About substitution tilings with statistical circular symmetry

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Quasicrystals - The Silver Jubilee
Tel Aviv
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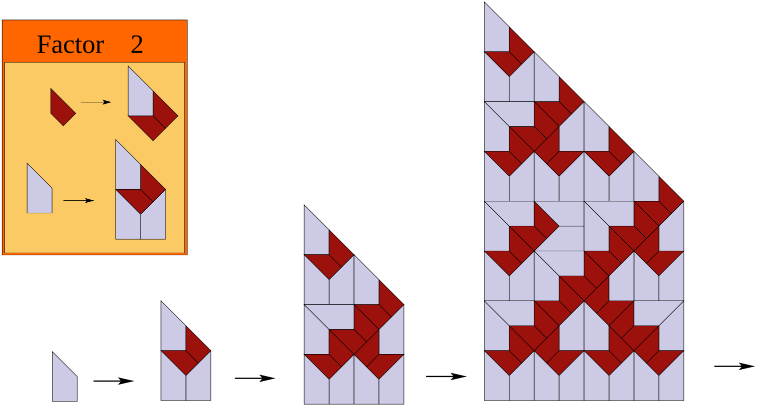
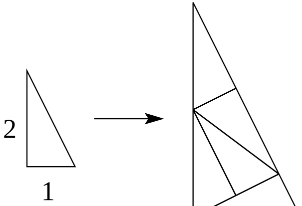
1. Examples of tilings with statistical circular symmetry
2. Diffraction of...
3. (Dynamics of...)

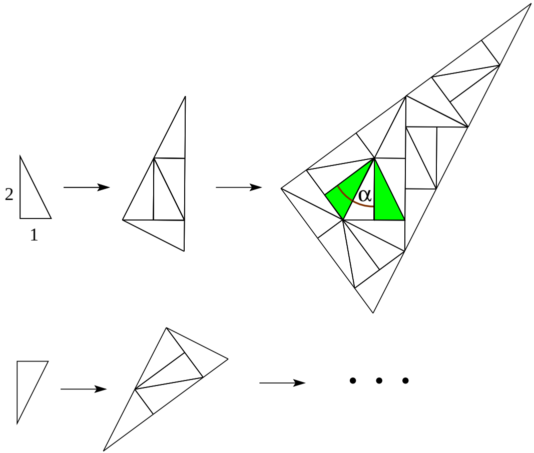
Substitution tilings:



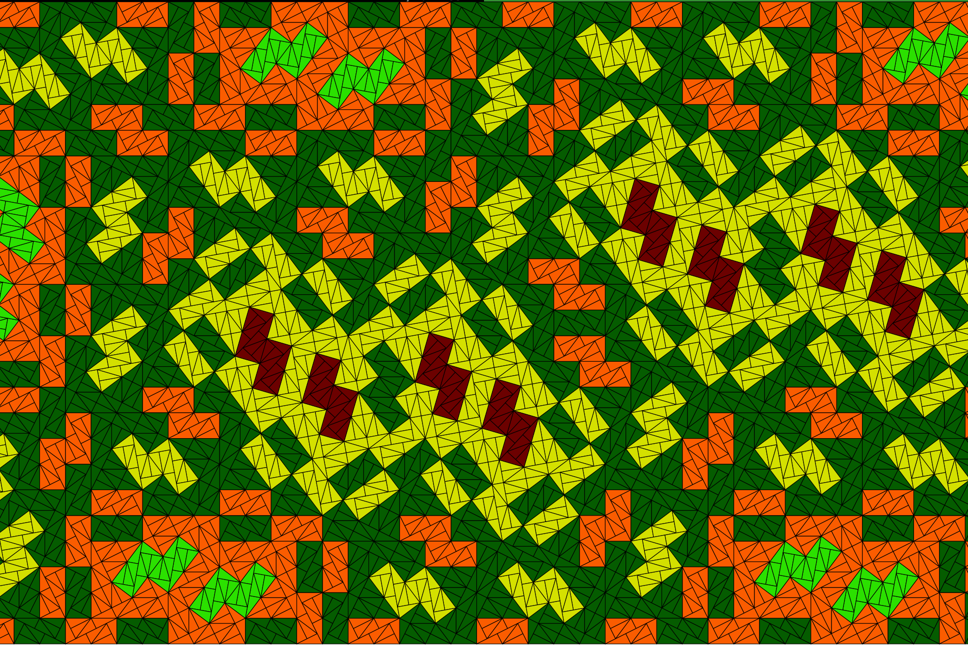
Usually, tiles occur in finitely many different orientations.

