Mathematical Quasicrystals and Inductive Rotation Tilings

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Wolfgang Pauli Lectures
Wien
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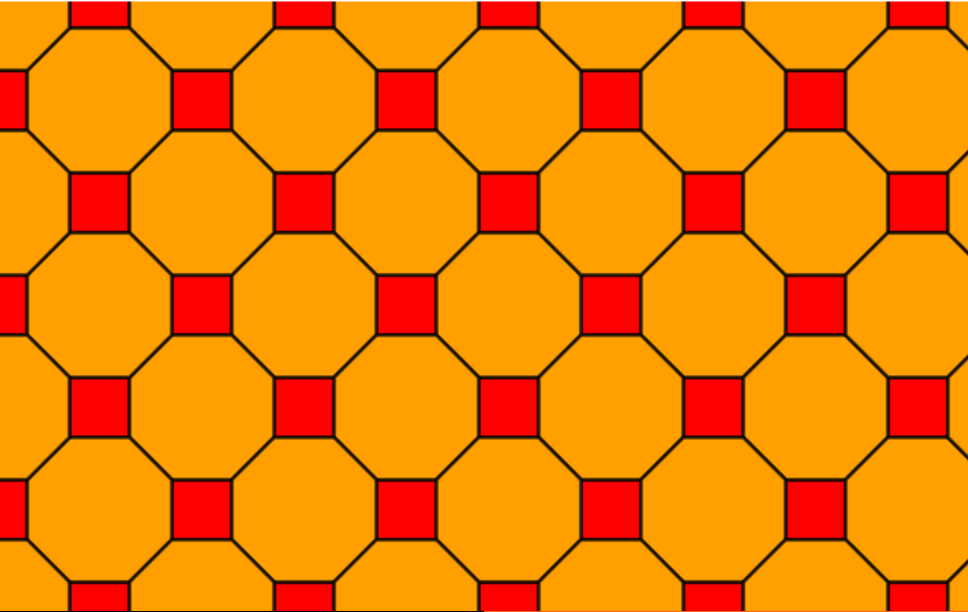
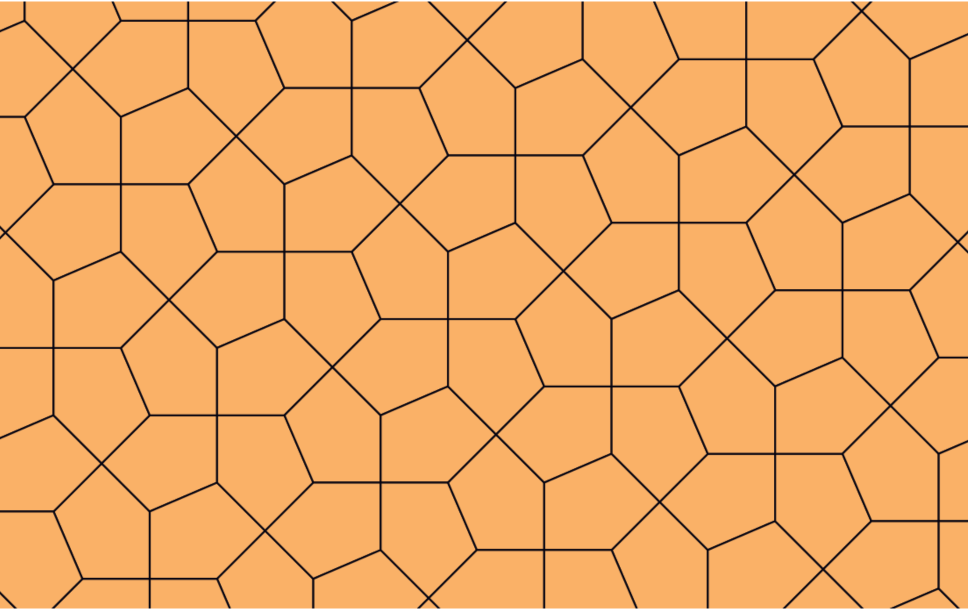
1. Tilings and quasicrystals

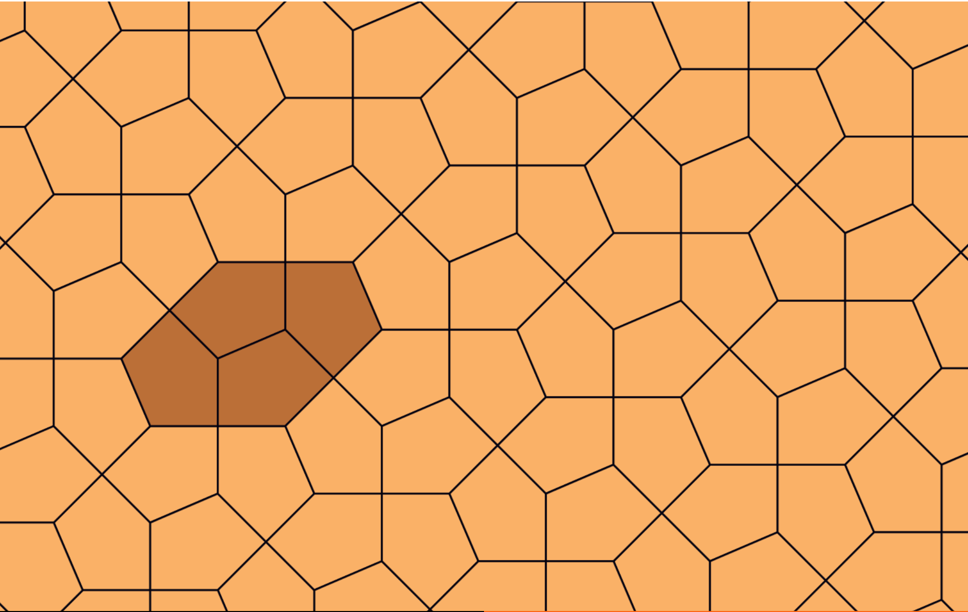
Tilings



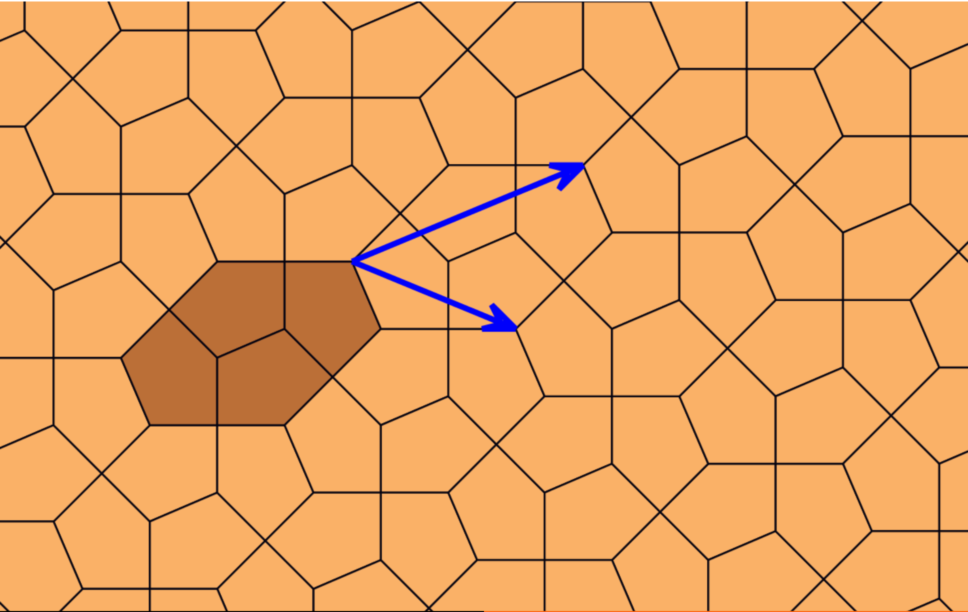
Tilings (=Tesselations = Parkette...)



