# Dynamical properties of almost repetitive patterns

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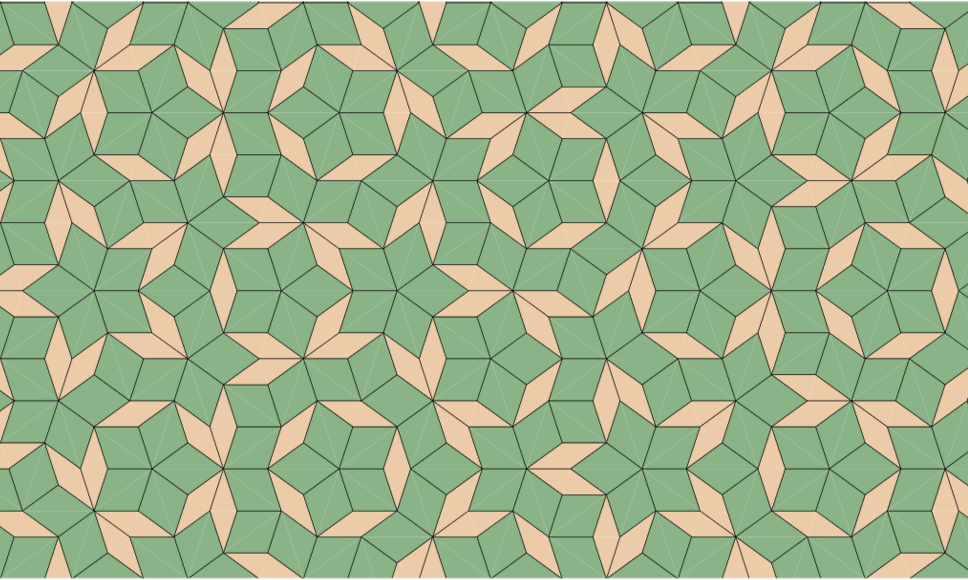
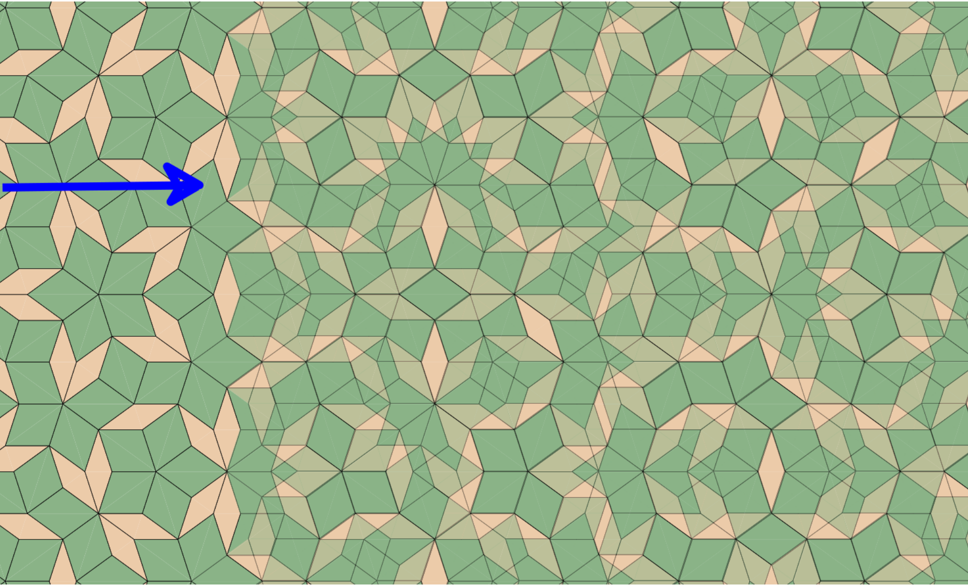
Joint work with Christoph Richard

