i.e., self-similar and primitive):

A substitution σ is *primitive*, if there is $n \in \mathbb{N}$ such that M_σ^n has positive entries only.

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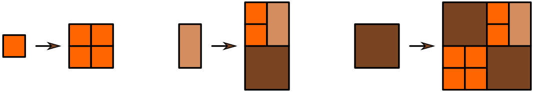

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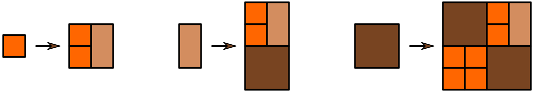

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- If θ is the substitution factor, then $\lambda = \theta^d$ is the largest eigenvalue (*Perron-Frobenius eigenvalue*) of the substitution matrix (d the dimension).
- The left eigenvector corr. to λ contains the relative d-dimensional volumes of the prototiles.
- The right eigenvector corr. to λ contains the relative frequencies of the prototiles.
- Hence the matrix also determines the density of the tiles (once we choose a size, e.g. smallest tile T has volume $\mathrm{vol}_d(T) = 1$)

There are also one-dimensional substitution tilings.

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The set of all substitution tilings (the *hull*) generated by a substitution σ is denoted by $\mathbb{X}_\sigma$.

More formally,

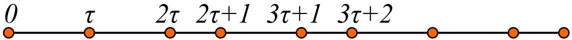
$$\mathbb{X}_\sigma = \{\mathcal{T} \text{ tiling} \mid \exists k, i \in \mathbb{N}, t \in \mathbb{R}^d \; \forall P \subset \mathcal{T} \text{ finite } t + P \subset \sigma^k(T_i)\}$$

Even more formally, let $\sigma(\mathcal{T}) = \mathcal{T}$ be a fixed point of σ, then

$$\mathbb{X}_\sigma = \overline{\{t + \mathcal{T} \mid t \in \mathbb{R}\}},$$

where the closure is taken in the local topology.

Translate the Fibonacci tiling into a Delone set Λ_{Fib}:



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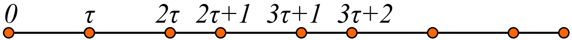

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A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution* if for all other eigenvalues λ' of M_σ holds: $0 < \lambda' < 1$.

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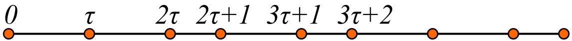

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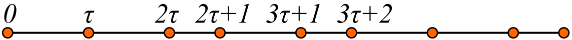

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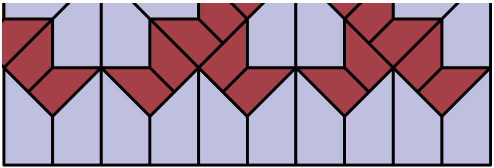
Hence for *all* Delone sets Λ from tilings $\mathcal{T} \in \mathbb{X}_{Fib}$ we have $\Lambda \overset{\mathrm{bd}}{\sim} \mathbb{Z}$. Hence $\mathbb{X}_{Fib}$ contains only one equivalence class.

In the sequel we consider mainly Delone sets arising from substitution tilings in $\mathbb{R}^2$.

If $\mathcal{T}$ is a tiling then $\Lambda_{\mathcal{T}}$ always denotes the Delone set obtained from $\mathcal{T}$ by putting a point in each tile (e.g. in the center).



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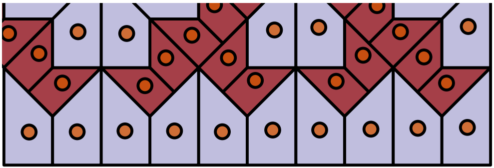

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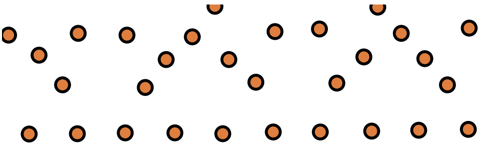

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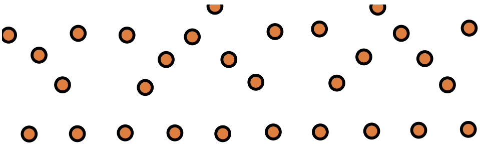
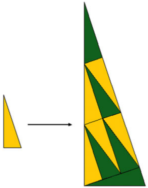


What can we say about Delone sets arising from primitive substitution tilings in general?
(arbitrary dimension, non-Pisot factor,...)