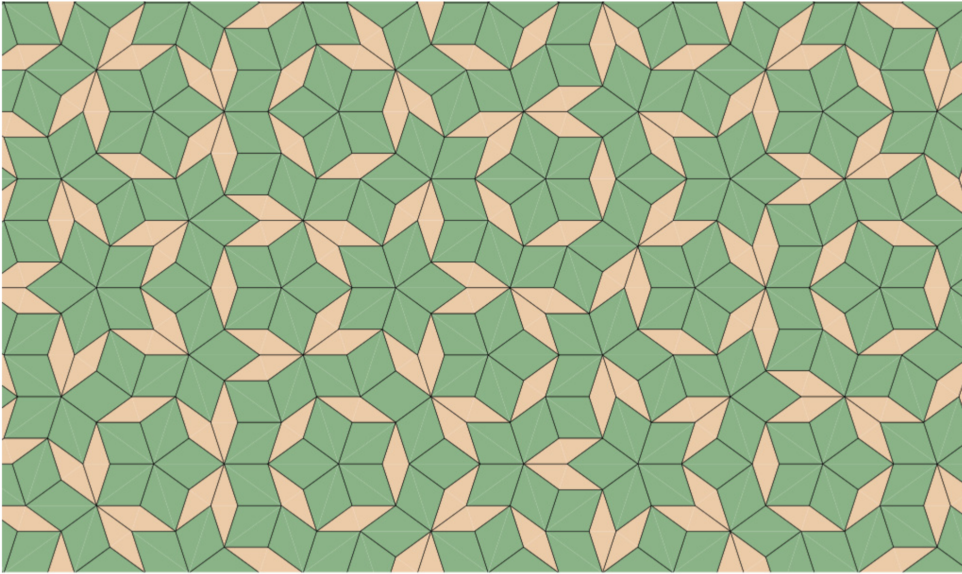
Tilings



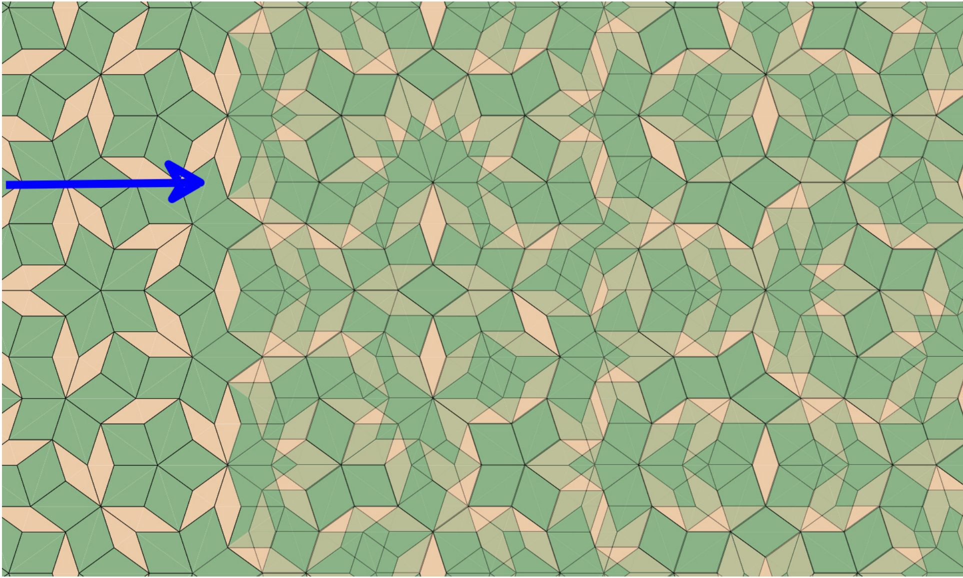
Periodic Tilings



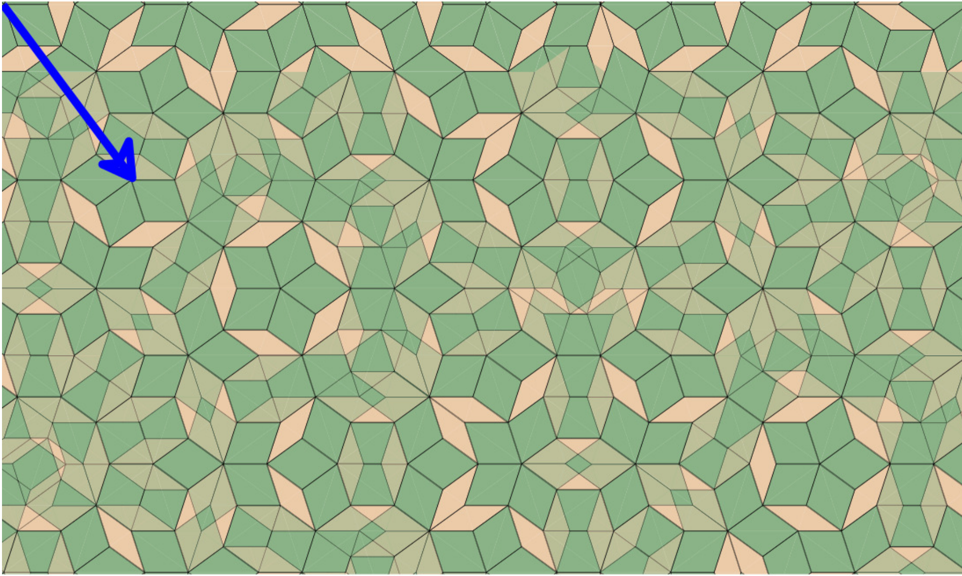
Aperiodic Tiling



Aperiodic

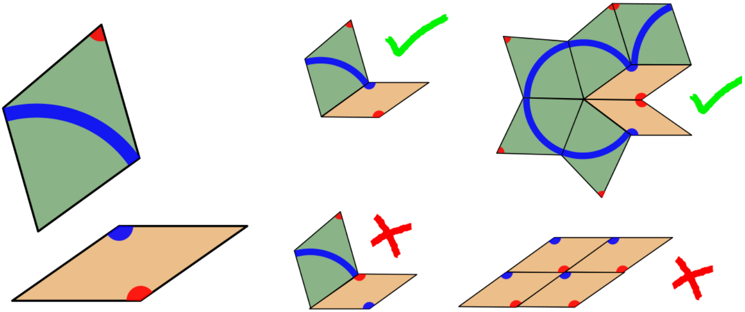


Aperiodic



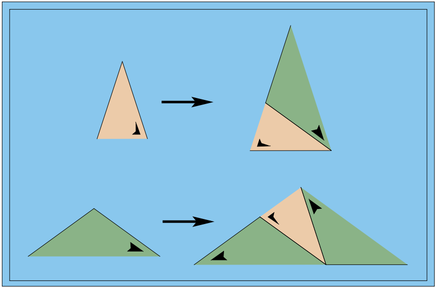
Local rules

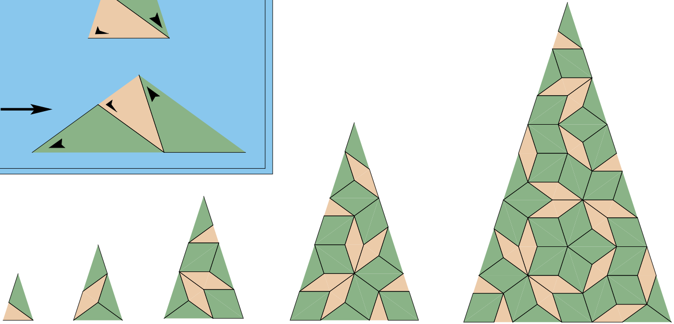
Penrose tiling: can be generated by local rules
(forces aperiodicity)



