About harmonious Delone sets

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1. Character theory
2. Delone sets, cut-and-project sets
3. Some proofs

Character theory

- G a locally compact abelian group
- *algebraic character*: Homom.
  $\chi : G \to U(1) := \{z \in \mathbb{C} \; : \; |z| = 1\}$
- $\widehat{G}_{alg}$: the group of all algebraic characters of G.
  $\mathrm{id} : x \mapsto 1; \quad \chi\overline{\chi} = 1$
- The group of *continuous* characters: the *dual group* $\widehat{G}$ of G.

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- The group of *continuous* characters: the *dual group* $\widehat{G}$ of G.

Let G_d be G with the discrete topology. Then *each* homom $\chi : G \to U(1)$ is cont., thus $\widehat{G}_{alg}$ is the dual group of G_d.

Facts about Pontryagin duality:

(a) $\widehat{\widehat{G}} \cong G$. Thus notation: $\langle \chi, x \rangle := \chi(x)$

(b) $\alpha : G_1 \to G_2$ cont. homom., then
$\widehat{\alpha} : \widehat{G}_2 \to \widehat{G}_1$ is a homom.
(where $\widehat{\alpha}$ is given by $\langle \widehat{\alpha}(\chi), x \rangle = \langle \chi, \widehat{\alpha}(x) \rangle$)

(c) $\alpha(G_1)$ dense in G_2 iff $\widehat{\alpha}$ is 1-1

(d) $\widehat{\alpha}(\widehat{G}_2)$ dense in $\widehat{G}_1$ iff α is 1-1

Bohr compactification:

Let G_d be as above: G with the discrete topology,
and $i : G_d \to G$ the identity.
i is cont. and 1-1, thus, by (c)&(d), $\widehat{i} : \widehat{G} \to \widehat{G}_{alg}$ has dense image:

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F compact in G_d, iff F is finite. The top on $\widehat{G}_{alg}$ is the top of uniform convergence on compact=finite sets F. That is, for each $\chi_0 \in \widehat{G}_{alg}$:

$$\forall \varepsilon > 0 \quad \exists \chi \in \widehat{G} : |\chi(x) - \chi_0(x)| < \varepsilon \quad \text{for all } x \in F$$

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In general, this uniform approximation property does not hold for infinite F. But for certain sets, it does!

Definition

A Delone set $F \subset \mathbb{R}^d$ is called harmonious, if (1) *holds for F.*

It follows a characterization of harmonious Delone sets.

Delone sets, cut-and-project sets

Definition

$\Lambda \subset \mathbb{R}^d$ is a Delone set, if

(i) $\exists r > 0 : \quad \forall x \in \mathbb{R}^d : \quad \#(B_r(x) \cap \Lambda) \le 1$ *(uniformly discrete)*

(ii) $\exists R > 0 : \quad \forall x \in \mathbb{R}^d : \quad \#(B_R(x) \cap \Lambda) \ge 1$ *(relatively dense)*

(ii) can also be stated as: $\Lambda + B_R(0) = \mathbb{R}^d$.

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In particular, Delone sets are infinite.

Ex.: point lattices in $\mathbb{R}^d$, for instance $\mathbb{Z}^d$.

Theorem (Meyer, Lag.)

Let Λ be relatively dense. TFAE:

(i) *Λ is harmonious*

(ii) *$\Lambda - \Lambda$ is Delone*

(iii) *$\Lambda - \Lambda \subseteq \Lambda + F$, with F finite*

(iv) *Λ^{ε} is relatively dense for all $\varepsilon > 0$*

(v) *Λ is a subset of a cut-and-project set*

Definition

(ε-dual:) $\Lambda^{\varepsilon} = \{k \in \mathbb{R}^d \ : \forall x \in \Lambda : \ |e^{2\pi i x \cdot k} - 1| \leq \varepsilon\}$