But first a remark: the examples above are all aperiodic crystals.
But...



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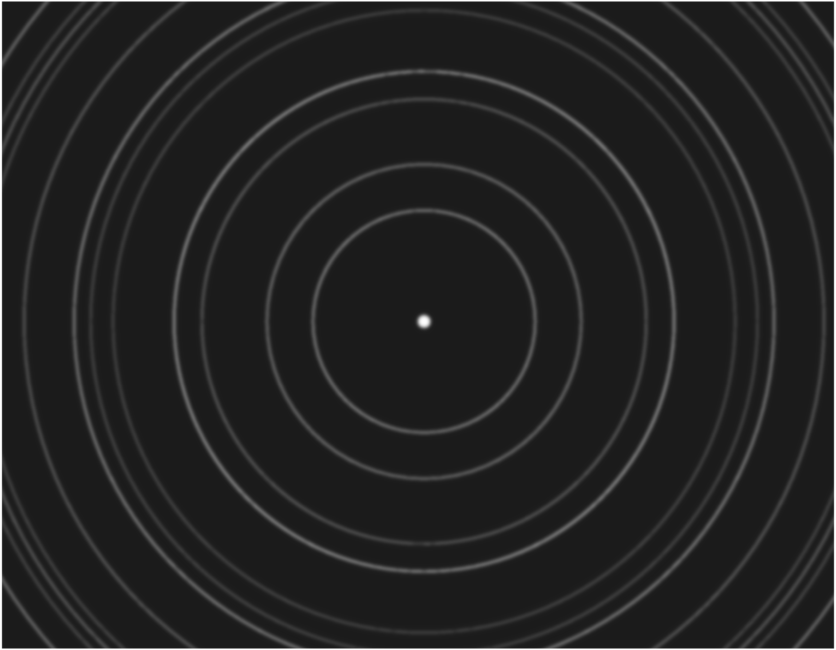