Not always. Conway's Pinwheel substitution (1991):

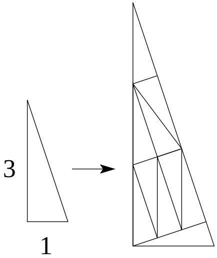




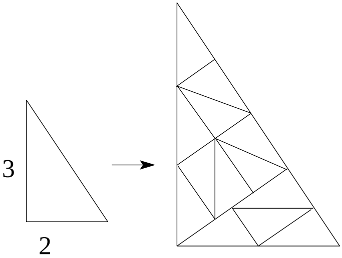
The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.



Obvious generalizations: Pinwheel (n, k)



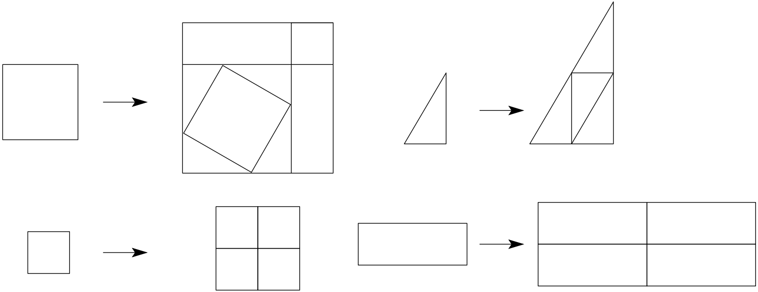
$n = 3, k = 1$



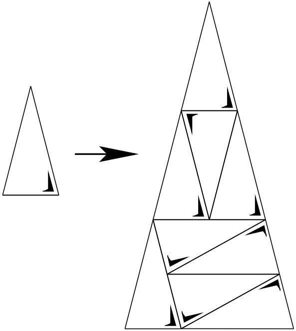
$n = 3, k = 2$

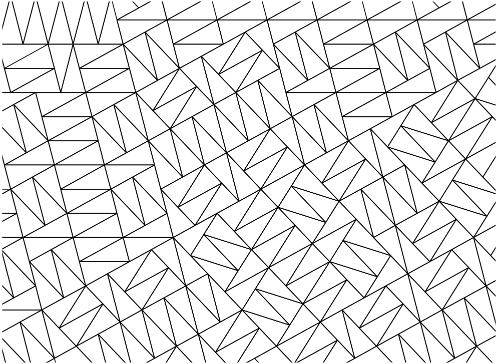
etc.

Cesi's example (1990):



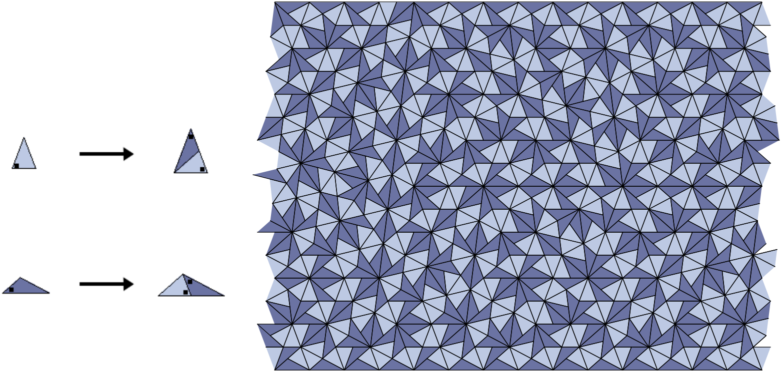
Unknown ($<$ 1996, communicated to me by Danzer):