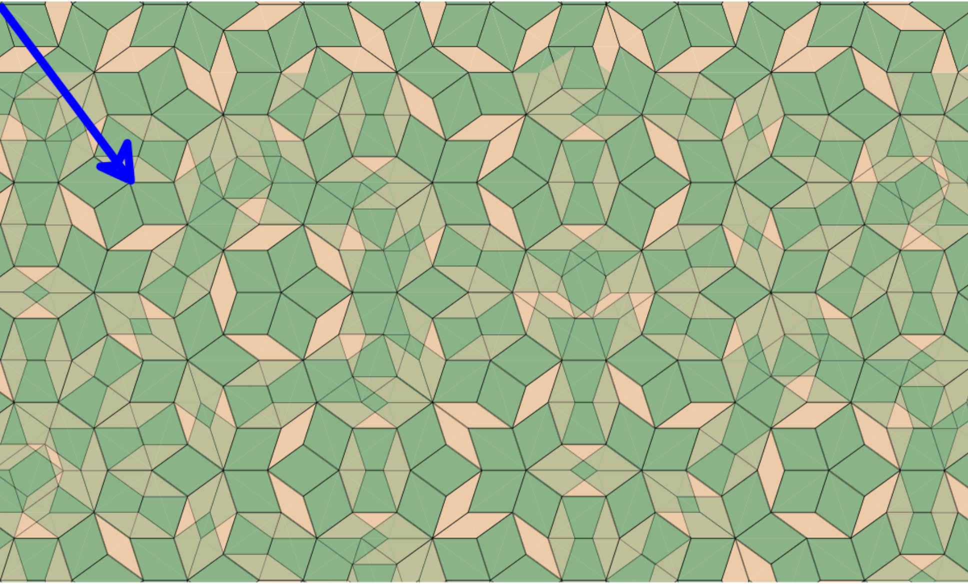
**Overview:**

1. Aperiodic tilings and Delone sets
2. Tiling dynamical systems of aperiodic tilings and Delone sets
3. Results on dynamical systems from substitution tilings
4. Tilings without finite local complexity (FLC)
5. Results on dynamical systems from non-FLC tilings

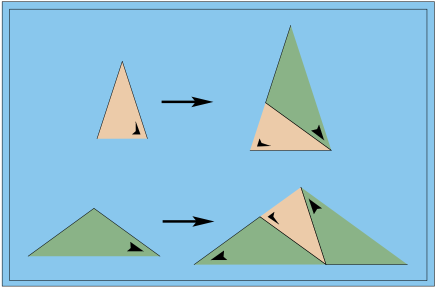
# 1. Aperiodic tilings and Delone sets

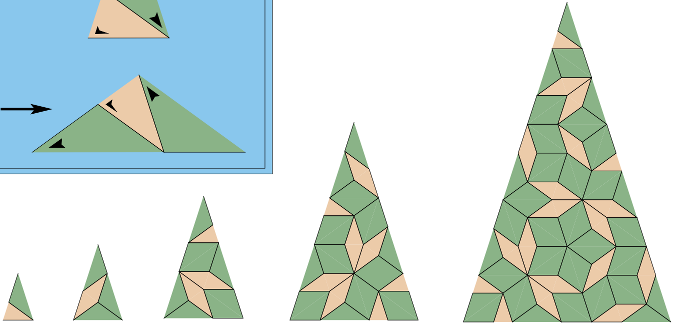






Substitution tilings:





Let $P$ be a tiling of the plane $\mathbb{R}^2$. (I.e., $P$ is a covering of $\mathbb{R}^2$ as well as a packing)

- $P$ is *aperiodic*: $P + t = P$ implies $t = 0$.
- An *r-patch* is a set of the form $B_r(x) \cap P$ (where $B_r(x)$ is the open ball of radius $r$ about $x$)

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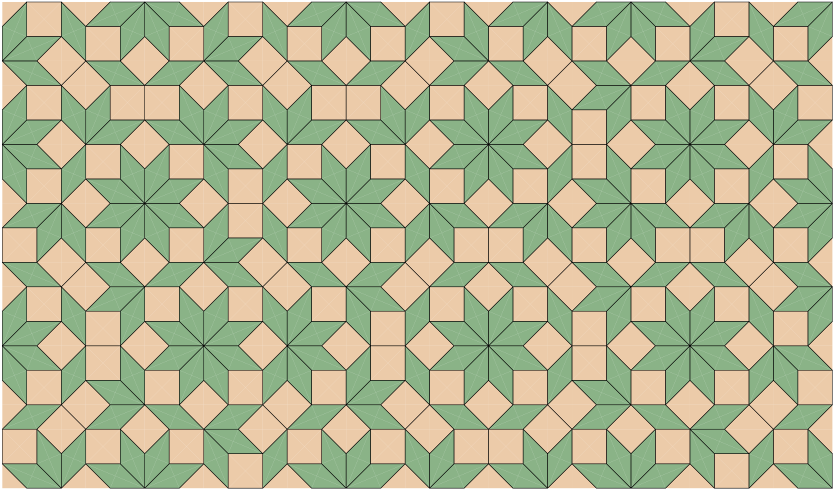
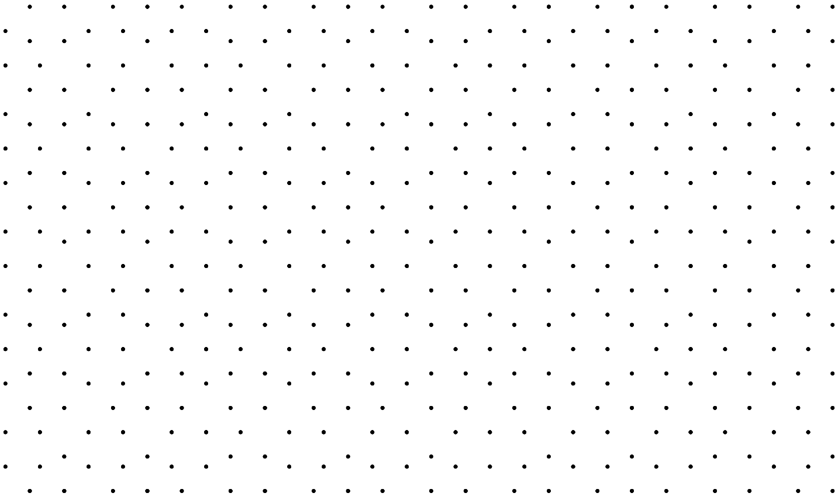
Sometimes it is beneficial to consider discrete point sets rather than tilings.