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Definition

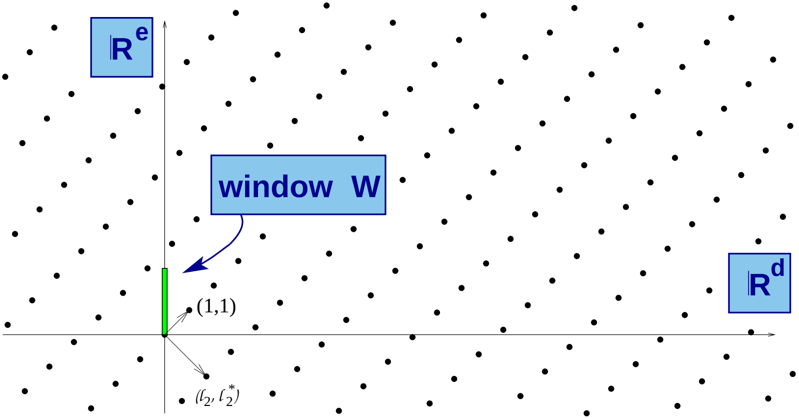
(ε-dual:) $\Lambda^{\varepsilon} = \{k \in \mathbb{R}^d \, : \, \forall x \in \Lambda : \, |e^{2\pi i x \cdot k} - 1| \leq \varepsilon\}$

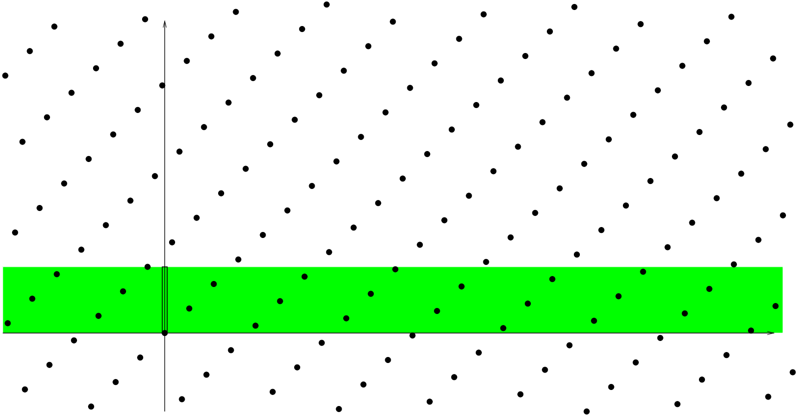
Ex.: If Λ is a lattice, then $\Lambda - \Lambda = \Lambda$.
So Λ fulfills (ii), and also (iii) with $F = \{0\}$.

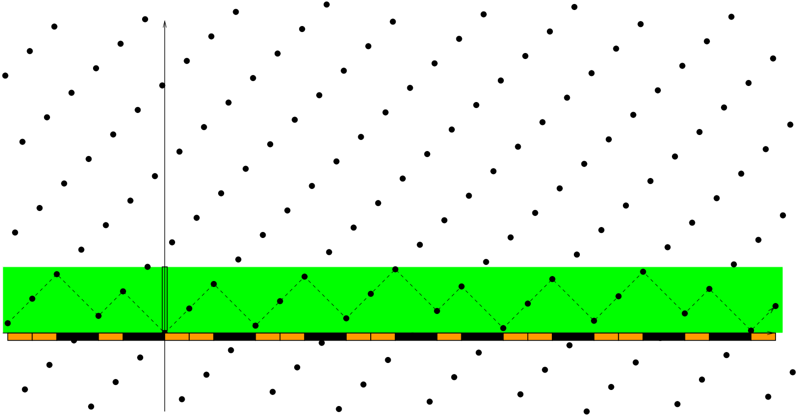
$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^e & \xrightarrow{\pi_2} & \mathbb{R}^e \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

- Γ a *lattice* in $\mathbb{R}^d \times \mathbb{R}^e$
- π_1, π_2 *projections*
  - $\pi_1|_\Lambda$ injective
  - $\pi_2(\Gamma)$ dense
- The *window* W *compact*

Then $\Lambda = \{\pi_1(x) \,|\, x \in \Gamma, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set*.


$\mathbb{R}^e$
window W
$\mathbb{R}^d$
(1,1)
(ℓ_2, ℓ_2^*)





Some proofs

(vi) $\Rightarrow$ (i): Let Λ^ε be relatively dense.

Then there is a compact $K \subset \widehat{\mathbb{R}^d} \cong \mathbb{R}^d$, s.t.

$$\Lambda^\varepsilon \oplus K = \widehat{\mathbb{R}^d}$$

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$$\Lambda^\varepsilon \oplus K = \widehat{\mathbb{R}^d}$$

Let $V^\varepsilon(\Lambda) = \{\mu \in \widehat{\mathbb{R}^d}_{alg} : \forall x \in \Lambda : |\mu(x) - 1| \leq \varepsilon\}$.