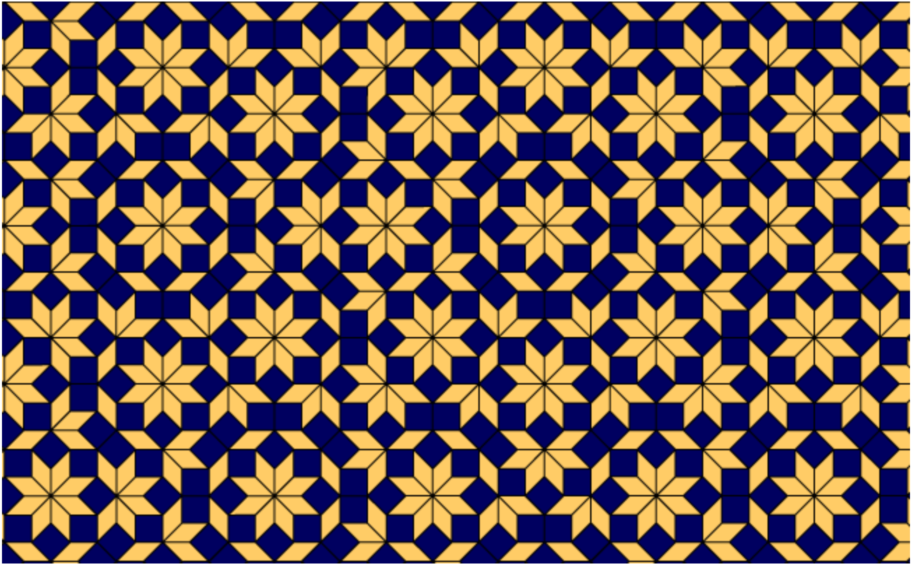
Substitution tilings:

Penrose tiling: can also be generated by a substitution rule

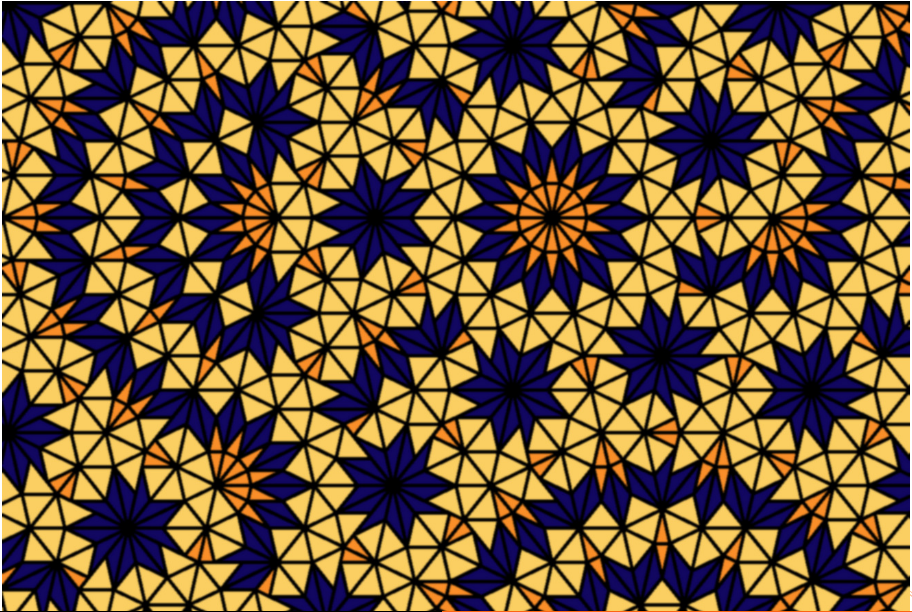




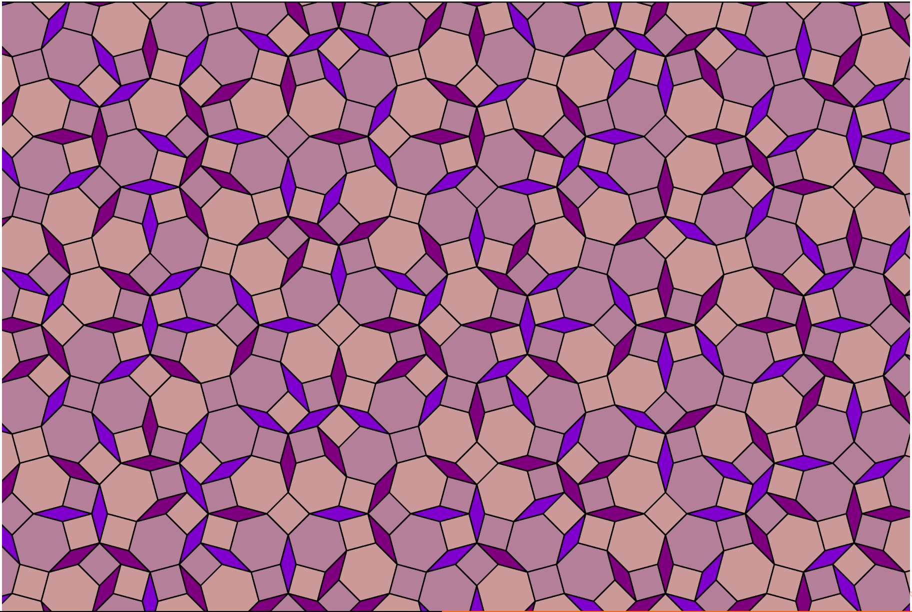
Other examples: Ammann-Beenker tiling



Other examples: Buffalo tiling



Other examples: Socolar's 12-fold tiling



...where are we:

Aperiodic tilings with a high degree of local and global order can be generated by

- Local matching rules
- Substitution rules
- (Cut and project method, slightly complicated)

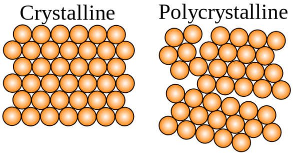
Penrose: 1972, others in the 70s and 80s.

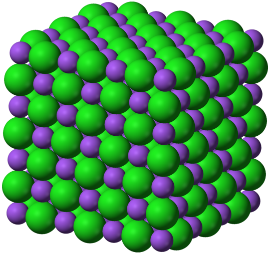
1982/84: Shechtman et al discovered quasicrystals

Quasicrystals

...no rigorous definition (at least not mathematically)

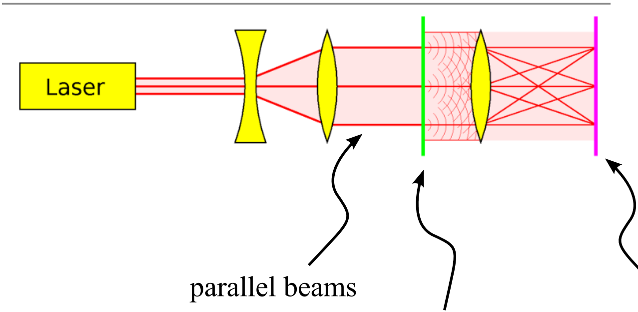
Crystal: Roughly / physics: Crystal $\leftrightarrow$ periodic structure.





Crystallic structure shows up in a diffraction experiment:

Physical diffraction experiment:

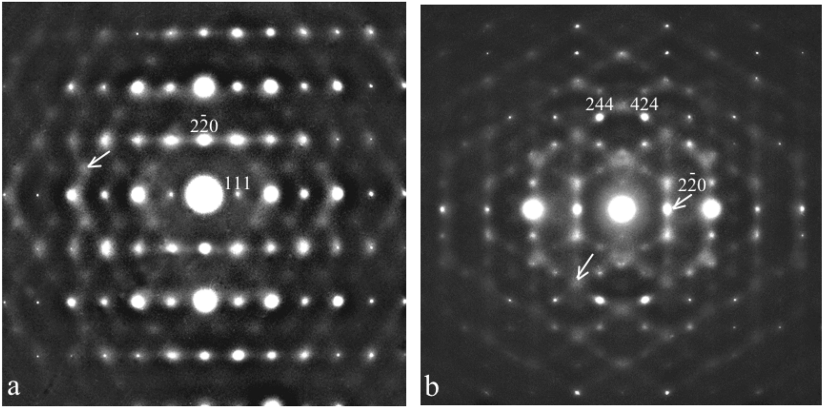


parallel beams
(laser, X-ray,...)

sample
(crystal, quasicrystal...)

