Introduction to the Mathematics of Quasiperiodic Structures

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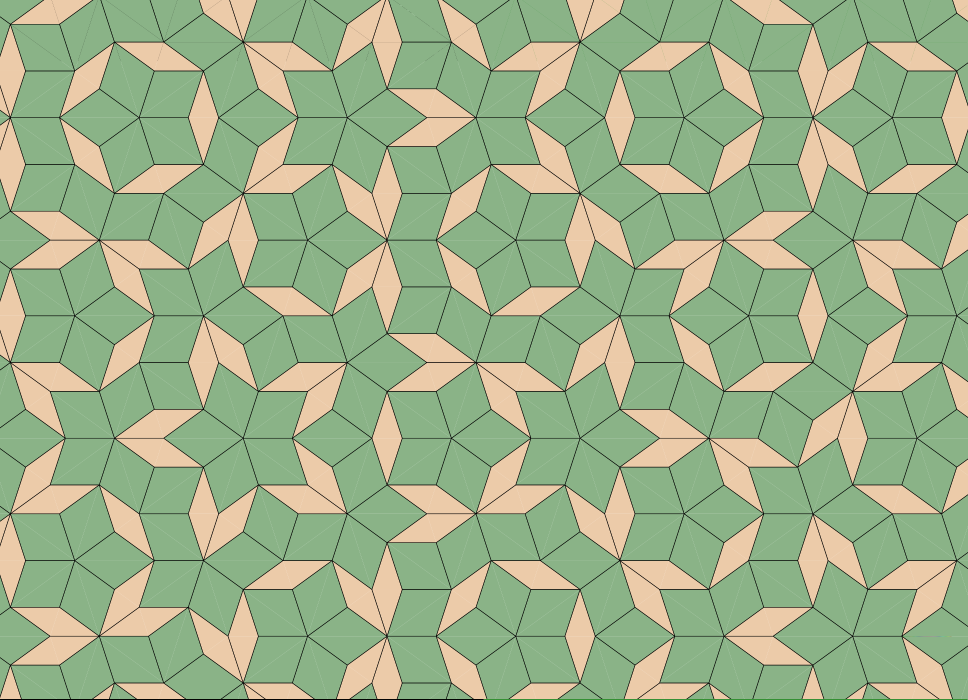
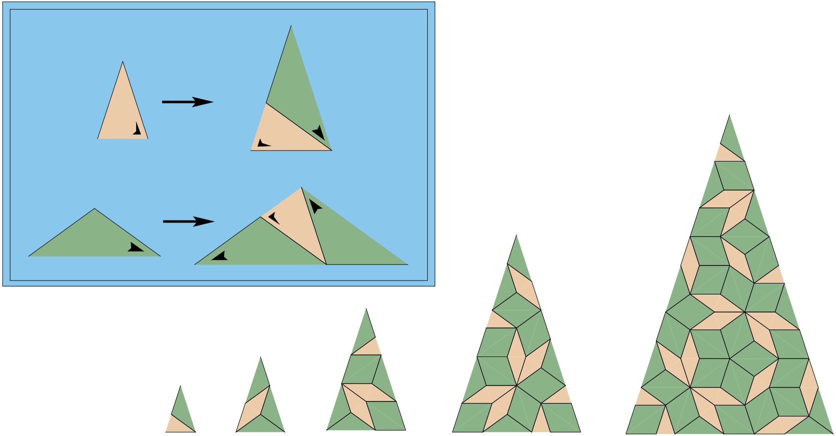
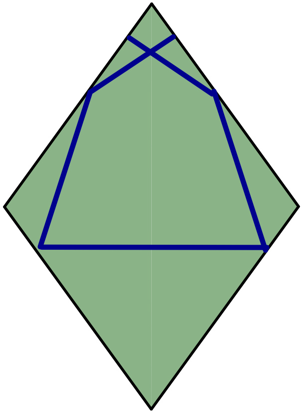
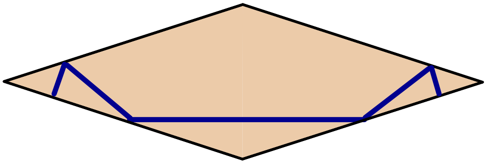
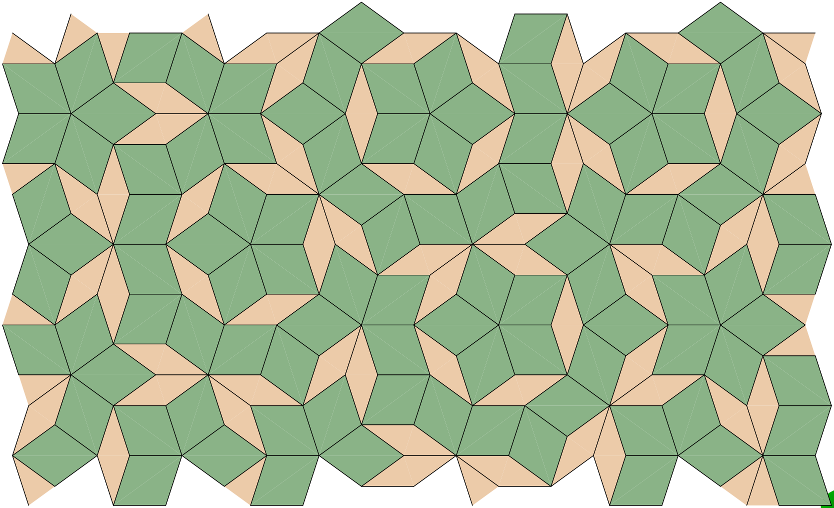
1. Examples
2. Diffraction
3. Cut-and-project method