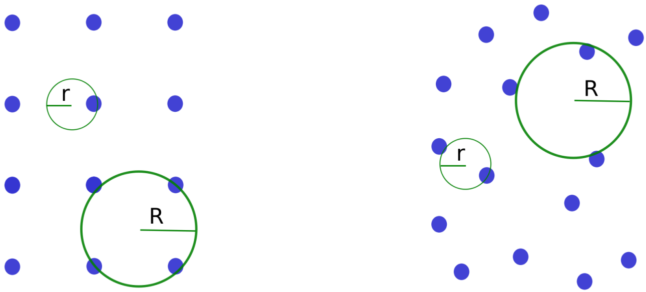
This can be achieved by decorating the tiles with points appropriately.





More formally, a *Delone set* (in $\mathbb{R}^2$) is a set $P$ that is

- uniformly discrete. I.e., there is $r > 0$ such that every ball of radius $r$ contains at most one point of $P$
- relatively dense. I.e., there is $R > 0$ such that every ball of radius $R$ contains at least one point of $P$




Here we assume we can freely switch between tilings and Delone sets.

# 2. Tiling dynamical systems of aperiodic tilings and Delone sets

Given a tiling $P$ (or a Delone set $P$) we can define a (topological) dynamical system $(X_P, G)$, where

- $G$ is a group acting on $P$
- $X_P = \overline{\{gP \mid g \in G\}}$ (the *hull* of $P$, aka tiling space)

$G$ can be for instance the group of all translations in $\mathbb{R}^2$, or the group of all (rigid) Euclidean motions $E(2) = SO(2) \rtimes \mathbb{R}^2$.

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The closure $\overline{\cdot}$ is taken in an appropriate topology.

E.g. the *local matching topology*; given by the metric

$$d_{LM}(P,P') := \min\Big\{ \tfrac{1}{\sqrt{2}}, \inf\big\{\varepsilon > 0 \,|\, \exists x, x' \in B_\varepsilon(0) \text{ such that } (xP) \cap B_{1/\varepsilon}(0) = P' \cap B_{1/\varepsilon}(0) \text{ and } P \cap B_{1/\varepsilon}(0) = (x'P') \cap B_{1/\varepsilon}(0)\big\}\Big\}.$$

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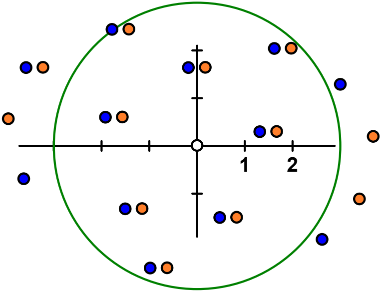


Here $d_{LM}(P, P') < \frac{1}{3}$.

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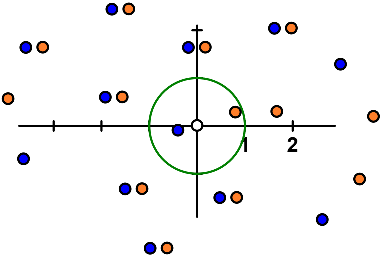


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# 3. Results on dynamical systems from substitution tilings

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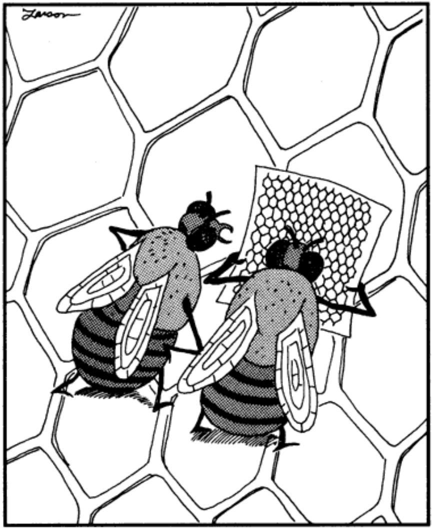
**Goal:** Relate geometric properties of the tiling (or Delone set) $P$ with properties of the dynamical system $(X_P, G)$.

**Geometric properties** can be:

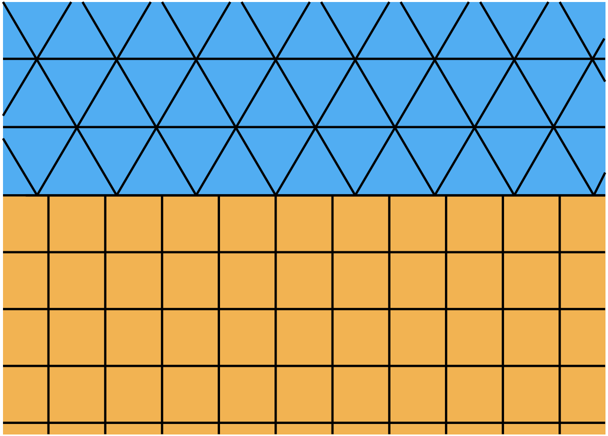
- $P$ is of *FLC* w.r.t $G$: (finite local complexity) for each $r > 0$ there are only fintely many $r$-patches (w.r.t. to $G$ actions)
- *Repetitive*: For each $r$ there is $R$ such that each $R$-patch contains copies (i.e, translates, or congruent copies) of each $r$-patch.
- *UPF*: (Uniform patch frequency) Well defined frequencies of patches (independent of averaging sequence)

**Example:** FLC, and repetitive, and UPF:



"Face it, Fred—you're lost!"