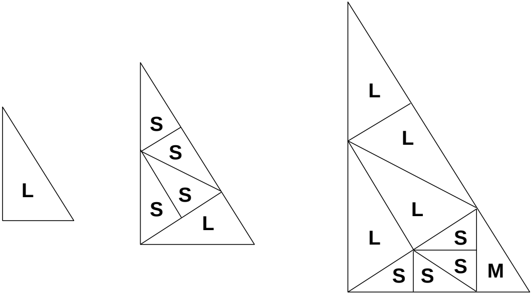


(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):

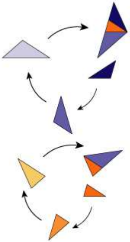


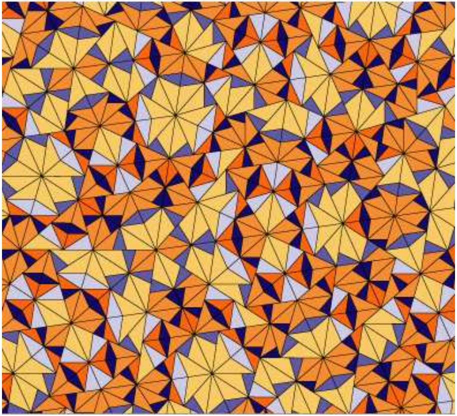
Sadun's generalized Pinwheels (1998):



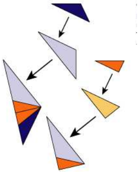
Yields infinitely many proper tile-substitutions.

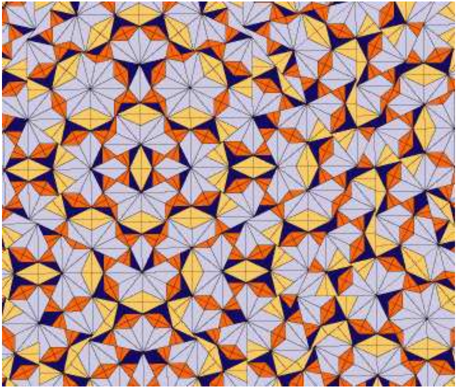
Harriss' Cubic Pinwheel (2004 ± 1):



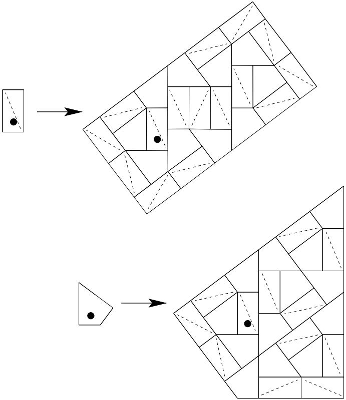


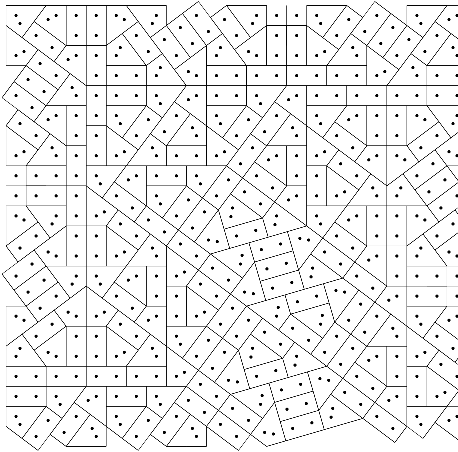
Harriss' Quartic Pinwheel (2004 ± 1):