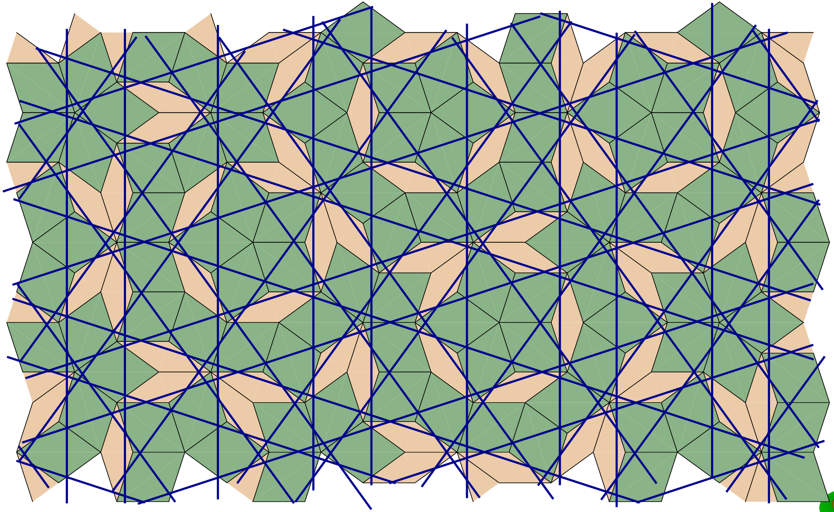


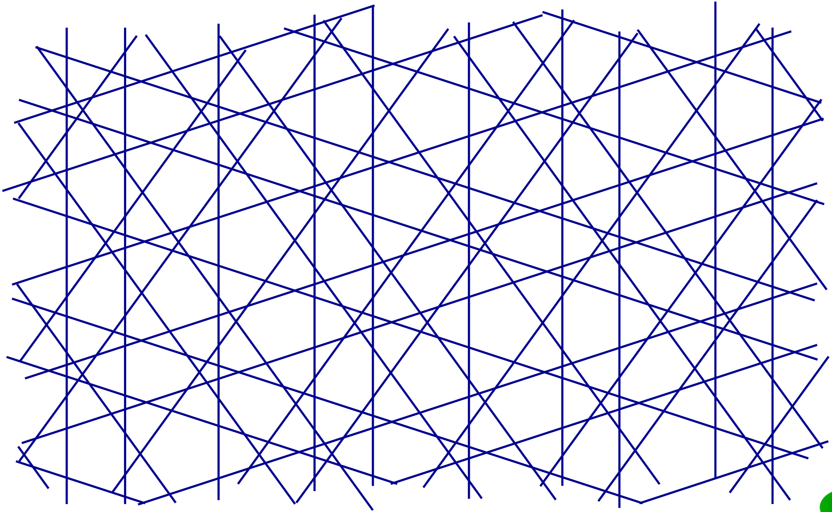












For more on the Penrose tiling, read chapter 10 in

Grünbaum, Shephard: Tilings and Patterns

For more examples & facts, visit the Tilings Encyclopedia

`tilings.math.uni-bielefeld.de`

2 Diffraction

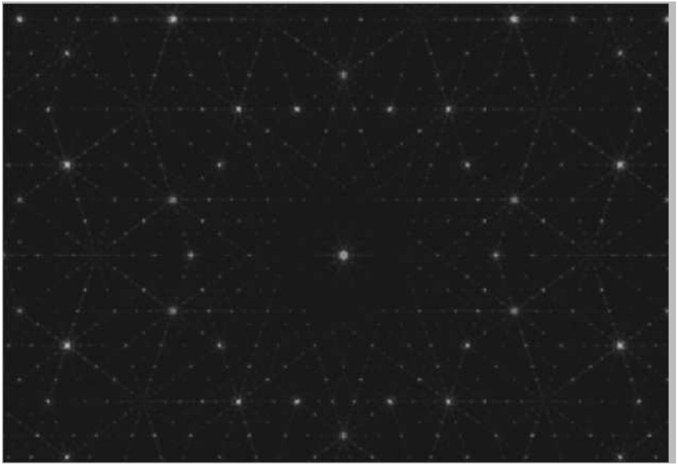
Mathematical description:

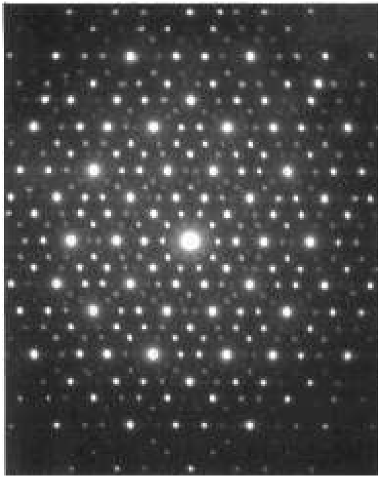