**Example:** Neither FLC, nor repetitive, nor UPF:



**Goal:**

Relate geometric properties of $P$ with ergodic properties of $X_P$.

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- *UPF*: (Uniform patch frequency) Well defined frequencies of patches (independent of averaging sequence)

- $X_P$ *compact* (w.r.t. local matching topology)
- $(X_P, \mathbb{R}^2)$ *minimal*. I.e., for every $P$ the orbit $\{P + t \mid t \in \mathbb{R}^2\}$ is dense in $X_P$.
- $(X_P, \mathbb{R}^2)$ *uniquely ergodic*. I.e., there is a unique $\mathbb{R}^2$-invariant probability measure on $X_P$.

**Classical results:** (many due to Boris Solomyak)

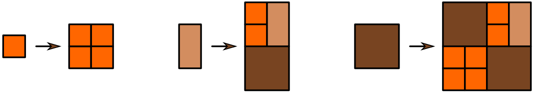
W.r.t. $\mathbb{R}^2$-actions:

- $P$ FLC $\Rightarrow$ $X_P$ compact (Radin-Wolff 92?)
- $P$ repetitive $\Leftrightarrow (X_P, \mathbb{R}^2)$ minimal (Solomyak 97?) (essentially Gottschalk's Theorem)
- $P$ UPF $\Rightarrow (X_P, \mathbb{R}^2)$ uniquely ergodic (Solomyak 97)
- $P$ UPF $\Leftrightarrow (X_P, \mathbb{R}^2)$ uniquely ergodic (Lee-Moody-Sol. 02)

...or w.r.t. substitution tilings:

A substitution $\sigma$ is *primitive*, if for any tile $T$ there is $k \geq 1$ such that $\sigma^k(T)$ contains all tile types.

Nonprimitive substitution:



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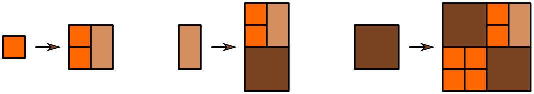

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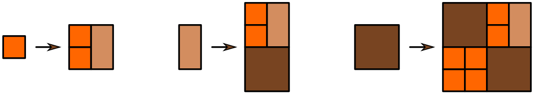
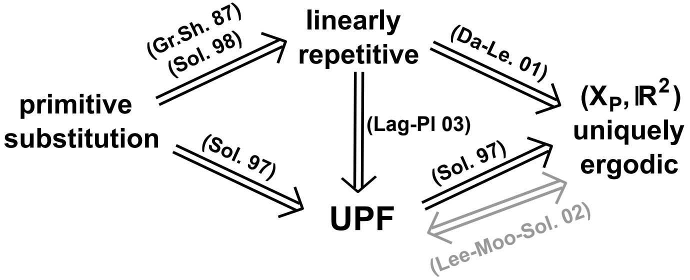


### Theorem (Solomyak 1997)

*Let $P$ be a primitive substitution tiling with FLC w.r.t. $G = \mathbb{R}^2$. Then $(X_P, \mathbb{R}^2)$ is uniquely ergodic.*

**Overview:** Classical case, $P$ FLC w.r.t. $\mathbb{R}^2$



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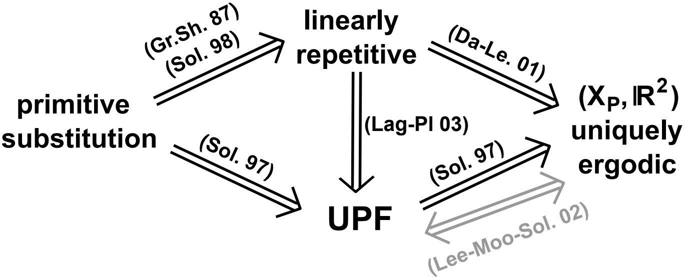


**Recall:**

*Repetitive*: For each $r$ there is $R$ such that each $R$-patch contains copies (i.e, translates, or congruent copies) of each $r$-patch.

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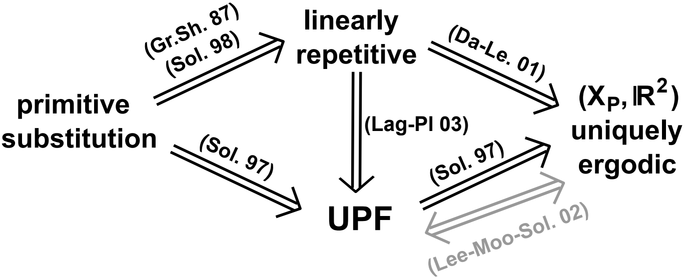


**Recall:**

*Repetitive*: For each $r$ there is $R$ such that each $R$-patch contains copies (i.e, translates, or congruent copies) of each $r$-patch.