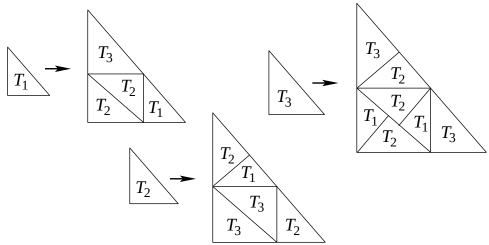


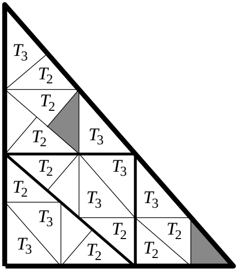
Kite domino (equivalent with pinwheel):





Pythia (m,j), here: $m = 3, j = 1$.





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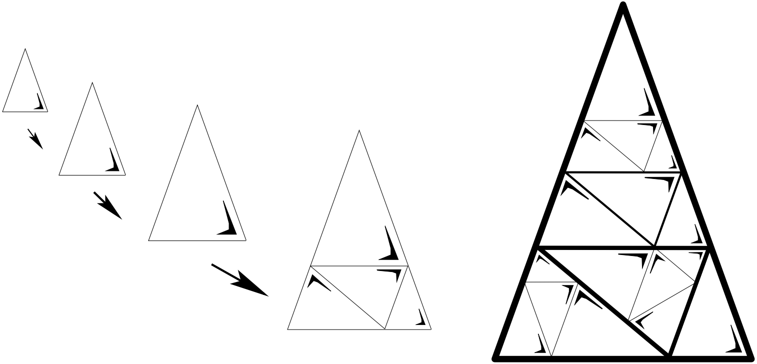
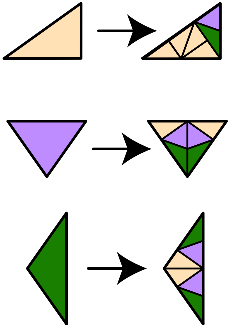
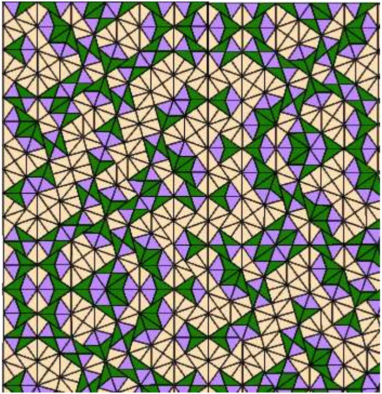