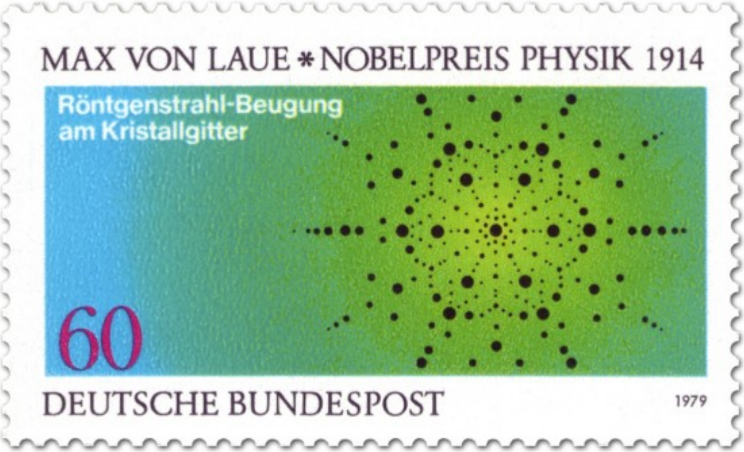
diffraction image
(on screen, film...)

Diffraction image of a crystal (physical experiment)

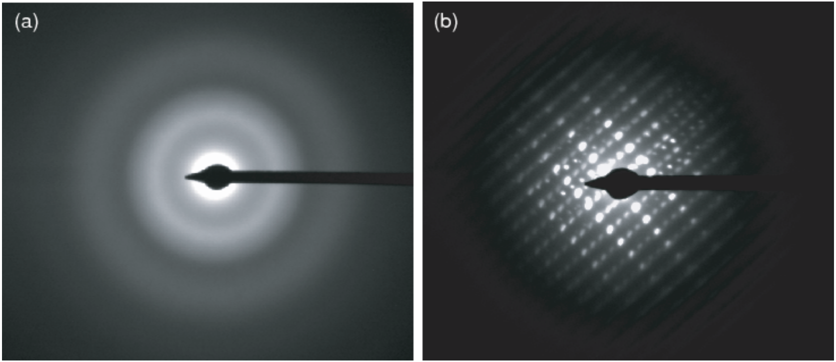


Yun Liu and Ray L. Withers: Local Structure of Relaxor Dielectric Ceramics, in "Advances in Ceramics - Electric and Magnetic Ceramics, Bioceramics, Ceramics and Environment", Costas Sikalidis (ed.), ISBN 978-953-307-350-7.

Crystal diffraction: Noble prize 1914



Diffraction image (physical experiment)



Not a crystal (amorphous) crystal

Martin, I.W. et al.: Effect of heat treatment on mechanical dissipation in Ta2O5 coatings Class.Quant.Grav. 27 (2010) 225020 arXiv:1010.0577

Mathematical computation of diffraction:

Rather than doing an experiment, one may also compute a diffraction image mathematically.

- Tiling $\rightsquigarrow$ discrete point set Λ.
- $\gamma_\Lambda = \lim_{r\to\infty} \frac{1}{\mathrm{vol}\, B_r} \sum_{x,y\in\Lambda\cap B_r} \delta_{x-y}$.
- Fouriertransform $\widehat{\gamma}_\Lambda$ is the diffraction spectrum.

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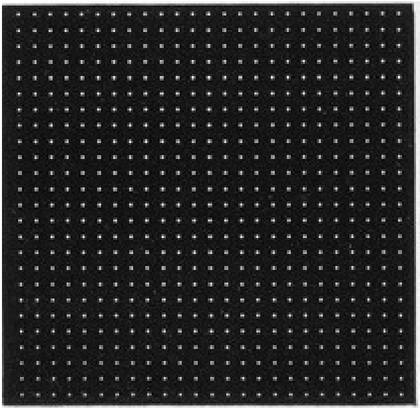
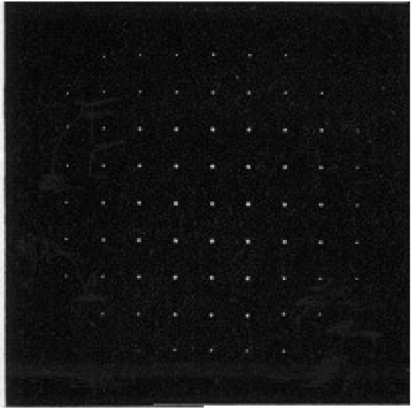
↑ **for impressional reasons only** ↑

Crystal diffraction (mathematical computation):

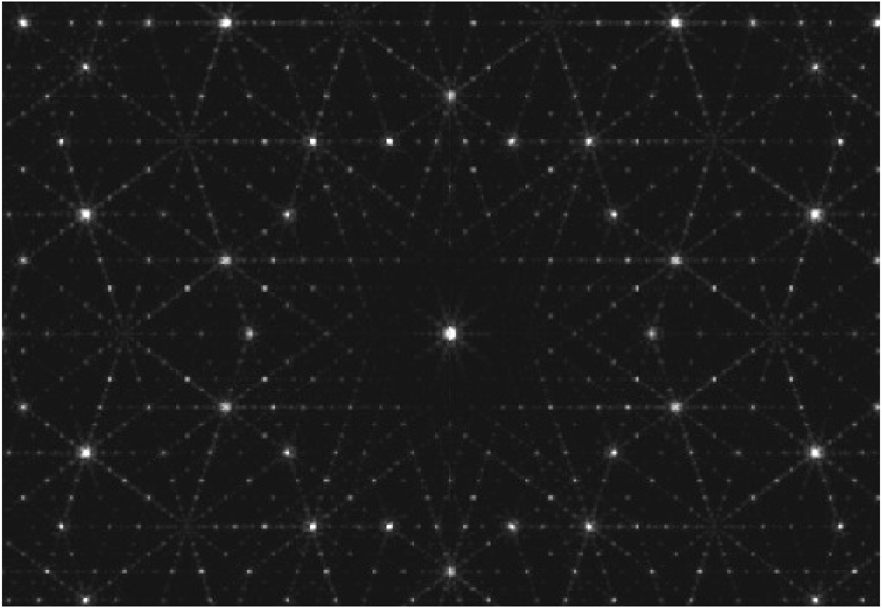




Crystal pattern its diffraction image

What if we compute the diffraction image of a Penrose tiling?

Diffraction of Penrose tiling (mathematical)



The diffraction image shows bright spots! And no diffuse parts.

Like a crystal! But...

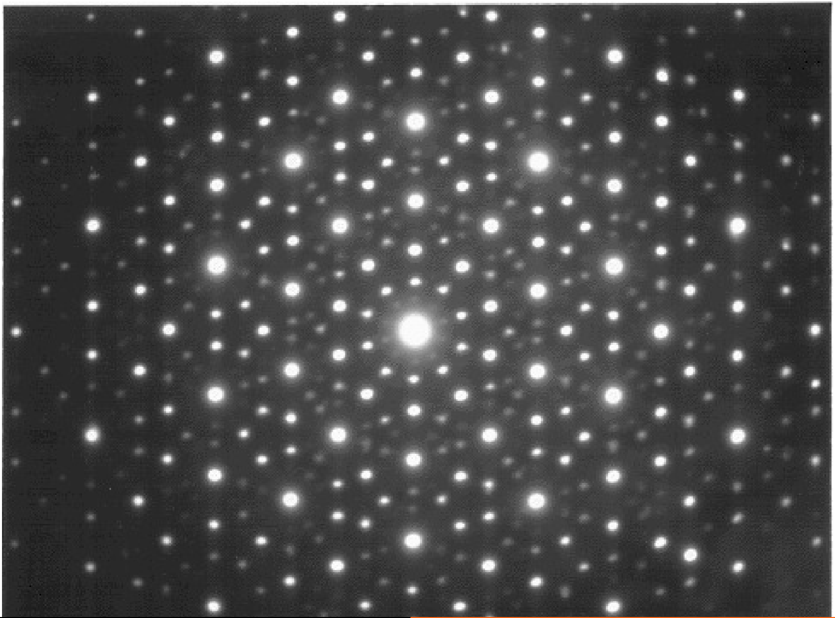
The diffraction image shows bright spots! And no diffuse parts.