- Tiling $\rightsquigarrow$ discrete point set Λ.
- Autocorrelation $\gamma_\Lambda = \lim_{r\to\infty} \frac{1}{\mathrm{vol}\, B_r} \sum_{x,y\in\Lambda\cap B_r} \delta_{x-y}$.
- Fouriertransform $\widehat{\gamma}_\Lambda$ of the autocorrelation is the *diffraction spectrum*.

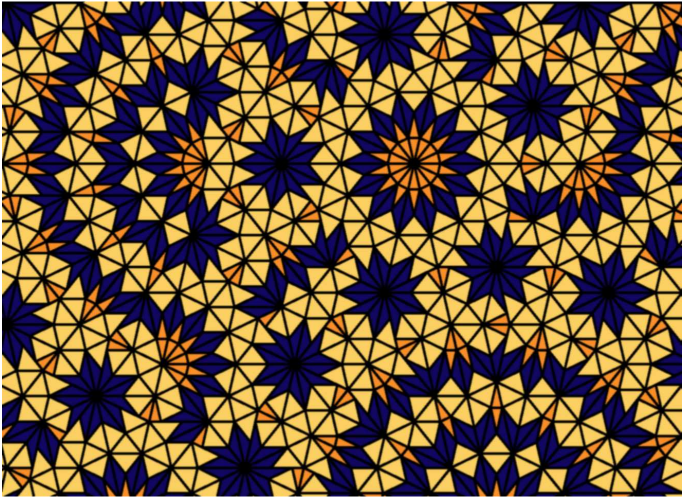
Since $\widehat{\gamma} := \widehat{\gamma}_\Lambda$ is again a measure, it decomposes into three parts:

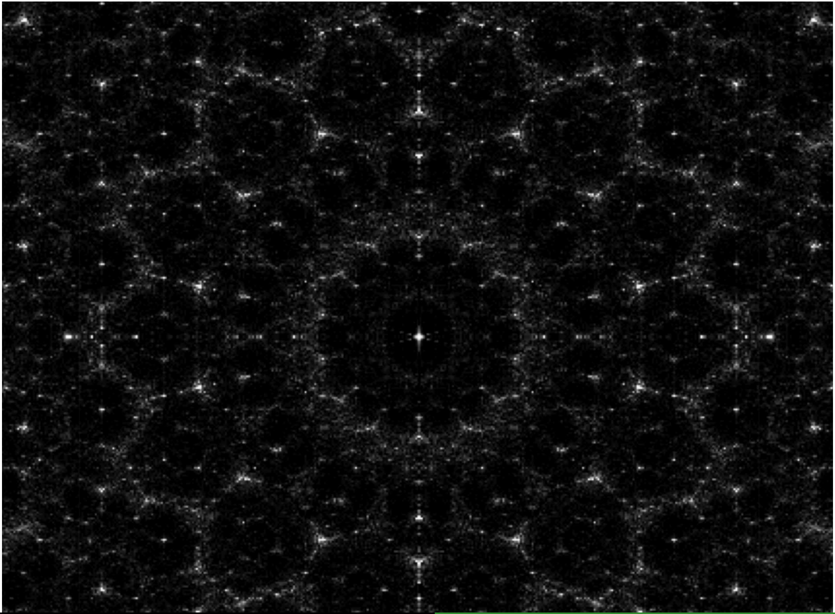
$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}$$

(pp: pure point, ac: absolutely continuous, sc: singular continuous)









For crystals:

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \cancel{\widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}}$$

For crystals: Λ is a lattice.

$$\gamma = \sum_{x \in \Lambda} \delta_x$$

By Poisson's summation formula:

$$\widehat{\gamma} = \mathrm{dens}(\Lambda) \sum_{x \in \Lambda^*} \delta_x = \widehat{\gamma}_{pp}$$

(where Λ^* is the *dual* lattice: $\Lambda^* = \{ z \,|\, \forall x \in \Lambda : \; z.x \in \mathbb{Z} \}$.)

Thus (ideal, infinite, perfect) crystals have pure point diffraction.

For quasicrystals:

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \cancel{\widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}}$$

An ideal (mathematical, infinite) quasicrystal has also pure point diffraction:

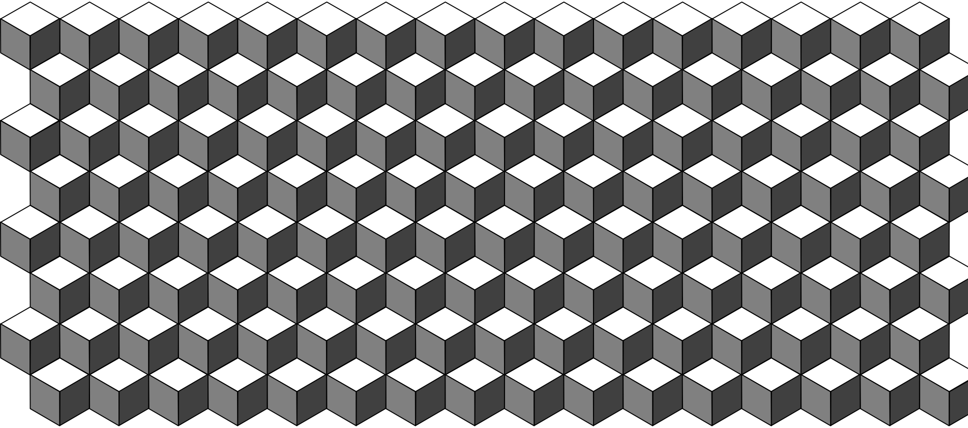
$$\widehat{\gamma} = \widehat{\gamma}_{pp}$$