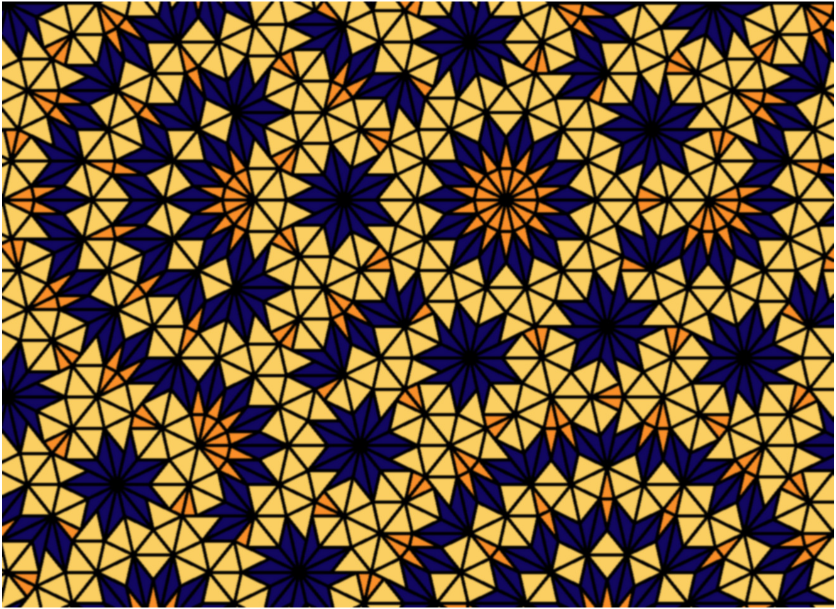

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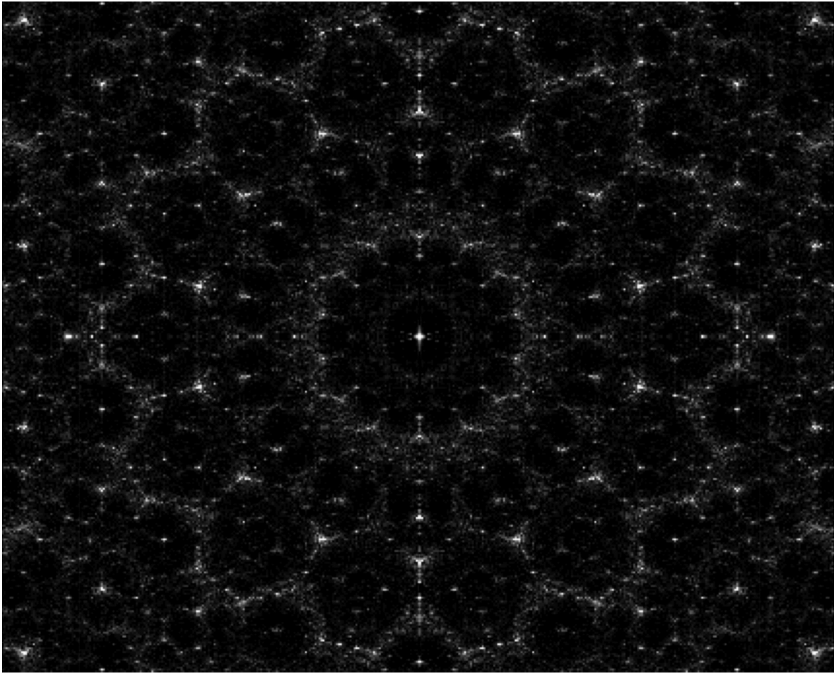

Most general result on $\overset{\mathrm{bd}}{\sim}$-equivalence of substitution tilings:

Theorem (Yaar Solomon '14)

Let σ be a primitive substitution in $\mathbb{R}^d$ with substitution matrix M_σ. Let $s \geq 2$ be the minimal index so that the eigenvalue λ_s of M_σ has an eigenvector whose sum of coordinates is non-zero. Then for any Delone set $\Lambda_{\mathcal{T}}$ corresponding to a tiling $\mathcal{T} \in \mathbb{X}_\sigma$:

(I) *If $|\lambda_s| < \lambda_1^{\frac{d-1}{d}}$ then $\Lambda_{\mathcal{T}} \overset{\mathrm{bd}}{\sim} \mathbb{Z}^d$.*

(II) *If $|\lambda_s| > \lambda_1^{\frac{d-1}{d}}$ then $\Lambda_{\mathcal{T}} \overset{\mathrm{bd}}{\nsim} \mathbb{Z}^d$.*

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Here λ_1 is the largest eigenvalue.

Essentially, λ_s is the second largest eigenvalue. Also shown:

(IV) If there is no such λ_s then $\Lambda_{\mathcal{T}} \overset{\mathrm{bd}}{\sim} \mathbb{Z}^d$.