Clearly, $\Lambda^\varepsilon = V^\varepsilon(\Lambda) \cap \widehat{\mathbb{R}^d}$.

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Let $V^\varepsilon(\Lambda) = \{\mu \in \widehat{\mathbb{R}^d}_{alg} : \forall x \in \Lambda : |\mu(x) - 1| \le \varepsilon\}$.
Clearly, $\Lambda^\varepsilon = V^\varepsilon(\Lambda) \cap \widehat{\mathbb{R}^d}$.
Since $\widehat{\mathbb{R}^d}$ dense in $\widehat{\mathbb{R}^d}_{alg}$, we have $V^\varepsilon(\Lambda) \oplus K$ dense in $\widehat{\mathbb{R}^d}_{alg}$.
Since $V^\varepsilon(\Lambda) \oplus K$ closed,

$$V^\varepsilon(\Lambda) \oplus K = \widehat{\mathbb{R}^d}_{alg}$$

In particular, for all $\varepsilon > 0$:

$$V^\varepsilon(\Lambda) \oplus \widehat{\mathbb{R}^d} = \widehat{\mathbb{R}^d}_{alg}$$

$$V^{\varepsilon}(\Lambda) \oplus \widehat{\mathbb{R}^d} = \widehat{\mathbb{R}^d}_{alg} \qquad (2)$$

Now, for $\varepsilon > 0$ and $\chi_0 \in \widehat{\mathbb{R}^d}_{alg}$, we have

$$\chi_0 = \chi \oplus \mu \quad \text{for some } \mu \in V^{\varepsilon/2}(\Lambda), \chi \in \widehat{\mathbb{R}^d},$$

and for all $x \in \Lambda$ holds:

$$|\chi_0(x) - \chi(x)| = |\chi \oplus \mu(x) - \chi(x)| = |\mu(x) - 1| \leq \varepsilon/2 < \varepsilon$$

□

$$V^{\varepsilon}(\Lambda) \oplus \widehat{\mathbb{R}^d} = \widehat{\mathbb{R}^d}_{alg} \qquad (2)$$

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<u>(i) $\Rightarrow$ (iv)</u> : Use (2), construct K (slightly tricky).

(i)&(iv) $\Rightarrow$ (iii): Let Λ^ε be relatively dense.

Let $\varepsilon > 0$, $\chi \in \Lambda^\varepsilon$. If $x_1, x_2 \in \Lambda$, then $x_1 - x_2 \in \Lambda - \Lambda$, and

$$|\chi(x_1 - x_2) - 1| \leq |\chi(x_1) - 1| + |\chi(-x_2) - 1| \leq 2\varepsilon$$

thus $\Lambda^\varepsilon \subset (\Lambda - \Lambda)^{2\varepsilon}$.

Since Λ^ε is relatively dense for all $\varepsilon > 0$, so is $(\Lambda - \Lambda)^\varepsilon$.

Lemma

Λ *rel. dense* $\Rightarrow$ Λ^ε *unif. discrete for all* $\varepsilon < 1$.

Lemma

Λ rel. dense $\Rightarrow \Lambda^\varepsilon$ unif. discrete for all $\varepsilon < 1$.

By Pontryagin duality: Λ^ε rel. dense implies $\Lambda^{\varepsilon\varepsilon}$ unif. discrete for all $\varepsilon < 1$.

With $\Lambda^{\varepsilon\varepsilon} \subset \Lambda$ (using $\mathbb{R}^d \cong \widehat{\widehat{\mathbb{R}^d}}$) follows: Λ uniformly discrete.

Altogether: Λ^ε rel. dense $\Rightarrow (\Lambda - \Lambda)^\varepsilon$ rel. dense
$\Rightarrow \Lambda - \Lambda$ unif. discrete $\Rightarrow \Lambda - \Lambda$ Delone. □

Scheme:

$$(iii) \iff (iv) \iff (v) \quad : \quad \textit{Meyer } 1972$$

$$(ii) \iff (iii) \quad : \quad \textit{Lagarias } 1996$$

$$(i) \iff (iv) \quad : \quad \textit{Moody } 1996$$

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Thank you.