E.g., for the Penrose pattern, we know everything about its diffraction: intensities and positions of the Bragg peaks.

Also for other examples, with 8-fold, 12-fold or icosahedral symmetry.

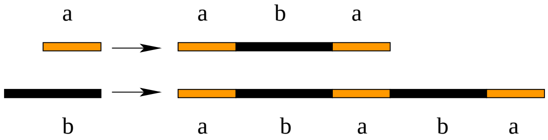
In contrast, for several patterns we know very few about their diffraction.

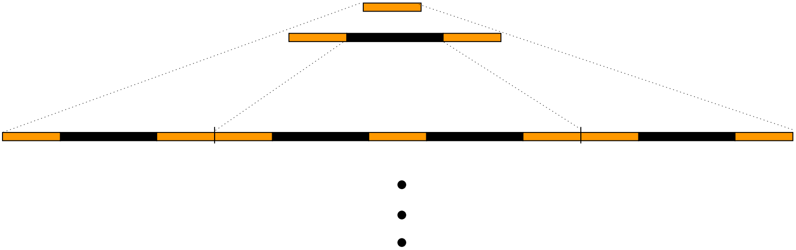
3 Cut and project method

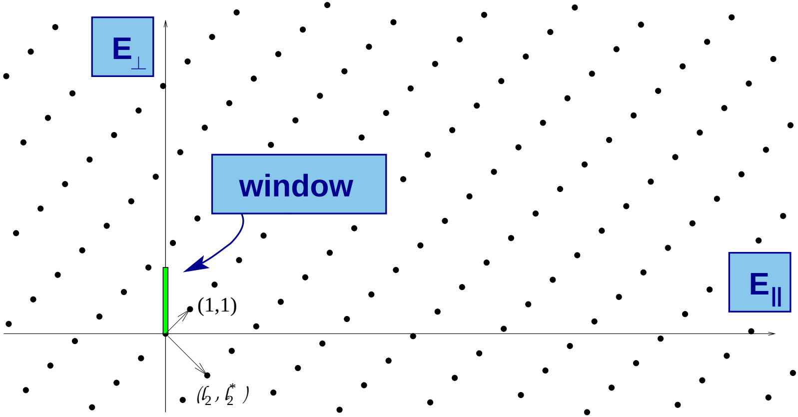


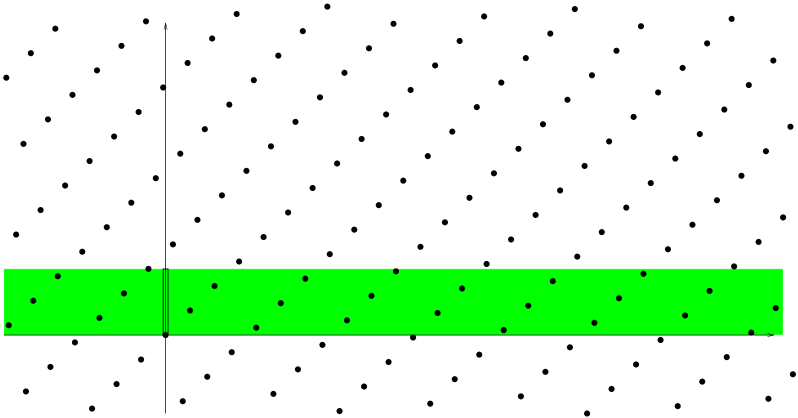
3 Cut and project method

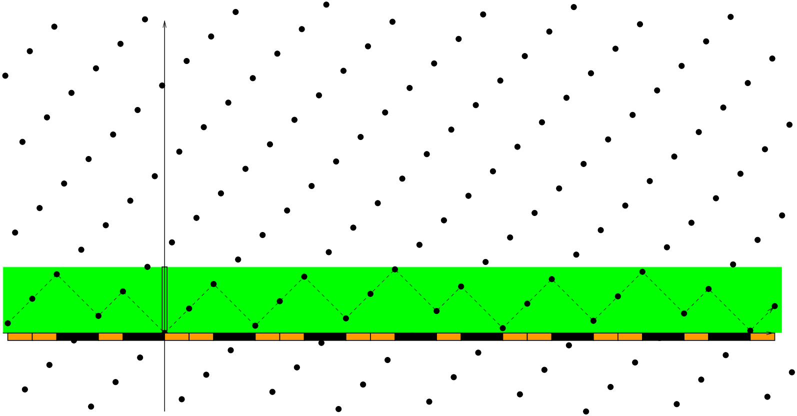
First in dimension 1.

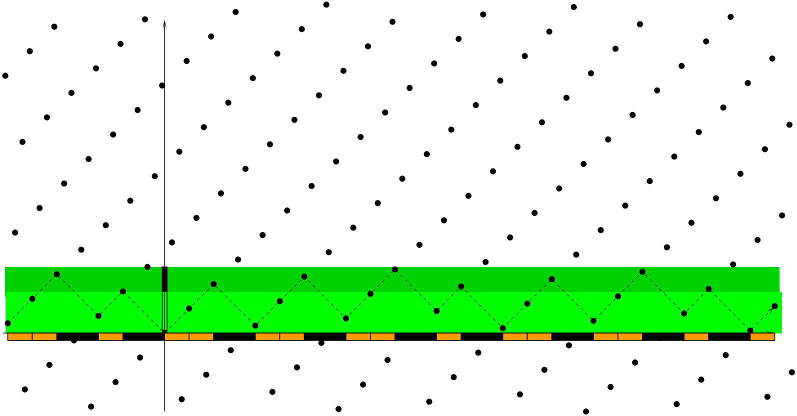

a
a b a
b
a b a b a




E⊥
window
(1,1)
(ℓ_2, ℓ_2^*)
E||







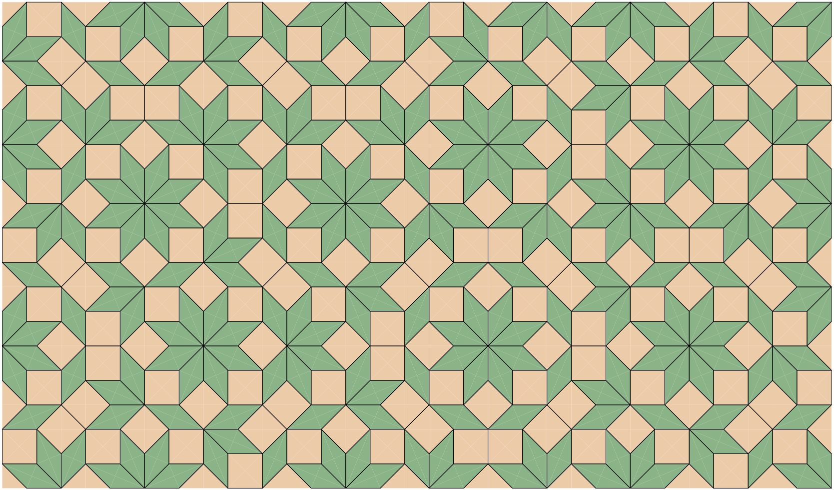
We know:

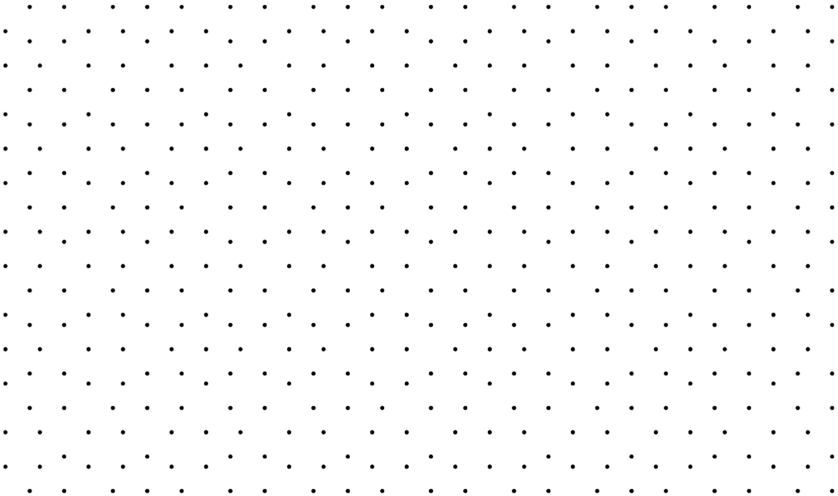
Theorem (Hof 95, Schlottmann 2000)

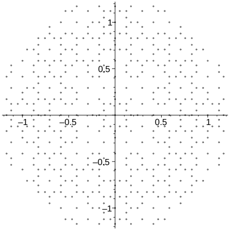
Each cut-and-project pattern, with compact & regular window, is pure point diffractive.

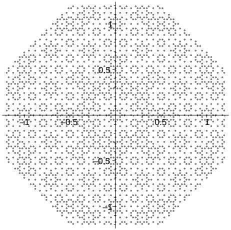
Cut and project for 2dim patterns:

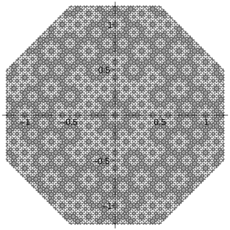
The Ammann Beenker tiling is obtained by projection from $\mathbb{R}^4$.
The window is a regular octagon.



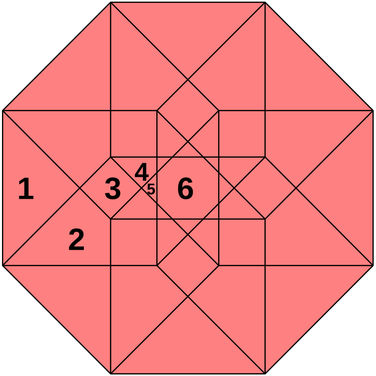


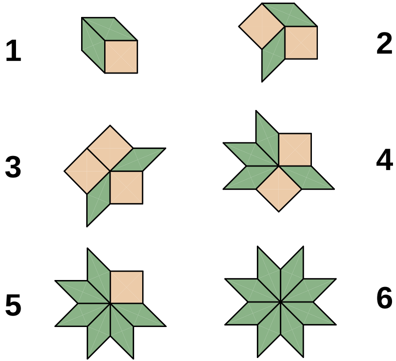






[picture removed]


1
2
3
4
5
6


1
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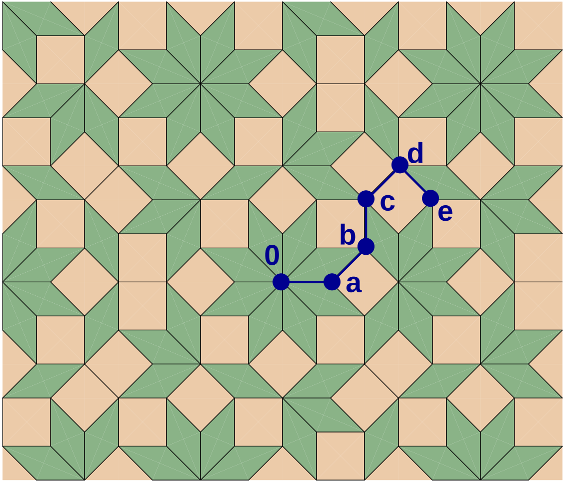
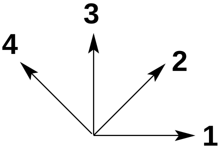