Dale Walton: several single examples





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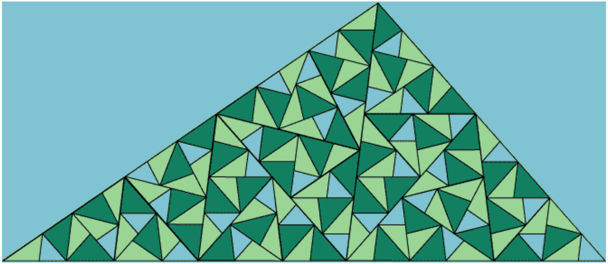
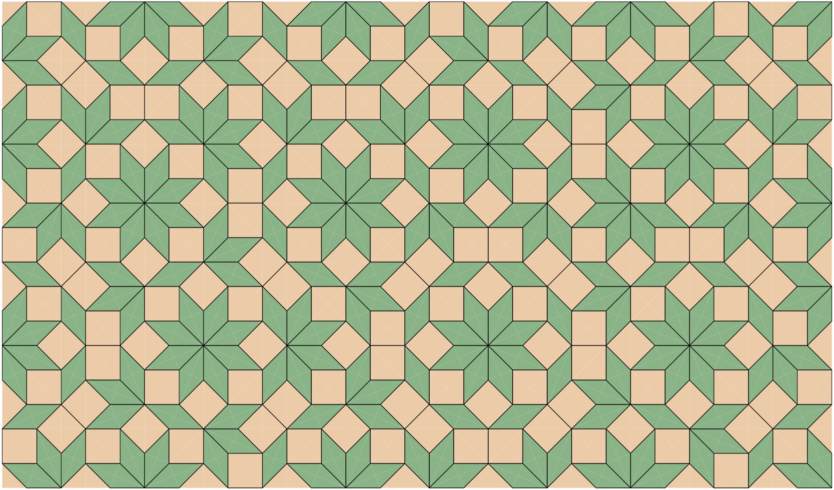
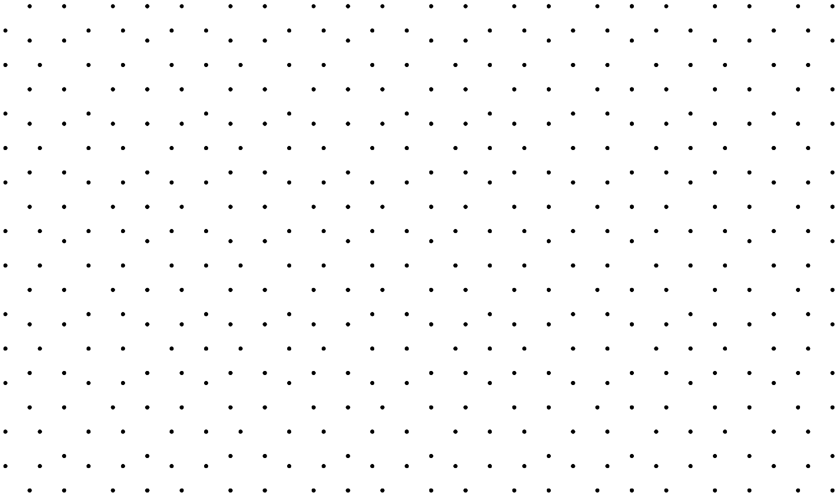
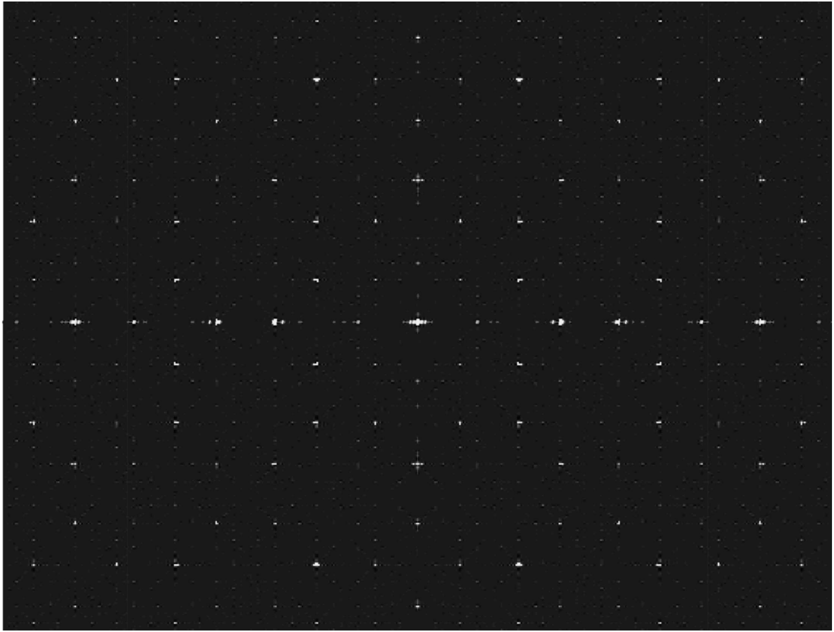


2. Diffraction







Mathematical description:

- Tiling $\rightsquigarrow$ discrete point set Λ.
- Autocorrelation $\gamma_\Lambda = \lim_{r \to \infty} \frac{1}{\text{vol } B_r} \sum_{x,y \in \Lambda \cap B_r} \delta_{x-y}$.
- Fouriertransform $\widehat{\gamma}_\Lambda$ of the autocorrelation is the *diffraction spectrum*.