*Linearly repetitive*: repetitive, and $R = O(r)$.
I.e., there are $a, b$ such that $R = ar + b$.

# 4. Tilings without FLC

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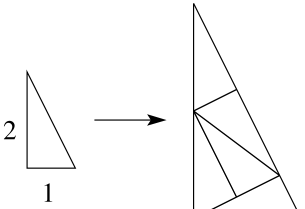
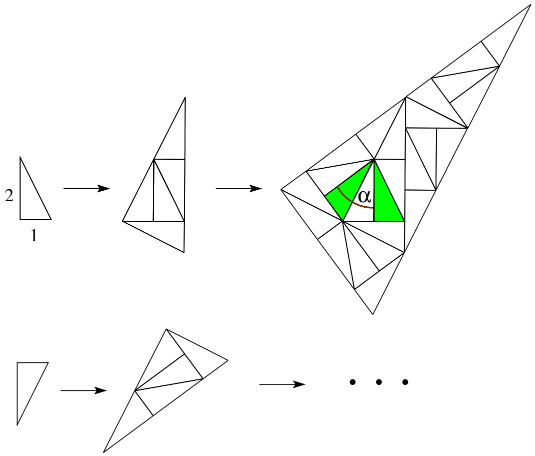
How can a pattern *not* have finite local complexity?

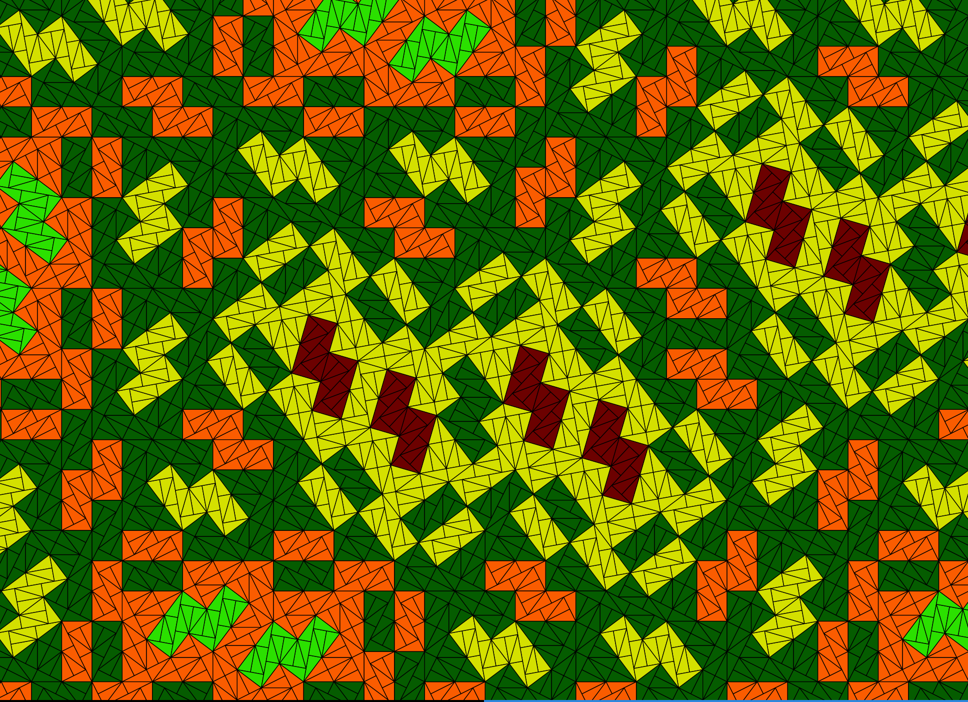
**Possibility 1**: Infinitely many orientations

E.g. Conway's and Radin's pinwheel substitution (1991):






The angle $\alpha$ is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.

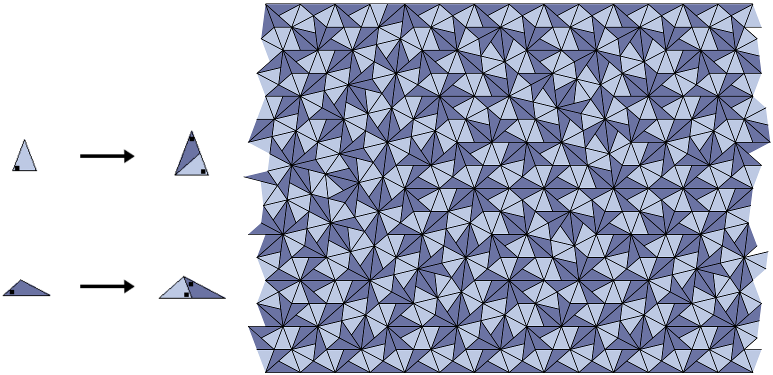


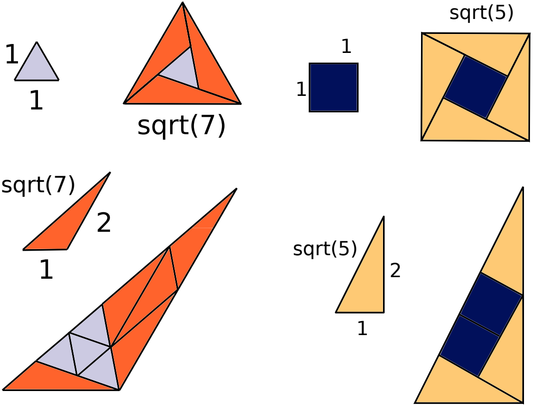
Here the orientations of the tilings are dense on the circle.