$$|\lambda_s| \overset{?}{\lesseqgtr} \lambda_1^{\frac{d-1}{d}}$$

- In dimension $d = 1$: $|\lambda_s| \lesseqgtr {\lambda_1}^0 = 1$
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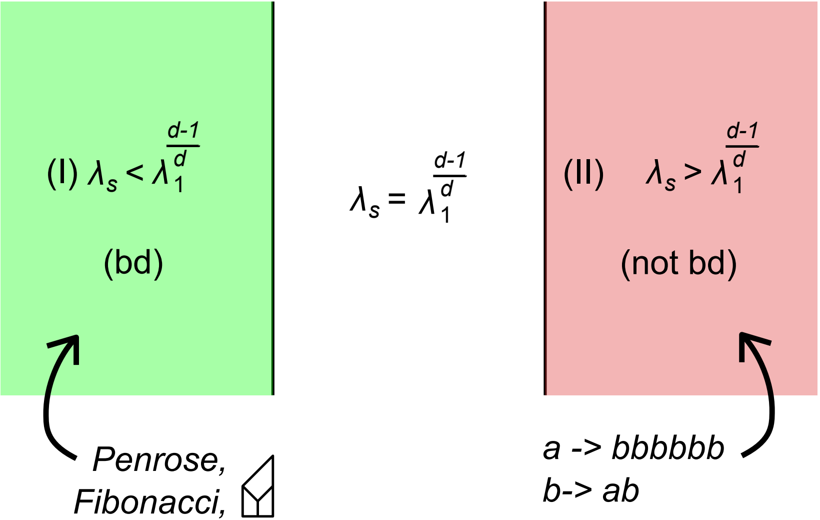
Let's draw a map of the situation...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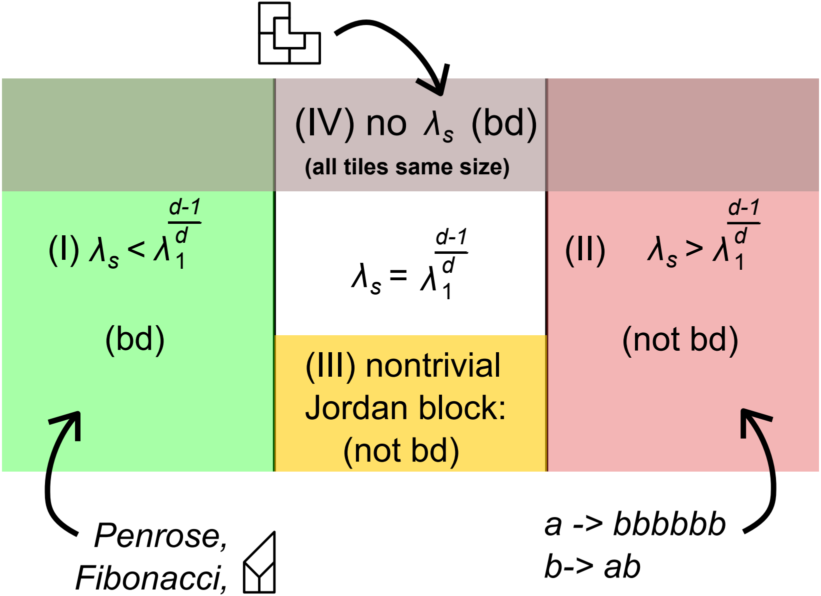
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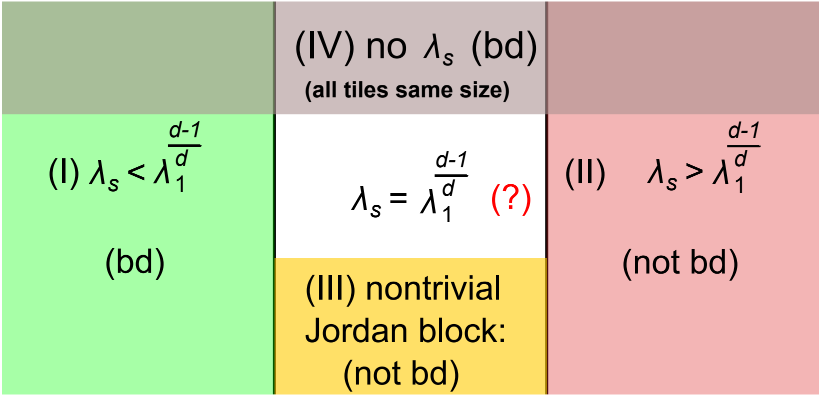
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(not bd)

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Theorem (F-Smilansky-Solomon 2019+)

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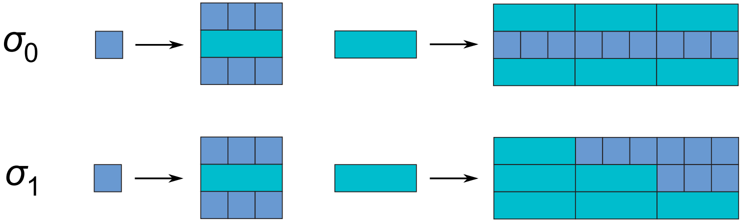

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The Master Theorem (used in Solomon, F-Garber, ...):

Theorem (Laczkovich 1992)

Let Λ be a Delone set in $\mathbb{R}^d$. $\Lambda \overset{\mathrm{bd}}{\sim} \mathbb{Z}^d$ if and only there is $c > 0$ such that for all unions P of lattice cubes holds:

$$\left| \#(\Lambda \cap P) - \mathrm{dens}(\Lambda)\, \mathrm{vol}_d(P) \right| \leq c\, \mathrm{vol}_{d-1}(\partial P)$$

(Proof makes use of the infinite version of Hall's Marriage Theorem by Rado)

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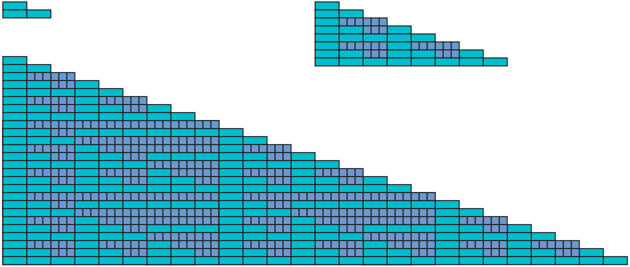
We use this result as follows:

Careful choice of a sequence of patches P_n in $\mathcal{T}' \in \mathbb{X}_{\sigma_1}$:



...and careful counting of the number of tiles in P_n:

$$\#(\Lambda \cap P_n) = 9^n - (n+1)3^n$$

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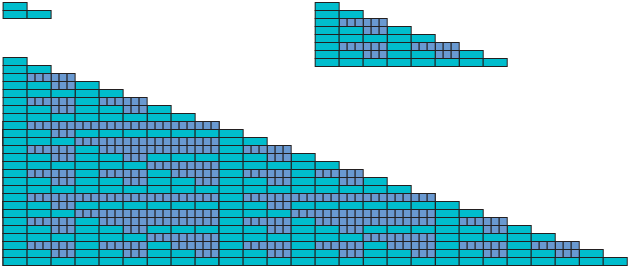

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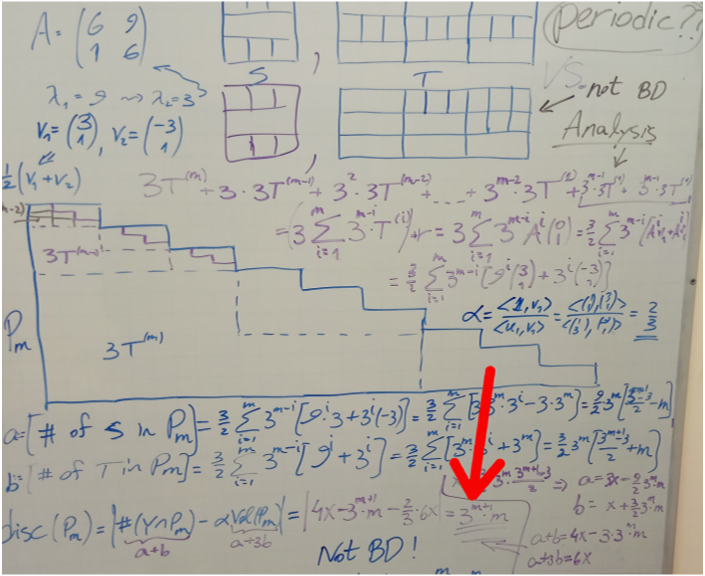
...yields

$$\left|\#(\Lambda \cap P_n) - \mathrm{vol}_2(P)\right| = \left|9^n - (n+1)3^n - \frac{2}{3} \cdot \frac{3}{2}(9^n - 3^n)\right| = n\,3^n.$$



A = (6 9 / 1 6)
λ1 = 9 ⇝ λ2 = 3
V1 = (3 / 1), V2 = (−3 / 1)
1/2 (V1 + V2)
S
T
Periodic?
VS. not BD
Analysis
3T^(m) + 3 · 3T^(m−1) + 3^2 · 3T^(m−2) + … + 3^(m−2) · 3T^(2) + 3^(m−1) · 3T^(1) + 3^(m−1) · 3T^(1)
= 3 Σ_{i=1}^{m} 3^(m−i) T^(i) + r = 3 Σ_{i=1}^{m} 3^(m−i) A^i (0 / 1) = 3/2 Σ_{i=1}^{m} 3^(m−i) (A^i v1 + A^i v2)
= 3/2 Σ_{i=1}^{m} 3^(m−i) [9^i (3 / 1) + 3^i (−3 / 1)]
α = ⟨u1, v1⟩ / ⟨u1, v1⟩ = 2/3
3T^(m−1)
3T^(m)
Pm
a = [# of S in Pm] = 3/2 Σ_{i=1}^{m} 3^(m−i) [9^i · 3 + 3^i (−3)] = 3/2 Σ_{i=1}^{m} [3 · 3^m · 3^i − 3 · 3^m] = 9/2 3^m [(3^(m+1) − 3)/2 − m]
b = [# of T in Pm] = 3/2 Σ_{i=1}^{m} 3^(m−i) [9^i + 3^i] = 3/2 Σ_{i=1}^{m} [3^m 3^i + 3^m] = 3/2 3^m [(3^(m+1) − 3)/2 + m]
⇒ a = 3X − 9/2 3^m m
b = X + 3/2 3^m m
disc(Pm) = |#(Y ∩ Pm) − α Vol(Pm)| = |4X − 3^(m+1) m − 2/3 6X| = 3^(m+1) m
a+b
a+3b
a+b = 4X − 3 · 3^m m
a+3b = 6X
Not BD!