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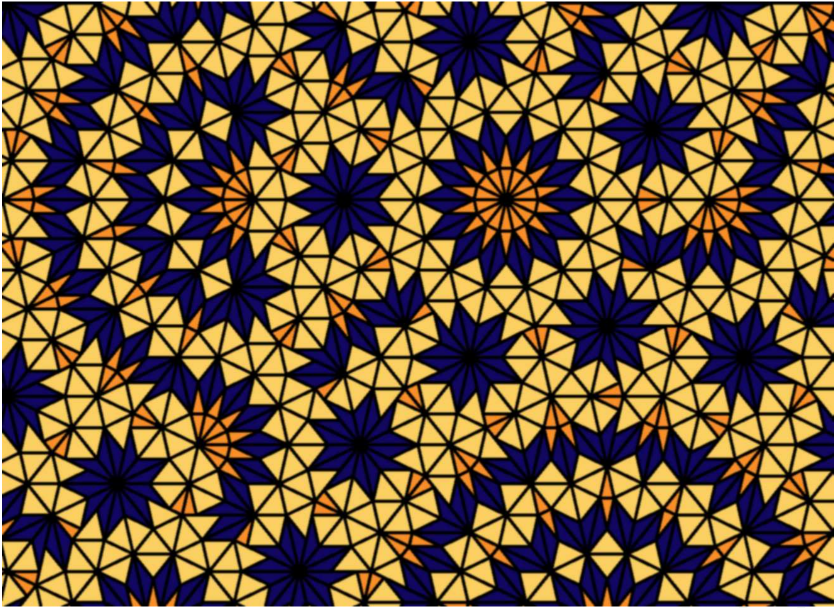
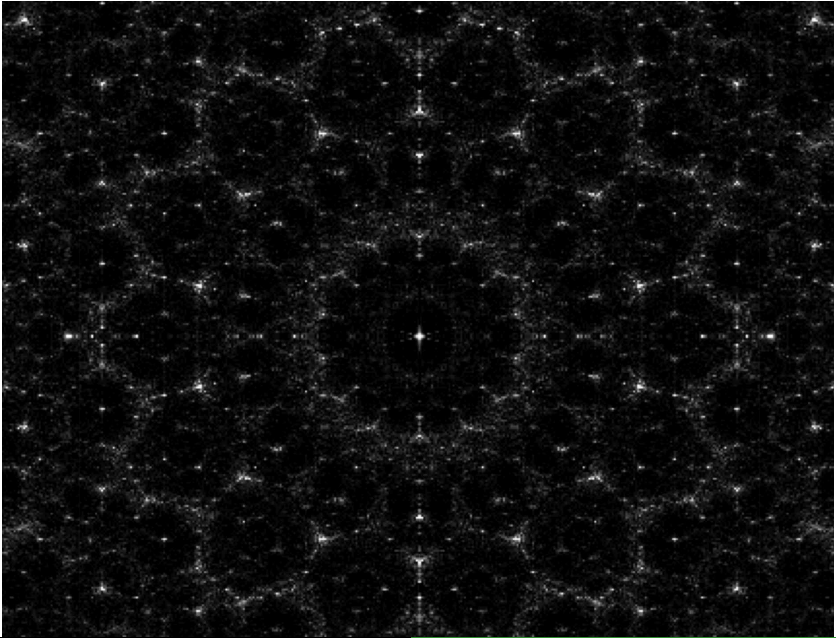
Since $\widehat{\gamma} := \widehat{\gamma}_\Lambda$ is again a measure, it decomposes into three parts:

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}$$

(pp: pure point, ac: absolutely continuous, sc: singular continuous)





$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac}$$

For an ideal (mathematical, infinite) quasicrystal:

$$\widehat{\gamma} = \widehat{\gamma}_{pp}$$

For primitive substitution tilings with infinitely many orientations (we've just seen many examples):

$$\widehat{\gamma} = \delta_0 + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac},$$

and $\widehat{\gamma}$ is circular symmetric.