Whenever this happens we say the tiling has *dense tile orientations*

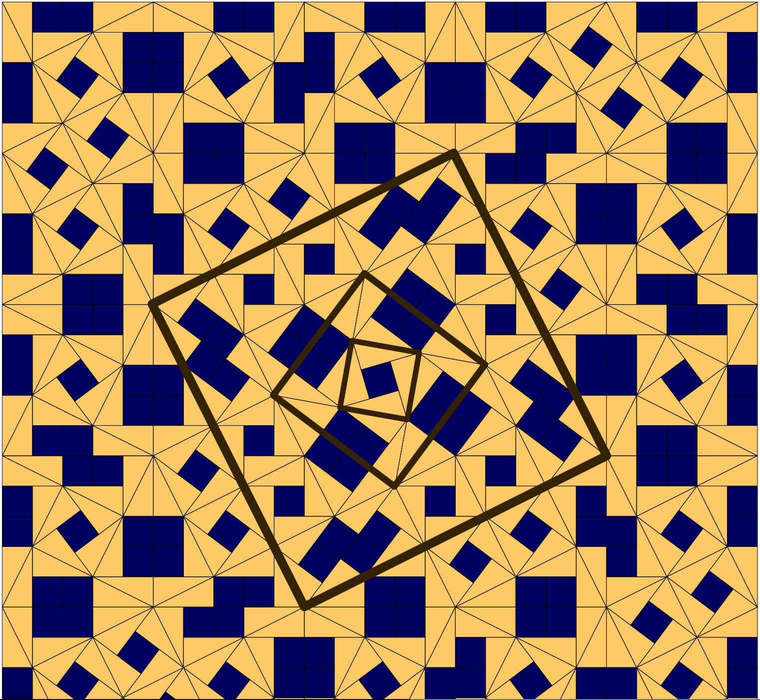
This happens a lot:

C. Goodman-Strauss, L. Danzer (ca. 1996):

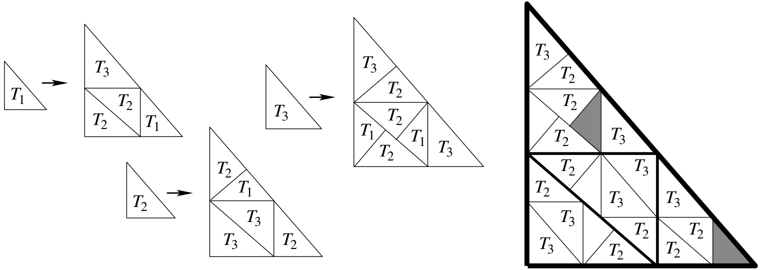




1
1
sqrt(7)
1
1
sqrt(5)
sqrt(7)
2
1
sqrt(5)
2
1

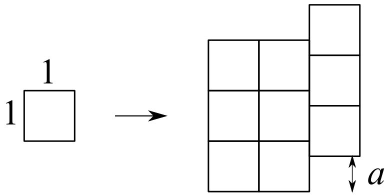


Pythia $(m, j)$, here: $m = 3$, $j = 1$.



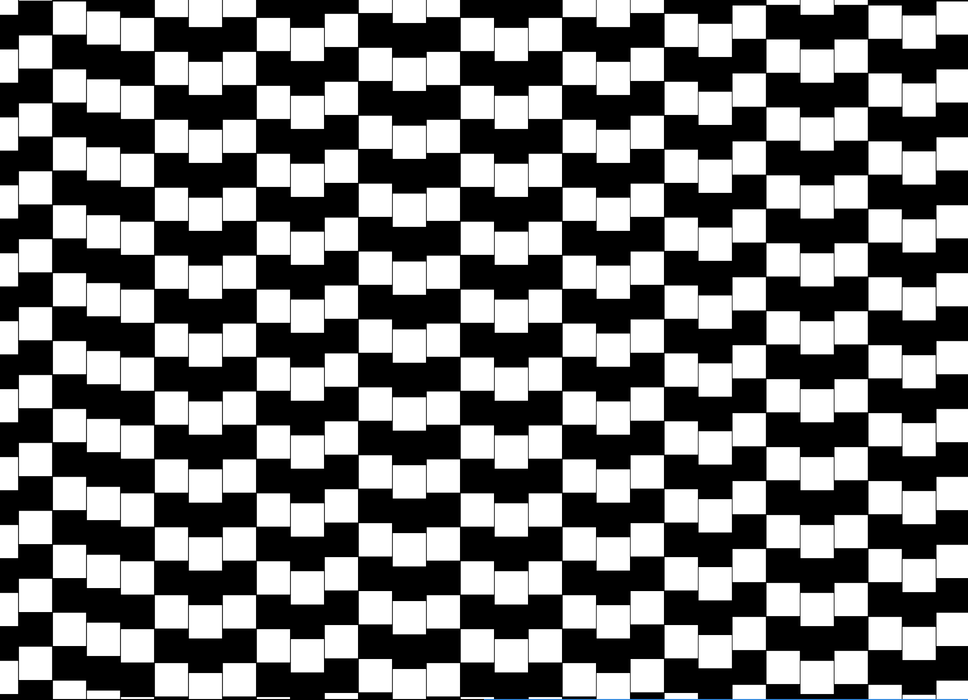

All these are not FLC w.r.t. $\mathbb{R}^2$. But all are FLC w.r.t. E(2).