Compare this value $n\,3^n$ with the length $\mathrm{vol}_1(\partial P_n) = 8 \cdot 3^n - 8$:

We need $c > 0$ such that

$$\left|\#(\Lambda \cap P_n) - \mathrm{vol}_2(P)\right| = n\,3^n \leq c(8 \cdot 3^n - 8) \quad \text{for all } n,$$

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Hence by Laczkovich $\Lambda_{\mathcal{T}'} \overset{\mathrm{bd}}{\not\sim} \mathbb{Z}^2 \overset{\mathrm{bd}}{\sim} \Lambda_{\mathcal{T}}$ $\square$

So we know that in case (?) everything can happen. What next? Recall our motivating questions. Now they become:

- When is $\Lambda \overset{\text{bd}}{\not\sim} \Lambda'$?
- How many equivalence classes has $\mathbb{X}_\sigma$ for a primitive substitution σ?

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For both questions we would need a condition for $\Lambda \overset{\text{bd}}{\not\sim} \Lambda'$. Voila:

Theorem (F-Smilansky-Solomon 2019+)

Let Λ, Λ' be Delone sets in $\mathbb{R}^d$ (with uniform discreteness parameter $r \geq d^{\frac{1}{2}}$, say). If there is a sequence (P_n), each P_n a finite union of unit cubes, such that

$$\lim_{n\to\infty} \frac{|\#(\Lambda \cap P_n) - \#(\Lambda' \cap P_n)|}{\mathrm{vol}_{d-1}(\partial P_n)} = \infty$$

then there is no BD-map $g : \Lambda \to \Lambda'$.

Again the proof uses the infinite version of the Marriage Theorem.

This result is a partial answer to Question 1.

Ad Question 2: we could not do it for (classical) substitution tilings as above.

But there is a broader concept (kind of trendy in Bielefeld):

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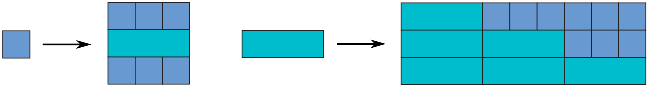
Ad Question 2: we could not do it for (classical) substitution tilings as above.

But there is a broader concept (kind of trendy in Bielefeld):

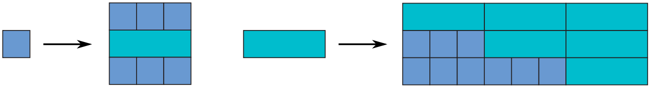
- mixed substitutions
- random substitutions

For both you need two (compatible) substitution rules:

σ_1

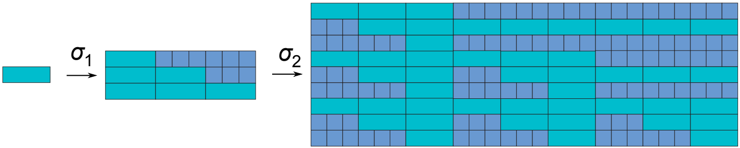


σ_2



The *mixed* substitution tilings $\mathbb{X}_{(\sigma_1,\sigma_2)}$ are obtained in the following way:

- Start with some prototile T.
- In each step toss a coin: $i \in \{1, 2\}$
- Apply σ_i to the entire patch



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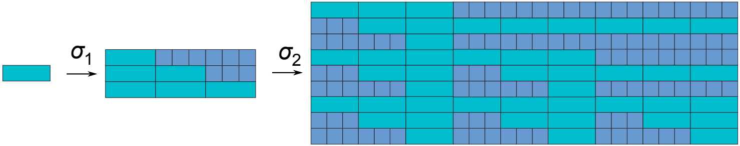


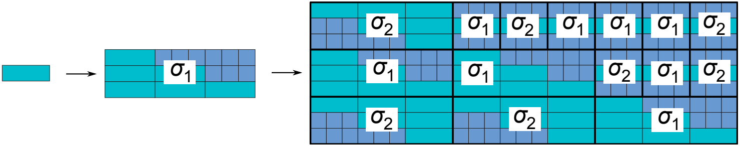
All subpatches of all these iterates are called *legal*

wrt the mixed substitution (σ_1, σ_2).