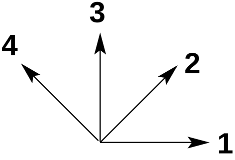
a: (1,0,0,0)

b: (1,1,0,0)

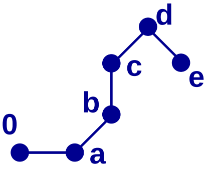
c: (1,1,1,0)

d: (1,2,1,0)

e: (1,2,1,−1)



In $E_{\parallel}$



a: (1,0,0,0)

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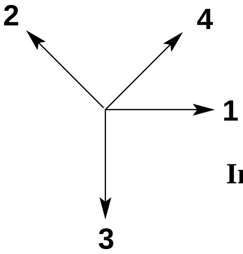
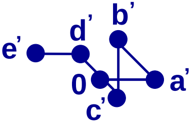


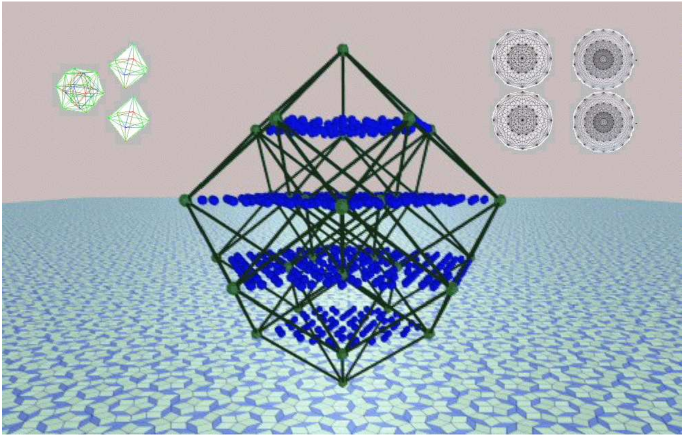
In $E_{\perp}$



For the Penrose pattern: slightly more complicated.

One *can* obtain it by projection from $\mathbb{R}^4$, but this requires some further techniques.

One obtains it by projection from $\mathbb{R}^5$ more easily.



Conclusion

The cut-and-project method is useful.

- To prove pure point diffractivity
- To determine quantities, e.g. intensities of Bragg peaks, density of points etc.
- More general, it is a powerful tool to analyze quasiperiodic structures.

Thank you.

$$
\begin{array}{ccccc}
E_{\|} & & & & E_{\perp} \\
\| & & & & \| \\
\mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^e & \xrightarrow{\pi_2} & \mathbb{R}^e \\
\cup & & \cup & & \cup \\
V & & \Lambda & & W
\end{array}
$$

- Λ a *lattice* in $\mathbb{R}^d \times \mathbb{R}^e$
- π_1, π_2 *projections*
  - $\pi_1|_\Lambda$ injective
  - $\pi_2(\Lambda)$ dense
- The *window* W *compact*
  - $\mathrm{cl}(\mathrm{int}(W)) = W$
  - $\mu(\partial(W)) = 0$

Then $V = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *model set*.

The *star map* : $\quad \star : \pi_1(\Lambda) \to \mathbb{R}^e,\ x^\star = \pi_2 \circ {\pi_1}^{-1}(x)$

Given a substitution tiling which *is* a cut-and-project tiling:

Tiling $\rightsquigarrow$ point set V; $\overline{V^\star} = W$
which is the *window*.