Like a crystal! But...

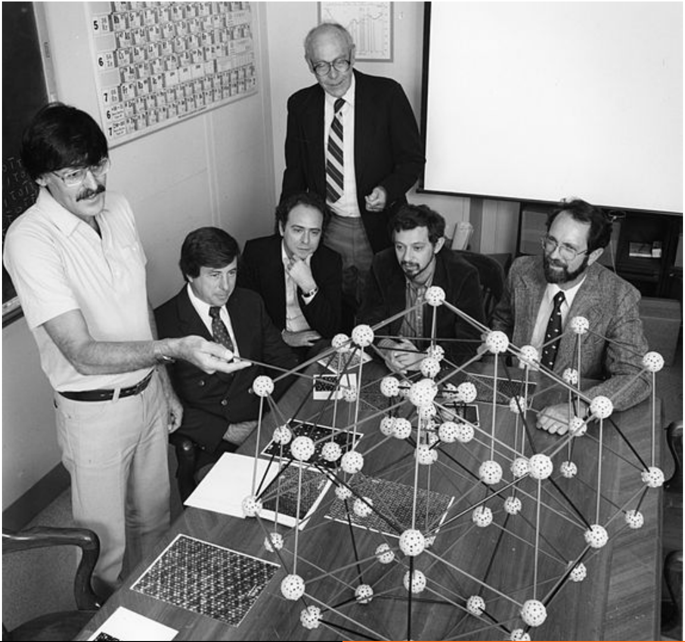
It shows also 10-fold rotational symmetry. Which is not possible for a crystal = periodic structure. Certainly not.

OK, this is just theory. But in 1982 Danny Shechtman saw something like this in a diffraction experiment:

Quasicrystal diffraction (physical experiment)



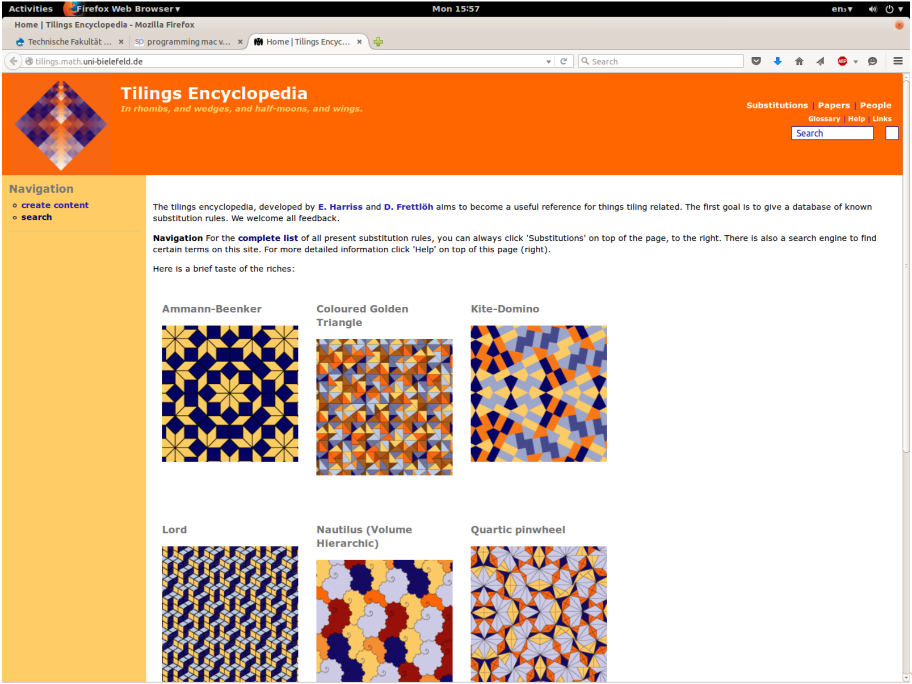
Quasicrystal diffraction: Noble prize 2011 for Danny Shechtman



From the Noble Prize Comitee:

After the discovery, Shechtman spent a long time convincing colleagues about the veracity of his interpretation (local icosahedral symmetry rather than twinning), and the two original papers on the discovery were published more than two years later. The achievement of Dan Shechtman is clearly not only the discovery of quasicrystals, but the realization of the importance of this result and the determination to communicate it to a skeptical scientific community.

The publication in 1984 triggered a hype on research on aperiodic strutures. I worked in this field for some time. Maybe my most important contribution:


Activities
Firefox Web Browser
Mon 15:57
Home | Tilings Encyclopedia - Mozilla Firefox
Technische Fakultät ...
programming mac v...
Home | Tilings Encyc...
tilings.math.uni-bielefeld.de
Search
Tilings Encyclopedia
In rhombs, and wedges, and half-moons, and wings.
Substitutions | Papers | People
Glossary | Help | Links
Search
Navigation
create content
search
The tilings encyclopedia, developed by E. Harriss and D. Frettlöh aims to become a useful reference for things tiling related. The first goal is to give a database of known substitution rules. We welcome all feedback.
Navigation For the complete list of all present substitution rules, you can always click 'Substitutions' on top of the page, to the right. There is also a search engine to find certain terms on this site. For more detailed information click 'Help' on top of this page (right).
Here is a brief taste of the riches:
Ammann-Beenker
Coloured Golden Triangle
Kite-Domino
Lord
Nautilus (Volume Hierarchic)
Quartic pinwheel

2. Inductive rotation tilings

In 2010 I received an email:

Terminal

ALPINE 2.02(1266)   MESSAGE TEXT ...ehl-zwei2010   Message 264 of 703 65% ANS

Sehr geehrter Herr Prof. Frettlöh,

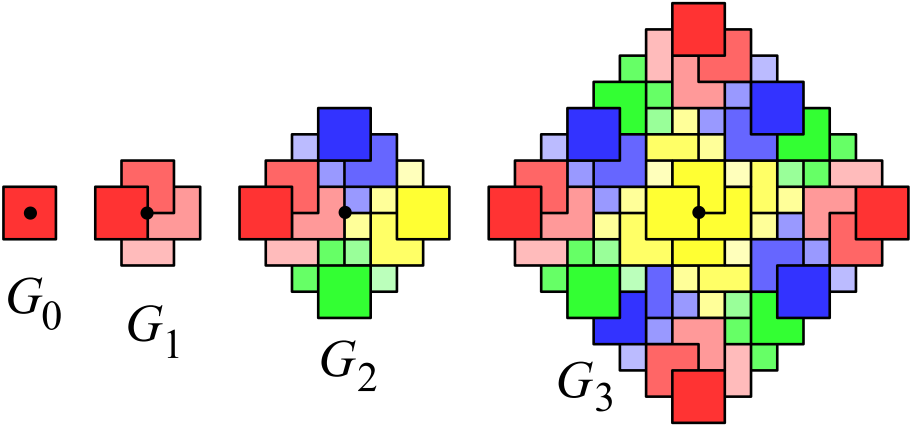
ich beschäftige mich in meiner künstlerischen Arbeit intensiv mit Parallelität und Kreislauf und habe folgendes System der Induktiven Rotation (siehe anbei) entdeckt um "lebendige" Muster / Parkettierungen / tilings zu kreieren. Dieser Entdeckung sind die wissenschaftlichen Veröffentlichungen meiner geometrischen Konstruktionen im mathematischen Journal ForumGeometricorum (links siehe anbei) vorangegangen, die mich mit Sicherheit stimuliert haben derartige Muster zu entwickeln.