**Possibility 2:** Fault lines

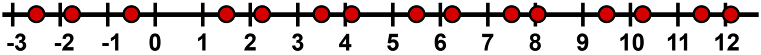



E.g. Kenyon's Non-FLC tiling:

If $a \notin \mathbb{Q}$ the corresponding tilings don't have FLC.



**Possibility 3:** Deformed lattices




Point $n \in \mathbb{Z} \setminus \{0\}$ is shifted by $2^{-k-1}$, if $n$ is divisible by $2^k$, and not by $2^{k+1}$

These Delone sets don't have FLC.

# 5. Results on dynamical systems from non-FLC tilings

In order to generalize the theorems above to the non-FLC tilings we have several choices:

**Recall:** The *hull* $X_P$ of a tiling $P$: closure of the $G$-orbit $GP$.

E.g. $G = \mathbb{R}^2$ (translations), or $G = E(2)$ (Euclidean motions)

Note that the pinwheel tiling (and its relatives) do not have FLC w.r.t. $\mathbb{R}^2$, but they *have* FLC w.r.t. $E(2)$.

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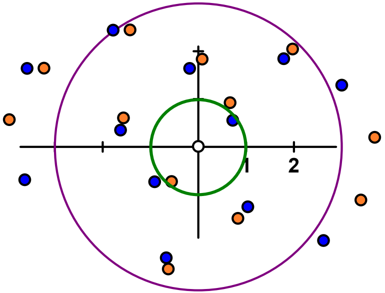
'Closure' w.r.t. an appropriate topology, e.g.

- local matching topology
- wiggle topology
- local rubber topology

- *local matching topology*: $P$ and $P'$ are $\varepsilon$-close: $P+x$ and $P'+y$ agree on $B_{1/\varepsilon}(0)$ for $|x|,|y|<\varepsilon$.
- *wiggle topology*: $P$ and $P'$ are $\varepsilon$-close: $R_\alpha P+x$ and $P'+y$ agree on $B_{1/\varepsilon}(0)$ for $|x|,|y|<\varepsilon$, with $R_\alpha$ a rotation by $|\alpha|<\varepsilon$.
- *local rubber topology*: $P$ and $P'$ are $\varepsilon$-close: $P$ and $P'$ agree on $B_{1/\varepsilon}(0)$, after moving each point *individually* by an amount $<\varepsilon$.




Here $d_{LM}(P,P')=\frac{1}{\sqrt{2}}$,
but $d_{LR}(P,P')<\frac{1}{3}$

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For primitive substitution tilings with FLC w.r.t. $\mathbb{R}^2$: All three metrics yield the same hull.

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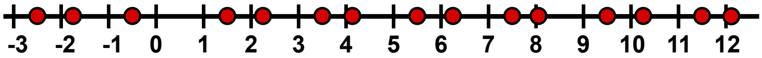
For primitive substitution tilings with FLC w.r.t. $\mathbb{R}^2$: All three metrics yield the same hull.

For primitive substitution tilings with FLC w.r.t. $E(2)$ (e.g. the pinwheel): the last two metrics yield the same hull.

In the deformed lattice example (or its 2-dim counterpart)




Point $n \in \mathbb{Z} \setminus \{0\}$ is shifted by $2^{-k-1}$, if $n$ is divisible by $2^k$, and not by $2^{k+1}$

Here the first two metrics yield the same hull, the third metric yields a different hull.

The meaning of *repetitive* comes in three distinct flavours, too:

A Delone set $P$ is...

- *Repetitive*, if for each $r > 0$ there is $R > 0$ such that each $R$-patch contains a translate of each $r$-patch of $P$
- *wiggle-repetitive*, if for each $r > 0$ and for each $\varepsilon > 0$ there is $R > 0$ such that each $R$-patch contains a translate of each $r$-patch of $P$, up to a rotation by $\alpha < \varepsilon$
- *almost repetitive*, if for each $r > 0$ and for each $\varepsilon > 0$ there is $R > 0$ such that each $R$-patch contains an $\varepsilon$-similar version of each $r$-patch

$\varepsilon$-similar: congruent after shifting each point individually by some distance $t < \varepsilon$

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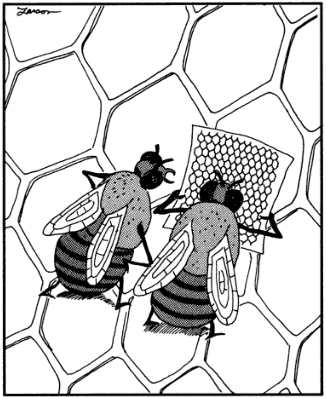
All three have a *linear* version, too: linearly repetitive, linearly wiggle-repetitive, linearly almost repetitive.