This follows from:

Theorem

Each primitive substitution tiling with infinitely many orientations is of statistical circular symmetry, i.e. the orientations are equidistributed on the circle.

Roughly spoken, each orientation occurs with the same probability.

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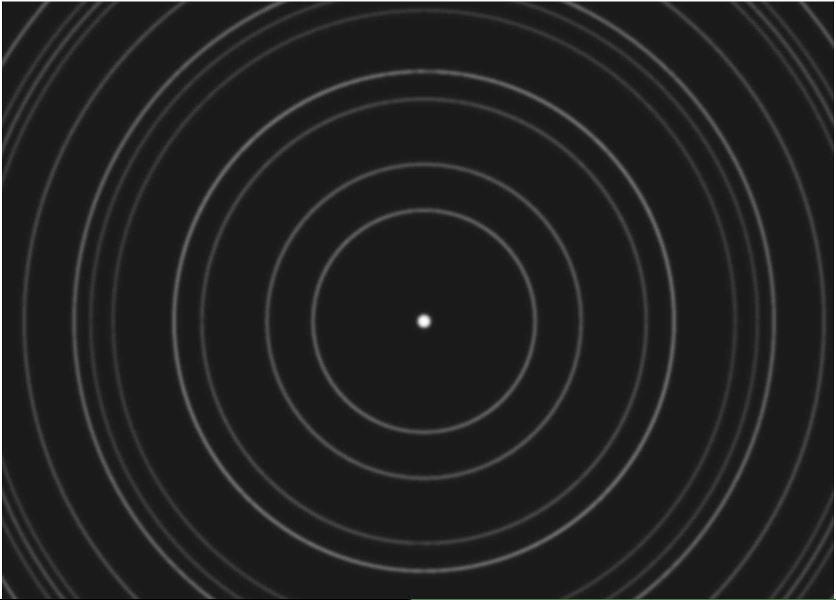
Continuous parts are still mysterious.

Pinwheel diffraction



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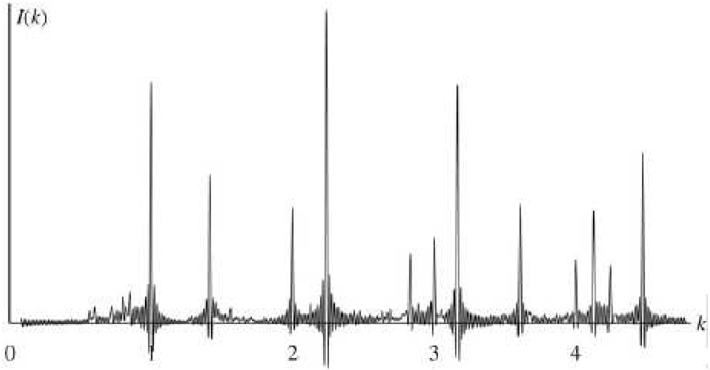

3. Dynamics / more about symmetry

The *hull* $\mathbb{X}_{\mathcal{T}}$ of a tiling:

The closure of $\{\mathcal{T} + t \,|\, t \in \mathbb{R}^d\}$ in an appropiate topology.

For primitive substitution tilings:
hull = LI-class = set of all tilings given by this substitution.

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In the pure point case, the dynamical spectrum of the dynamical system $(\mathbb{X}_{\mathcal{T}}, \mathbb{R}^d)$ yields information about the diffraction spectrum of $\mathcal{T}$. (Dworkin, Hof, Solomyak, Robinson, Schlottmann...).

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Maybe we can extend this to the continuous parts of the spectrum.