Da ich via Internet herausgefunden habe, dass Sie sich wissenschaftlich mit Parkettierungen aller Art beschäftigen und daran auch enzyklopädisch arbeiten, wollte ich Ihnen o.a. Methode mit der Bitte um Ihre Stellungnahme bzw. Einordnung, zur Kenntnis bringen.

Mit den besten Grüßen aus Wien,
Hofstetter Kurt, e.h.
http://www.sunpendulum.at/hofstetterkurt.html [www.sunpendulum.at]

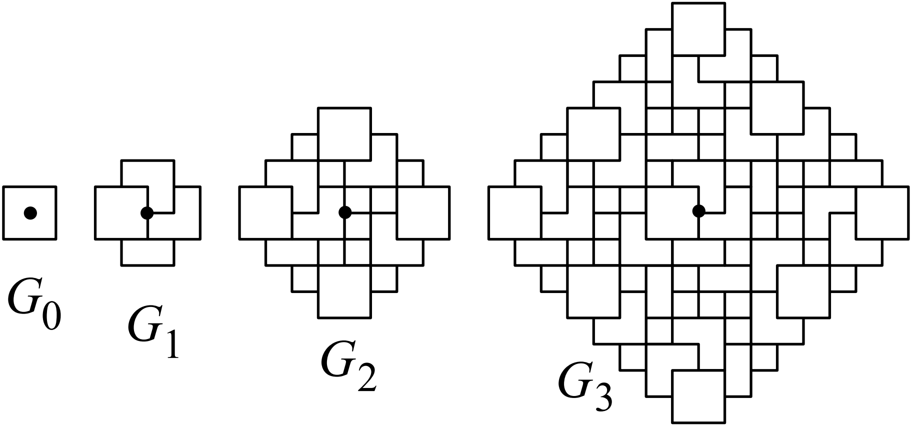
- scientific papers @ Forum Geometricorum ISSN 1534-1178:
2008, **A simple compass-only construction of the regular pentagon**
[forumgeom.fau.edu]

Inductive rotation



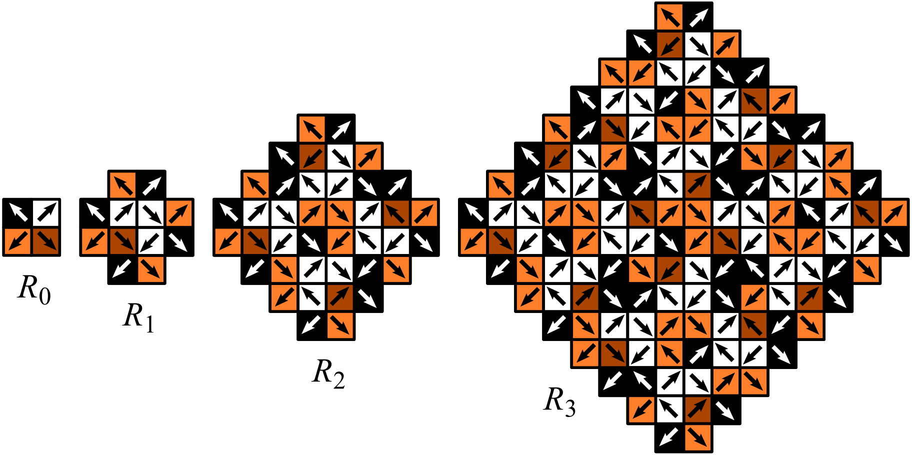
Repeating the iteration fills larger and larger regions.

Inductive rotation



A new construction for aperiodic tilings!?

Decorations of the underlying yields variants of the “naked” tiling (last slide): e.g. the “arrowed” tiling.



...and others: ⧅, ⊡, ⊟ . . . including periodic tilings: ⊞.

Due to some constraints we really got things started in 2014.

Questions:

- *Are* the tilings aperiodic?
- Do the tilings show pure point diffraction (i.e. bright spots, like quasicrystals)
- How are the tilings related (e.g. naked vs arrowed)
- ... (other more technical properties)

We were able to proof answers to 1 and 2 for the arrowed version. Answers: yes and yes.

Central results:

Theorem (1)

The arrowed tilings are limitperiodic.

Theorem (2)

The arrowed tilings can be generated by a substitution rule.

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Theorem (1)

The arrowed tilings are limitperiodic.

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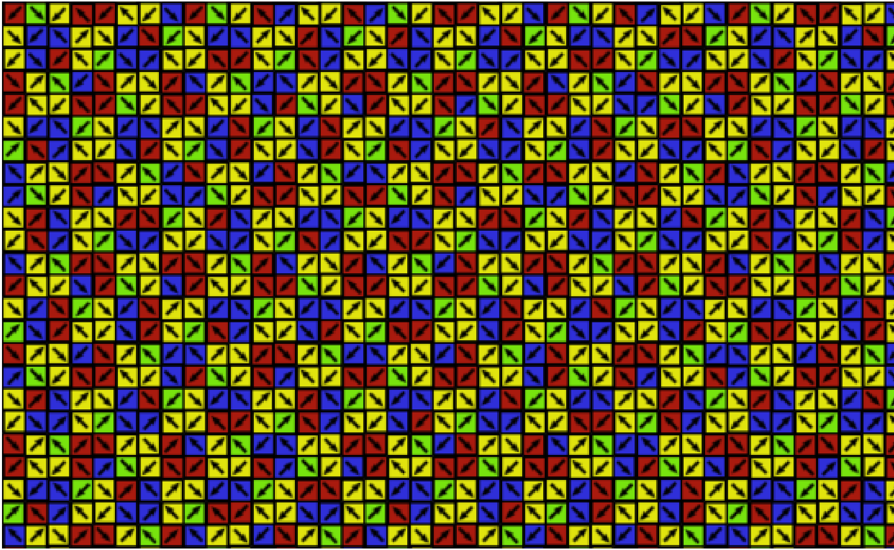
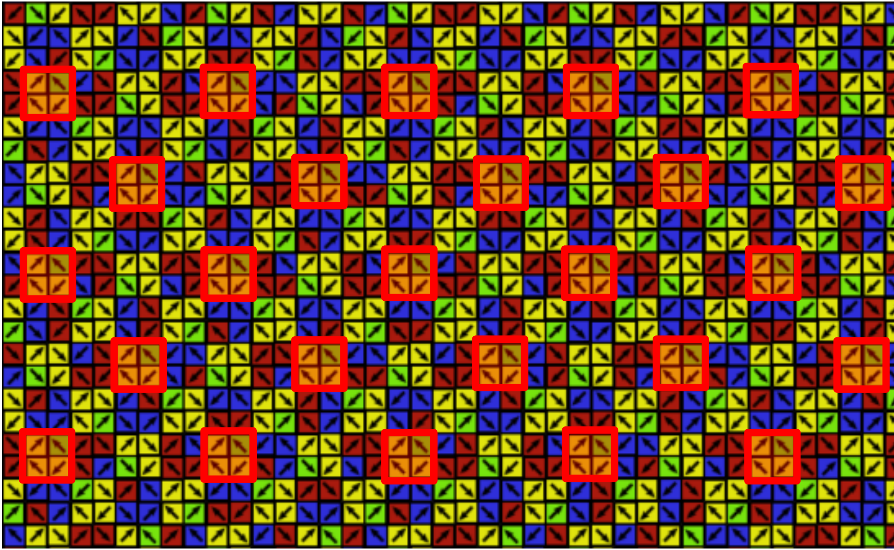
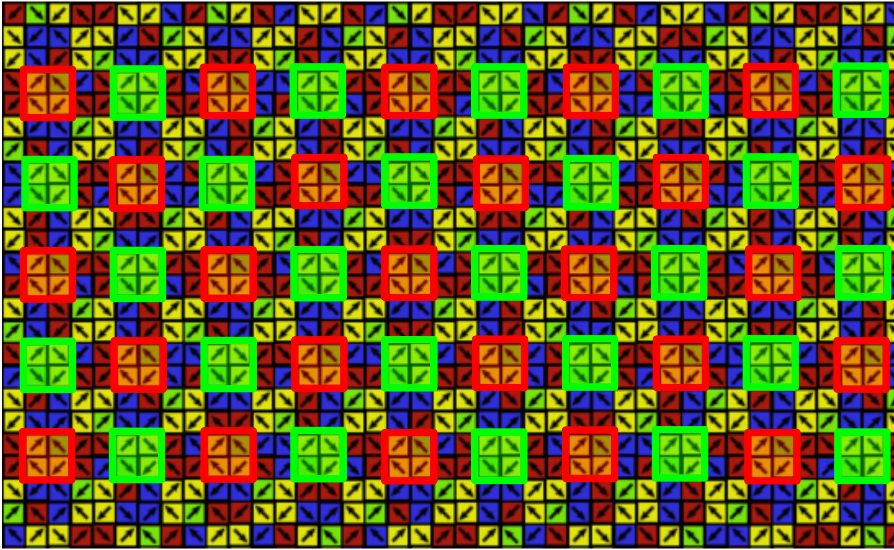
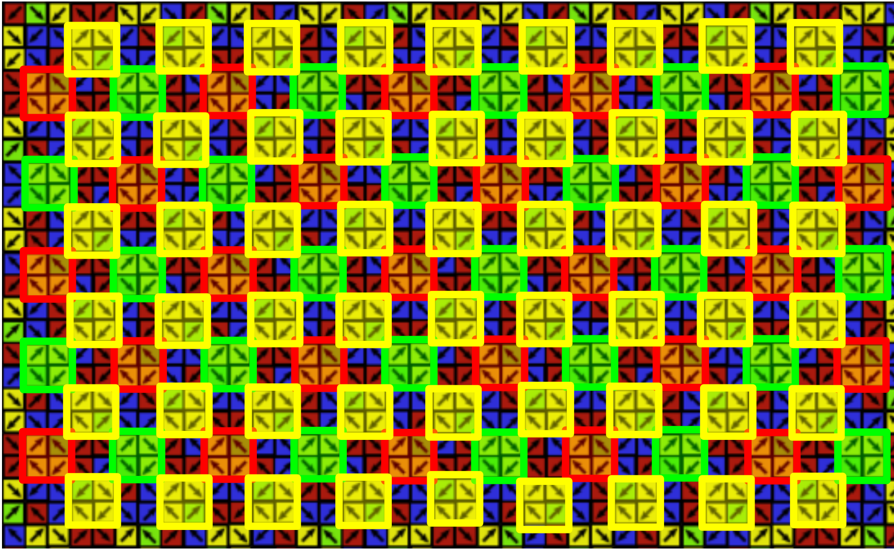
Remark: Theorem 2 is a pity, since this new method generates tilings that can also be obtained by the old method.