"Face it, Fred—you're lost!"

- repetitive: you are lost even if you have a compass
- wiggle-repetitive: you are lost only if you have no compass
- almost repetitive: you are lost only if you have bad eyesight

## Theorem (F-Richard 2014)

*Let $P$ be a primitive substitution tiling in $\mathbb{R}^2$ (a nice one, say, with convex tiles). Let $P$ have FLC w.r.t. $E(2)$. Then*

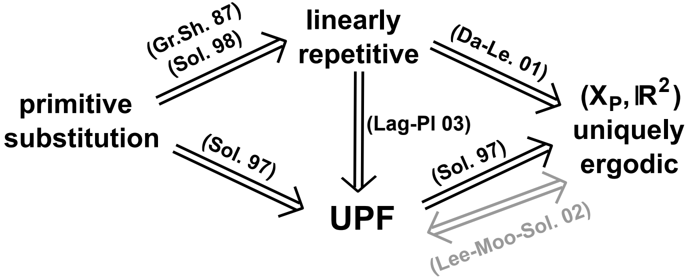
- *$P$ is linearly wiggle-repetitive*
- *$X_P$ is minimal*

## Theorem (F-Richard 2014)

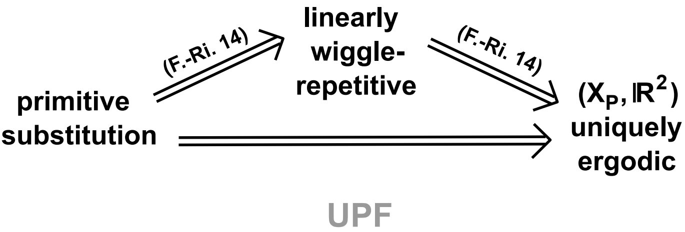
*Let $P$ be almost linearly repetitive. Then*

- *$(X_P, G)$ is minimal for both $G = \mathbb{R}^2$ and $G = E(2)$*
- *$(X_P, G)$ is uniquely ergodic for both $G = \mathbb{R}^2$ and $G = E(2)$*

**Recall** the classical case: $P$ FLC w.r.t. $\mathbb{R}^2$



Now we also know: $P$ FLC w.r.t. $E(2)$



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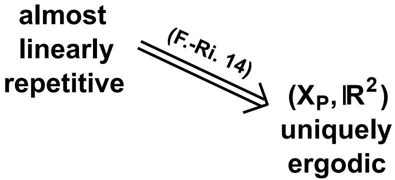
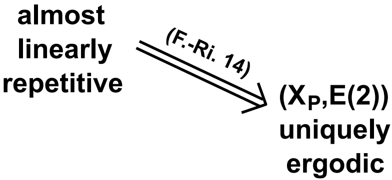
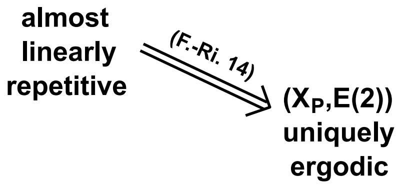


Now we also know: $P$ FLC w.r.t. $E(2)$

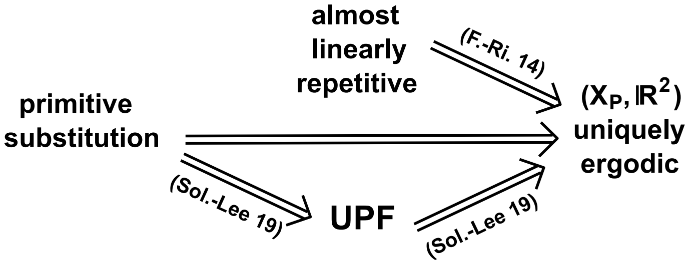


Moreover: $P$ not FLC at all

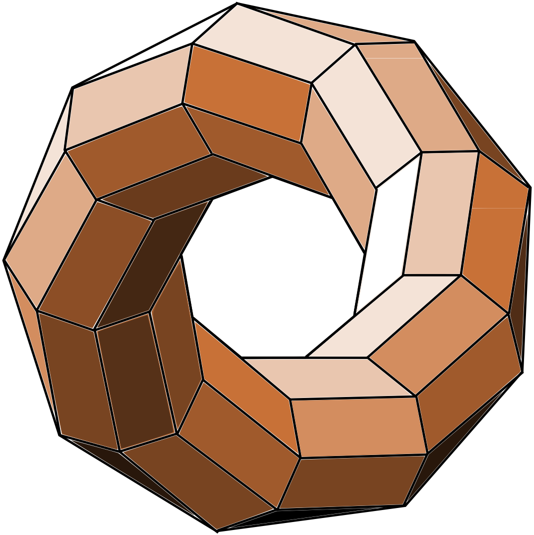



(Does not cover Kenyon's non-FLC example)

Moreover: $P$ not FLC at all




(*Does* cover Kenyon's non-FLC example)



THANK YOU!