All tilings with legal patches only are in the *mixed substitution hull* $\mathbb{X}_{(\sigma_1,\sigma_2)}$.

The *random* substitution tilings $\mathbb{X}_{\mathrm{rand}(\sigma_1,\sigma_2)}$ are obtained in the following way:

- Start with some prototile T.
- In each step toss several coins: $i_j \in \{1, 2\}$
- Apply σ_{i_j} to tile number j



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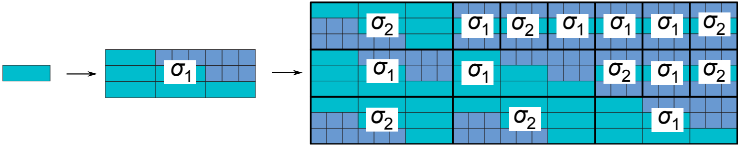

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Obviously

$$\mathbb{X}_{\sigma_1} \subset \mathbb{X}_{(\sigma_1,\sigma_2)} \subset \mathbb{X}_{\mathsf{rand}(\sigma_1,\sigma_2)}.$$

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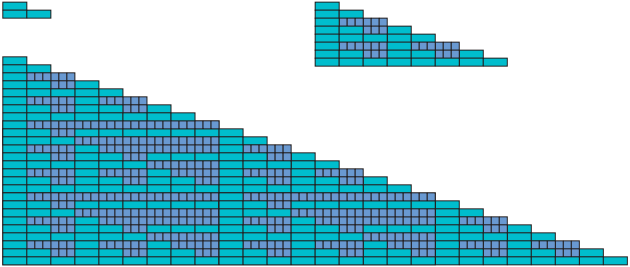
Again the proof relies on counting the tiles in the patch sequence



By adjusting the sequence $w = w_{i_1}\, w_{i_2}\, w_{i_3}\, \cdots \in \{1,2\}^{\mathbb{N}}$ by which we apply σ_1 and σ_2 we get for each different density of 1s in w different counts of the deficiency/surplus of tiles.

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Harvested from the proof of the last theorem:

Theorem (F-Smilansky-Solomon 2019+)

Let $n \in \mathbb{N}$. There is a substitution matrix — namely, $\left(\begin{smallmatrix} 6 & 9 \\ 1 & 6 \end{smallmatrix}\right)^n$ — such that there are $n+1$ different tile substitutions $\varrho_0, \varrho_1, \ldots, \varrho_n$ producing tilings $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ with

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The ϱ_i are appropriate compositions of σ_1 and σ_2:

$$\begin{aligned}
\varrho_0 &= \sigma_1 \circ \sigma_1 \circ \cdots \circ \sigma_1 \circ \sigma_1 \\
\varrho_1 &= \sigma_2 \circ \sigma_1 \circ \cdots \circ \sigma_1 \circ \sigma_1 \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
\varrho_{n-1} &= \sigma_2 \circ \sigma_2 \circ \cdots \circ \sigma_2 \circ \sigma_1 \\
\varrho_n &= \sigma_2 \circ \sigma_2 \circ \cdots \circ \sigma_2 \circ \sigma_2
\end{aligned}$$

All results of this talk in:

[FSS] Dirk Frettlöh, Yotam Smilansky, Yaar Solomon: Pairwise non-BD sets arising from substitution tilings, submitted, `arxiv:1907.01597`

[S] Yaar Solomon: A simple condition for bounded displacement, *Journal of Mathematical Analysis and Applications* 414 (2014) 134-148.

[L] Miklós Laczkovich: Uniformly spread discrete sets in $\mathbb{R}^d$, *Journal of the London Mathematical Society* 46 (1992) 39-57.

Further reading:

[FG1] Dirk Frettlöh, Alexey Garber: Pisot substitution sequences, one dimensional cut-and-project sets and bounded remainder sets ..... *Indagationes Mathematicae* 29 (2018) 1114-1130.

[FG2] Dirk Frettlöh, Alexey Garber: Weighted 1x1 cut-and-project sets in bounded distance to a lattice, *Discrete and Computational Geometry* (2018).

⋆ ⋆

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Thank you!