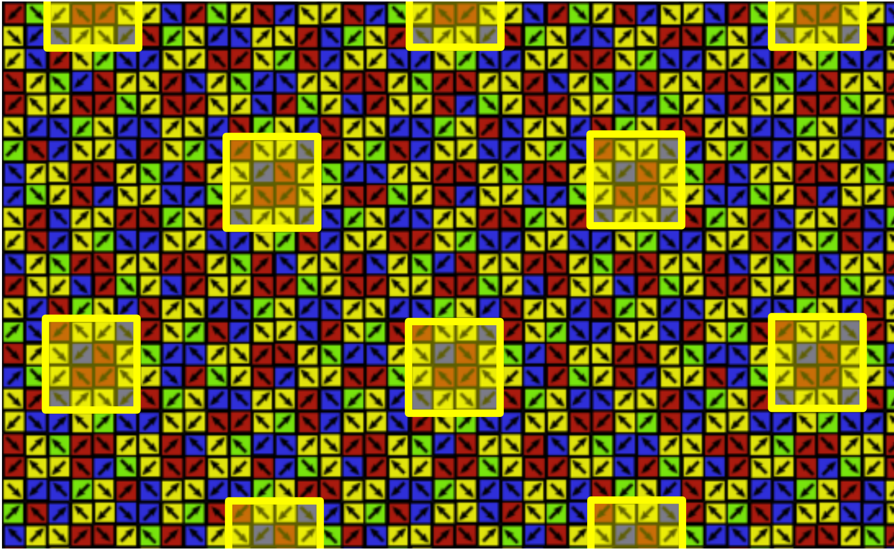
Anyway. What does "limitperiodic" mean?











...and so on. The entire tiling is the union of periodic sub-tilings.

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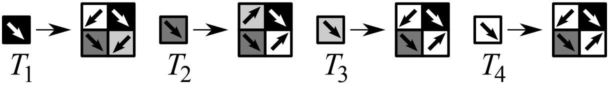
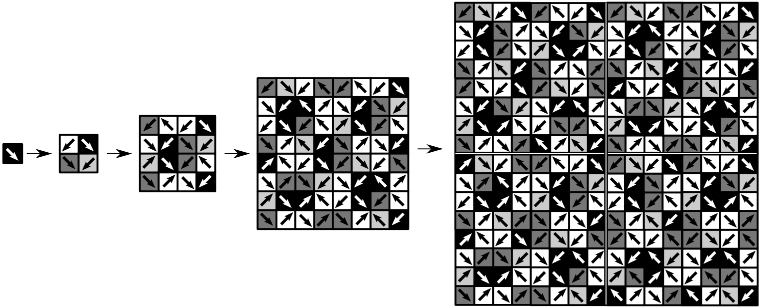
Once Theorem 1 is proven, this allows to prove Theorem 2.

The substitution rule:



...generates the same (infinite) tilings as Kurt's inductive rotation method.



Once Theorem 2 is proven, the well-developed machinery for substitution tilings can be applied. Yields e.g.

Theorem (3)

The arrowed tiling is aperiodic.

Theorem (4)

The arrowed tiling has pure point diffraction.

Theorem 3 can be proven by a “classical” result (1984) which says essentially:

If the substitution rule can be inverted uniquely (by local means) then the tiling is aperiodic.

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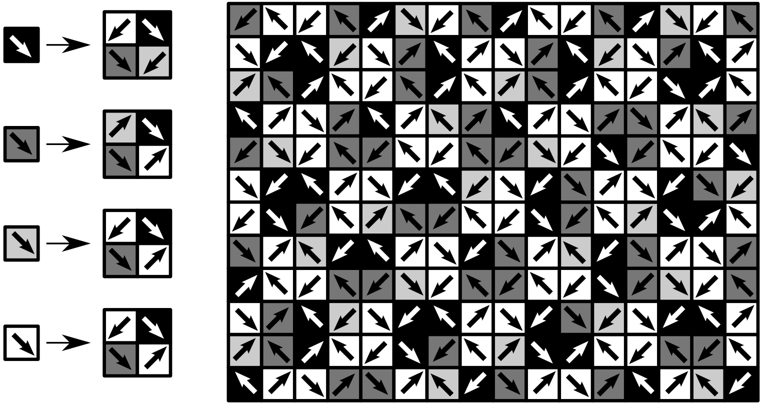
If the substitution rule can be inverted uniquely (by local means) then the tiling is aperiodic.

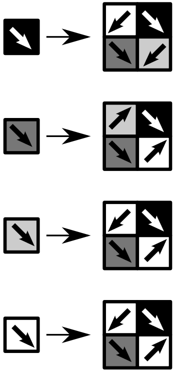
In even plainer words:

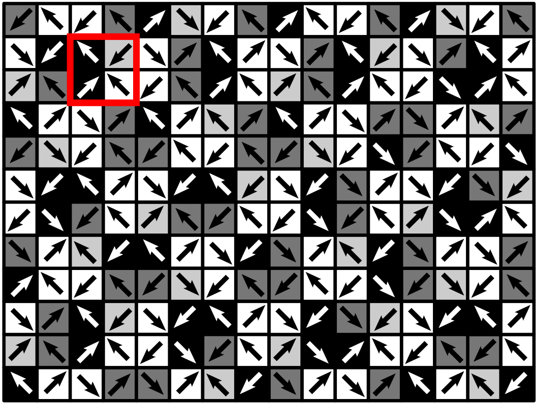
If one can identify the “previous generation” of the substitution tiling in a unique way then the tiling is aperiodic.

Here:



No.





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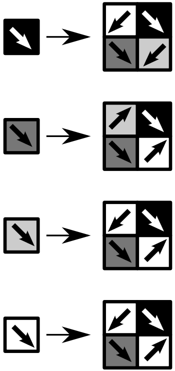

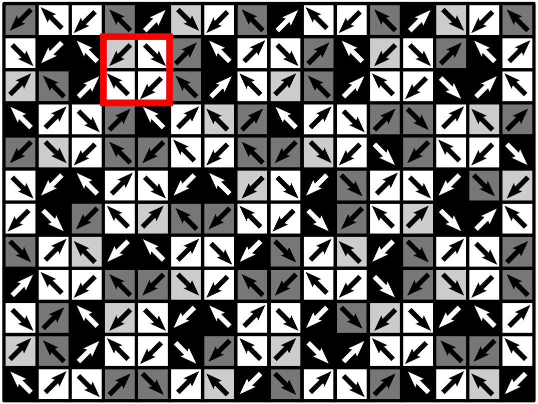

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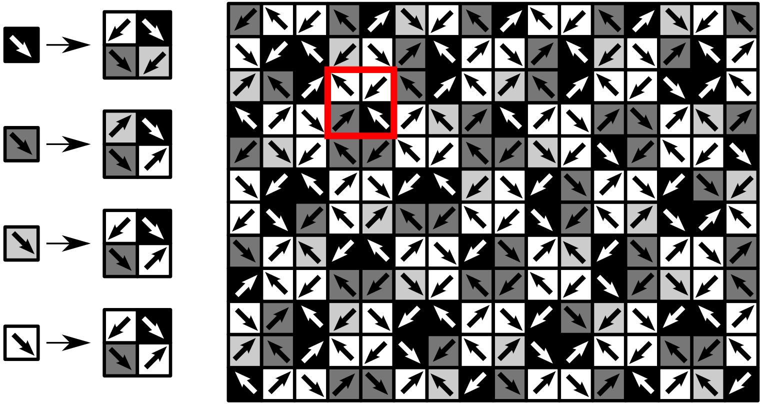

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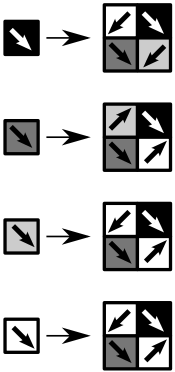
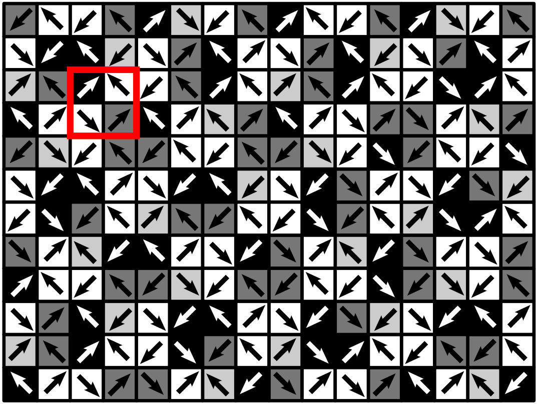
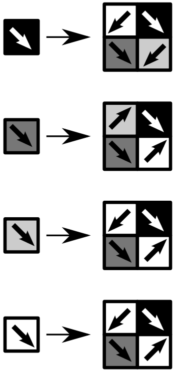
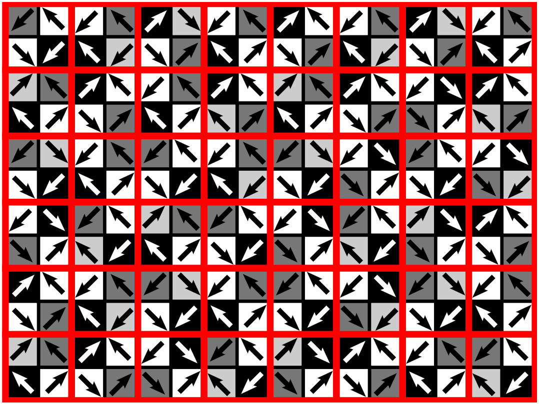




Yes.





Bingo!

In a similar manner we can use general results to compute e.g. the relative frequencies of the tiles in the tiling:

The relative frequencies of the tiles are the entries of the normalised eigenvector of the dominant eigenvalue of the substitution matrix

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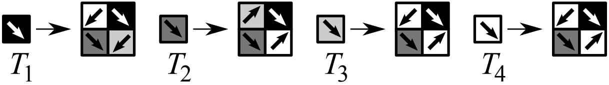

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

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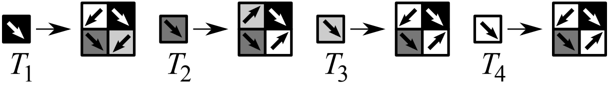
The relative frequencies of the tiles are the entries of the normalised eigenvector of the dominant eigenvalue of the substitution matrix



$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

Relative frequencies: **2:2:1:3**

Theorem 4 is obtained in a similar manner, using a general result applied to this situation (Lee-Moody-Solomyak 2003):

If [some pretty technical condition is fulfilled] then the tilings have pure point diffraction

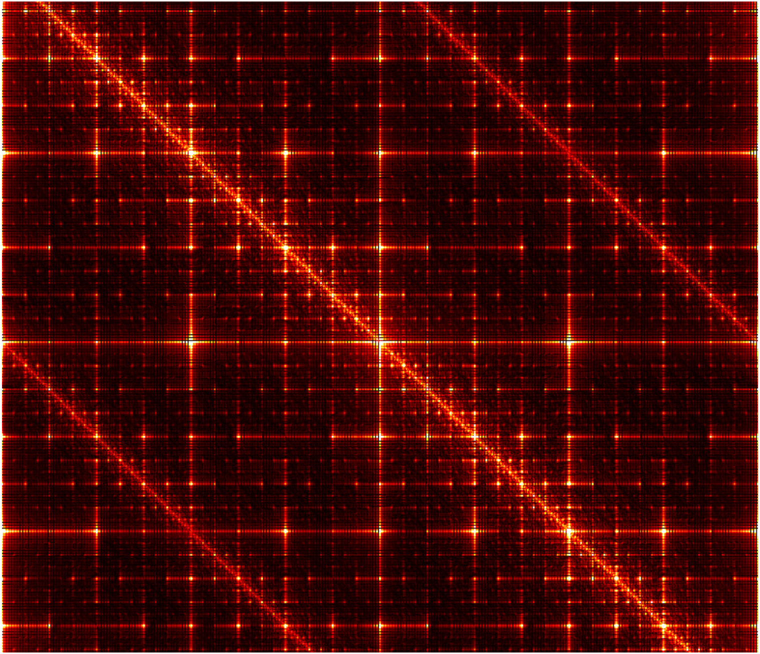
More in

D. Frettlöh, K. Hofstetter: Inductive rotation tilings, Proc. Steklov Inst. 288 (2015) 269-280; arXiv:1410.0592

The article mentioned answers some questions. Several further questions remain open. E.g.

- What about the naked version? (probably evrything the same)
- What about 2-step rotation and 5-step rotation? (No idea)

Let us conclude with a (rough) picture of the diffraction image of an inductive rotation tiling:



Thank you!